Wigner Measures and Effective Mass Theorems

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Murramarang, March 2018

Content:

The equation:

$$i\partial_t\psi^\varepsilon + \frac{1}{2}\Delta_x\psi^\varepsilon - \frac{1}{\varepsilon^2}V_{\rm per}\left(\frac{x}{\varepsilon}\right)\psi^\varepsilon - V(x)\psi^\varepsilon = 0, \ \ (t,x)\in\mathbb{R}\times\mathbb{R}^d,$$
$$\psi^\varepsilon_{|t=0} = \psi^\varepsilon_0 \in L^2(\mathbb{R}^d),$$

where $V_{\rm per}$ is a potential periodic with respect to $\mathbb{Z}^d$,

In this scaling (see [Poupaud & Ringhofer 96]), ε is the ratio between the mean spacing of the lattice and the scale of variation of the external potential.

$$\varepsilon \ll 1.$$

For $0<a<b$, $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^d)$, describe the limit as $\varepsilon\to 0$ of

$$\int_a^b\int_{\mathbb{R}^d}\varphi(x)|\psi^\varepsilon(t,x)|^2dxdt.$$

Schrödinger Equation in a Lattice : The equation

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Schrödinger Equation in a Lattice : Effective mass theory

Effective Mass Theory consists in showing situations where $\psi^\varepsilon(t)$ can be approximated by the solution of a Effective Mass Equation:

$$i\partial_t \phi(t,x) + \frac{1}{2}\langle M\, D_x, D_x\rangle \phi(t,x) - V(x)\phi(t,x) = 0.$$

where M is a $d \times d$ matrix called the effective mass tensor.

In the sense that

$$\int_a^b \int_{\mathbb{R}^d} \varphi(x) |\psi^\varepsilon(t,x)|^2 dt \sim \int_a^b \int_{\mathbb{R}^d} \varphi(x) |\phi(t,x)|^2 dt dx.$$

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Schrödinger Equation in a Lattice : Effective mass theory

$\Longrightarrow$ In the literature, two questions have been extensively addressed :

- Finding initial conditions for which the above analysis holds,
- Clarifying the dependence of M on the sequence of initial data.

$\Longrightarrow$ [Bensoussan, Lions & Papanicolaou 78], [Poupaud & Ringhofer 96], [Allaire & Piatniski 05], [Hoefer &Weinstein 11], [Barletti & Ben Abdallah 11].

- The semiclassical Schrödinger eq. has been widely studied in the limit $\varepsilon \to 0$

$$\begin{cases} i\varepsilon\partial_t v^\varepsilon(t,x) + \frac{\varepsilon^2}{2}\Delta_x v^\varepsilon(t,x) - V_{\rm per}\left(\frac{x}{\varepsilon}\right) v^\varepsilon(t,x) - \varepsilon^2 V(x) v^\varepsilon(t,x) = 0, \\ v^\varepsilon|_{t=0} = \psi_0^\varepsilon. \end{cases}$$

[Gérard], [GMMP] [Poupaud & Ringhofer], [Bechouche, Mauser & Poupaud], [Spohn & Teufel] [Panati, Spohn &Teufel], [Dimassi, Guillot & Ralston] [Allaire &Palombaro] [Carles &Sparber]

- Both problems are related through: $\psi^\varepsilon(t,x) = v^\varepsilon\left(\frac{t}{\varepsilon}, x\right)$.
  $\Longrightarrow$ Perform simultaneously the s.c. limit $\varepsilon \to 0$ with the limit $t/\varepsilon \to +\infty$.
  [Macia and his collaborators Anantharaman, Léautaud, Rivière, C.F.K.].

- Our goal : apply the aforementioned viewpoint to effective mass theory.
  $\Longrightarrow$ a generalized effective mass equation of Heisenberg type.

Schrödinger Equation in a Lattice : our purpose

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Floquet-Bloch theory : Bloch waves and energies

- We use the Ansatz : $\psi^\varepsilon(t,x) = U^\varepsilon\left(t, x, \frac{x}{\varepsilon}\right)$, where $U^\varepsilon(t,x,y)$ is assumed to be $\mathbb{Z}^d$-periodic in y and solves

$$i\varepsilon^2\partial_t U^\varepsilon(t,x,y) = P(\varepsilon D)U^\varepsilon(t,x,y) + \varepsilon^2 V(x)U^\varepsilon(t,x,y), \quad U^\varepsilon|_{t=0} = \psi_0^\varepsilon$$

$$\text{where } P(\xi) = \frac{1}{2}(\xi + D_y)^2 + V_{per}(y), \;\; y \in \mathbb{T}^d := \mathbb{R}^d \backslash \mathbb{Z}^d.$$

- The Bloch energies are the eigenvalues of the self-adjoint operator on the torus $P(\xi)$:

$$\lambda_1(\xi) \leq \lambda_2(\xi) \leq \cdots \leq \lambda_n(\xi) \to +\infty.$$

They are $2\pi\mathbb{Z}^d$ periodic and smooth in domain where they are of constant multiplicity.

- The Bloch waves are the orthonormal eigenfunctions of $P(\xi)$

$$P(\xi)\varphi_n(\xi,y) = \lambda_n(\xi)\varphi_n(\xi,y), \;\; n \in \mathbb{N}, \;\; y \in \mathbb{T}^d, \;\; \forall \xi \in \mathbb{R}^d.$$

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Floquet Bloch theory : Bloch decomposition

- Consider $(\Pi_n(\xi))_{n\in\mathbb{N}}$ a family of projectors on Bloch bands and

$$U_n^\varepsilon(t,x) := \Pi_n(\varepsilon D_x) U^\varepsilon(t,x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \Pi_n(\varepsilon\xi) U^\varepsilon(t,w,y) \mathrm{e}^{i\xi\cdot(x-w)} \frac{dwd\xi}{(2\pi)^d} dy,$$

$$\textit{so that } U^\varepsilon(t,x,y) = \sum_{n\in\mathbb{N}} U_n^\varepsilon(t,x,y).$$

- This construction leads to the following representation formula for the solution of the Schrödinger equation

$$\psi^\varepsilon(t,x) = \sum_{n\in\mathbb{N}} U_n^\varepsilon\left(t,x,\frac{x}{\varepsilon}\right).$$

- If $\Pi_n(\xi)$ is the projector on the eigenspace of $\lambda_n(\xi)$, an isolated eigenvalue,

$$U_n^\varepsilon(t,\cdot) = u_n^\varepsilon(t,\cdot) + \mathcal{O}(\varepsilon|t|),$$

where u_n^ε solves

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Floquet Bloch theory : Well prepared initial data

Assume $V = 0$ and take

$$\psi_0^\varepsilon(x) = \varphi_n\left(\frac{x}{\varepsilon}\right) a_0^\varepsilon(x), \quad a_0^\varepsilon(x) = \mathrm{e}^{\frac{i}{\varepsilon} x\cdot\xi_0} a(x), \quad a \in \mathcal{S}(\mathbb{R}^d).$$

Then

- $U_0^\varepsilon(x,y) = \varphi_n(y) a_0^\varepsilon(x)$,
- $U^\varepsilon(t,x,y) = \varphi_n(y) u^\varepsilon(t,x)$, with

$$u^\varepsilon(t,x) = \mathrm{Exp}\left[\frac{it}{\varepsilon^2}\lambda_n(\varepsilon D)\right] a_0^\varepsilon(x).$$

Therefore, assuming $\nabla\lambda(\xi_0) = 0$,

$$\int_a^b \int_{\mathbb{R}^d} \varphi(x) |\psi^\varepsilon(t,x)|^2 dt \sim \int_a^b \int_{\mathbb{R}^d} \varphi(x) |\Phi(t,x)|^2 dtdx.$$

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Quantifying the lack of dispersion : a more general question

- Consider equations of the form

$$\begin{cases} i\varepsilon^2\partial_t u^\varepsilon(t,x) = \lambda(\varepsilon D_x)u^\varepsilon(t,x) + \varepsilon^2 V(x)u^\varepsilon(t,x), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^d, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon. \end{cases} \tag{1}$$

- This equation ceases to be dispersive as soon as $\lambda(\xi)$ has critical points $\xi \neq 0$, and this is always the case if λ is a Bloch energy.

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Assumptions:

H0 The sequence (u_0^ε) is uniformly bounded in $L^2(\mathbb{R}^d)$ and ε-oscillating :

$$\limsup_{\varepsilon\to 0^+} \int_{|\xi|>R/\varepsilon} |\widehat{u_0^\varepsilon}(\xi)|^2 d\xi \underset{R\to+\infty}{\longrightarrow} 0.$$

H1 $V \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $\lambda \in \mathcal{C}^\infty(\mathbb{R}^d)$ grows at most polynomially; *i.e.* there exist $C, N > 0$ such that:

$$|\lambda(\xi)| \leq C(1+|\xi|)^N, \quad \forall \xi \in \mathbb{R}^d.$$

H2 The set $\Lambda := \left\{\xi \in \mathbb{R}^d : \nabla\lambda(\xi) = 0\right\}$ is a submanifold of $\mathbb{R}^d$ of codimension $0 < p \leq d$ and the Hessian $\nabla^2\lambda$ is of maximal rank over Λ. Moreover, each connected component of Λ is compact.

Remark

If all critical points of λ are non-degenerate, then Λ is a discrete set in $\mathbb{R}^d$. If moreover one has that λ is $\mathbb{Z}^d$-periodic, this set is finite modulo $\mathbb{Z}^d$.

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Theorem (Obstruction to smoothing effects in presence of critical points)

Assume H0 & H1 and that all critical points of λ are non-degenerate.
Then there exists a subsequence $(u_0^{\varepsilon_k})$ such that $\forall a < b$ and $\forall \phi \in \mathcal{C}_c(\mathbb{R}^d)$:

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^d} \phi(x)|u^{\varepsilon_k}(t,x)|^2 dxdt = \sum_{\xi\in\Lambda} \int_a^b \int_{\mathbb{R}^d} \phi(x)|u_\xi(t,x)|^2 dxdt,$$

where u_ξ solves the Schrödinger equation:

$$i\partial_t u_\xi(t,x) = \nabla^2\lambda(\xi) D_x \cdot D_x u_\xi(t,x) + V(x) u_\xi(t,x),$$

with initial data $u_\xi|_{t=0}$ which is the weak limit in $L^2(\mathbb{R}^d)$ of $\left(\mathrm{e}^{-i\xi/\varepsilon_k\cdot x} u_0^{\varepsilon_k}\right)$.
If $\Lambda = \emptyset$ then the right-hand side above is equal to zero.

Example : If $u_0^\varepsilon(x) = \frac{1}{\varepsilon^{d/4}}\rho\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right)\mathrm{e}^{i\xi_0/\varepsilon\cdot x}$, then $u_\xi = 0$ for all ξ and the Theorem yields that (u^ε) converge to zero in $L^2_{\rm loc}(\mathbb{R}\times\mathbb{R}^d)$.

Quantifying the lack of dispersion : non-degenerate case

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Quantifying the lack of dispersion : degenerate case

- When the non-degeneracy of the critical points is replaced by H2, we obtain a similar result which requires notations.
- Define the tangent bundle of Λ as the union of all tangent spaces to Λ,

$$T\Lambda := \{(z,\xi) \in \mathbb{R}^d \times \Lambda \,:\, z \in T_\xi\Lambda\}.$$

- The normal bundle of Λ is the union of linear subspaces normal to Λ:

$$N\Lambda := \{(y,\xi) \in \mathbb{R}^d \times \Lambda \,:\, y \in N_\xi\Lambda = (T_\xi\Lambda)^\perp\}.$$

  Every point $x \in \mathbb{R}^d$ can be uniquely written as $x = z + y$, where $z \in T_\xi\Lambda$ and $y \in N_\xi\Lambda$.
- Given a function $\phi \in L^\infty(\mathbb{R}^d)$, we write $m_\phi(z,\xi)$, where $z \in T_\xi\Lambda$, to denote the operator acting on $L^2(N_\xi\Lambda)$ by multiplication by $\phi(z+\cdot)$.
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  Every point $x \in \mathbb{R}^d$ can be uniquely written as $x = z + y$, where $z \in T_\xi \Lambda$ and $y \in N_\xi \Lambda$.
- Given a function $\phi \in L^\infty(\mathbb{R}^d)$, we write $m_\phi(z,\xi)$, where $z \in T_\xi \Lambda$, to denote the operator acting on $L^2(N_\xi \Lambda)$ by multiplication by $\phi(z + \cdot)$.
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Quantifying the lack of dispersion : degenerate case

- When the non-degeneracy of the critical points is replaced by H2, we obtain a similar result which requires notations.
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Quantifying the lack of dispersion : degenerate case

Theorem

Assume H0, H1 & H2*. Then there exist a subsequence* $(u_0^{\varepsilon_k})$*, a positive measure* $\nu \in \mathcal{M}_+(T\Lambda)$ *and a measurable fami. of s.-adj., positive, trace-class operators*

$$M_0 : T_\xi\Lambda \ni (z,\xi) \longmapsto M_0(z,\xi) \in \mathcal{L}^1_+(L^2(N_\xi\Lambda)), \quad \mathrm{Tr}_{L^2(N_\xi\Lambda)} M_0(z,\xi) = 1,$$

such that for every $a < b$ *and every* $\phi \in \mathcal{C}_c(\mathbb{R}^d)$ *one has:*

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^d} \phi(x)|u^{\varepsilon_k}(t,x)|^2 dxdt = \int_a^b \int_{T\Lambda} \mathrm{Tr}_{L^2(N_\xi\Lambda)} \left[m_\phi(z,\xi) M(t,z,\xi)\right] \nu(dz,d\xi)dt,$$

where $M(\cdot,z,\xi) \in \mathcal{C}(\mathbb{R}; \mathcal{L}^1_+(L^2(N_\xi\Lambda))$ *solves the following Heisenberg equation:*

$$i\partial_t M(t,z,\xi) + \left[\frac{1}{2}\Delta_{N_\xi\Lambda} + m_V(z,\xi), M(t,z,\xi)\right] = 0, \quad M|_{t=0} = M_0.$$

Quantifying the lack of dispersions : comments

- The measure ν and the family of operators $M_0(z,\xi)$, for $z \in T_\xi\Lambda$, only depend on the subsequence of initial data $(u_0^{\varepsilon_k})$.
- This result is the analogue of the previous Theorem : the sum over critical points is replaced by an integral with respect to a measure over $T\Lambda$, and the Schrödinger equation becomes a Heisenberg equation.
- When Λ is a set of isolated critical points: $T\Lambda = \{0\} \times \Lambda$ and the measure ν is simply

$$\nu = \sum_{\xi\in\Lambda} \alpha_\xi \delta_\xi, \quad \text{where} \quad \alpha_\xi = \| u_\xi|_{t=0} \|^2_{L^2(\mathbb{R}^d)}.$$

In addition, $N_\xi\Lambda = \mathbb{R}^d$ and $M(t,\xi)$ is the orth. proj. onto $u_\xi(t,\cdot)$.

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Strategy of the proof : phase space analysis

- Phase space analysis: Let $W(u^\varepsilon)$ be the Wigner transform of (u^ε),

$$W^\varepsilon(t,x,\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} \overline{u}^\varepsilon\left(t, x+\varepsilon\frac{v}{2}\right) u^\varepsilon\left(t, x-\varepsilon\frac{v}{2}\right) \mathrm{e}^{iv\cdot\xi} dv.$$

The Wigner transform plays the role of a generalised energy density since

$$|u^\varepsilon(t,x)|^2 = \int_{\mathbb{R}^d} W^\varepsilon(t,x,\xi)d\xi.$$

- Wigner measures of (u^ε) are positive measures $\mu(t)$ satisfying for some subsequence ε_k and for all $a<b$, $c\in\mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$\lim_{k\to\infty}\int_a^b\int_{\mathbb{R}^{2d}} c(x,\xi)W^{\varepsilon_k}(t,x,\xi)dxd\xi dt = \int_a^b\int_{\mathbb{R}^{2d}} c(x,\xi)\mu(t,dx,d\xi)dt.$$

- Besides, ε-oscillation $\Longrightarrow$

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Strategy of the proof : localisation of Wigner measures

Set for $\chi \in \mathcal{C}_0(\mathbb{R})$ and $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$I^\varepsilon(\chi, c) = \int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} \chi(t) c(x,\xi) W^{\varepsilon_k}(t,x,\xi) dx d\xi dt.$$

- Invariance of Wigner measure : Egorov's theorem $\Longrightarrow$

Proposition

Any μ_t is invariant by the flow $\phi_s^1 : (x,\xi) \mapsto (x + s\nabla\lambda(\xi), \xi)$.

- Localization of Wigner measures

Corollary

$$\mathrm{Supp}(\mu_t) \subset \{(x,\xi) \in \mathbb{R}^{2d}, \;\; \nabla\lambda(\xi) = 0\}.$$

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Strategy of the proof : Two scale observables

We add to the phase space $\mathbb{R}^{2d}$ a new variable $\eta \in \overline{\mathbb{R}^d}$.

[FK], [Nier], [Miller], [FK &Gérard], [Laser & Teufel], [Harris, Lukkarinen, Teufel & Theil], [Macia], [Anantharaman & Macia]

With $c = c(x, \xi, \eta) \in \mathcal{C}^\infty(\mathbb{R}^{3d})$ satisfying additional properties, which satisfy :

1. there exists a compact K such that for all $\eta \in \mathbb{R}^d$, $(x, \xi) \mapsto c(x, \xi, \eta)$ is a smooth function compactly supported in K;
2. there exists a function $c_\infty(x, \xi, \omega)$ defined on $R^{2d} \times \mathbf{S}^{d-1}$ and $R_0 > 0$ such that if $|\eta| > R_0$, then $c(x, \xi, \eta) = c_\infty(x, \xi, \eta/|\eta|)$.

Assume $\Lambda = \xi_0 + 2\pi\mathbb{Z}^d$. We associate with such c, the two-scale observable

$$c_\varepsilon^\sharp(x, \xi) = c\left(x, \xi, \frac{\xi - \xi_0}{\varepsilon}\right).$$

Remarks : 1) If $c \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, c is admissible.
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Strategy of the proof : Two scale Wigner measures

Theorem

There exist, $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0$, $\nu \in L^\infty(\mathbb{R}, \mathcal{M}^+(\mathbb{R}^d \times \mathbf{S}^{d-1})$, $\Phi \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^d))$ *such that*

$$I^{\varepsilon_n}(\chi, c^\sharp_{\varepsilon_n}) \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}} \chi(t)\left(a(x,\xi_0,D)\Phi(t),\Phi(t)\right)dt + \int_{\mathbb{R}} \chi(t)\langle a_\infty(\cdot,\xi_0,\cdot),\nu_t\rangle dt.$$

1. Φ *solves the effective mass equation*

$$i\partial_t\Phi = \mathrm{Hess}\,\lambda(\xi_0)D\cdot D\,\Phi + V_{ext}(x)\Phi, \quad \Phi(0) = \Phi_0,$$

   where Φ_0 *is a weak limit in* $L^2(\mathbb{R}^d)$ *of the sequence* $x \mapsto \mathrm{e}^{\frac{i}{\varepsilon}\xi_0\cdot x} u_0^\varepsilon(x)$.

2. ν^t *is invariant by the flow* $\phi^2_s : (x,\omega) \mapsto (x + s\mathrm{Hess}\,\lambda(\xi_0)\omega, \omega)$.

Corollary

If $\mathrm{Hess}\,\lambda(\xi_0)$ *is non degenerated, then* $\nu_t = 0$ *and* $\mu_t(x,\xi)\mathbf{1}_{\xi=\xi_0} = |\Phi(t,x)|^2 dx$.

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Back to effective mass theory : the assumptions

Let $I \subset \mathbb{N}$, a set of indices n such that the multiplicity of the Bloch energy $\lambda_n(\xi)$ is constant for every $\xi \in \mathbb{R}^d$

- Assume that $\mathrm{H2}$ holds for any λ_n, $n \in I$ and denote by Λ_n the set of critical points of λ_n.
- Assume $\psi_0^\varepsilon = \sum_{n \in I} \psi_{n,0}^\varepsilon, \quad \psi_{n,0}^\varepsilon = U_n^\varepsilon \left(0, x, \frac{x}{\varepsilon}\right)$, where $\widehat{U}_n^\varepsilon(0,\xi)$ is in the eigenspace of $\lambda_n(\xi)$.
- Assume that ψ_0^ε is strongly ε-oscillating :

$$\forall s \geq 0, \exists C_s > 0 : \quad \| \langle \varepsilon D_x \rangle^s \psi_0^\varepsilon \|_{L^2(\mathbb{R}^d)} \leq C_s, \quad \text{uniformly in } \varepsilon.$$

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Back to effective mass theory : the key points

Lemma (Control on the rests)

If ψ_0^ε is strongly ε-oscillating, then as N goes to $+\infty$,

$$\limsup_{\varepsilon\to 0} \sum_{n\in I, |n|>N} \| U_n^\varepsilon(0)\|_{H^s_\varepsilon(\mathbb{R}^d, H^s(\mathbb{T}^d))} \to 0.$$

Lemma (Control of the evolution of U^ε)

$$\|U^\varepsilon(t,\cdot)\|_{H^s_\varepsilon(\mathbb{R}^d;H^s(\mathbb{T}^d))} \leq \|U^\varepsilon(0)\|_{H^s_\varepsilon(\mathbb{R}^d;H^s(\mathbb{T}^d))} + C_s\varepsilon|t|,$$

Lemma (Restriction to the diagonal)

For $V \in L^2(\mathbb{R}^d_x; H^s(\mathbb{T}^d_y))$, set $L^\varepsilon V(x) := V\left(x, \frac{x}{\varepsilon}\right)$. Then L^ε maps $L^2(\mathbb{R}^d_x; H^s(\mathbb{T}^d_y))$ into $L^2(\mathbb{R}^d)$ and is uniformly bounded in ε.

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Back to effective mass theory : combining the key points

Lemma

Set $\psi_n^\varepsilon(t) = L^\varepsilon \Pi_{\lambda_n} U^\varepsilon(t)$, *then*

$$\limsup_{\varepsilon\to 0} \left\| \psi^\varepsilon(t) - \sum_{n=0}^{N} \psi_n^\varepsilon(t) \right\|_{L^2(\mathbb{R}^d)} \to 0.$$

$$\begin{aligned}
\|\psi^\varepsilon(t) - \sum_{n=0}^{N} \psi_n^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} &\leq \|L^\varepsilon U^\varepsilon(t) - \sum_{n=0}^{N} L^\varepsilon U_n^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} \\
&\leq \|U^\varepsilon(t) - \sum_{n=0}^{N} U_n^\varepsilon(t)\|_{L^2(\mathbb{R}^d, H^s(\mathbb{T}^d))} \\
&\leq \|U^\varepsilon(0) - \sum_{n=0}^{N} U_n^\varepsilon(0)\|_{L^2(\mathbb{R}^d, H^s(\mathbb{T}^d))} + C\varepsilon|t|
\end{aligned}$$

Back to effective mass theory : application of the Theorem

- $\psi^\varepsilon(t,x) \sim \sum_{n\in I,\, n\leq N} \psi^\varepsilon_n(t,x)$ with $\psi^\varepsilon_n(t) = L^\varepsilon \Pi_{\lambda_n} U^\varepsilon\,(t)$ and
  - $(\psi^\varepsilon_n(t))$ is ε-oscillating for all $t\in\mathbb{R}$.
  - ψ^ε_n solves

$$i\varepsilon^2\partial_t\psi^\varepsilon_n(t,x) = \lambda_n(\varepsilon D_x)\psi^\varepsilon_n(t,x) + \varepsilon^2 V(x)\psi^\varepsilon_n(t,x) + \varepsilon^2 f^\varepsilon_n(t,x),$$

with $\exists C>0,\ \ \forall t\in\mathbb{R},\ \ ||f^\varepsilon_n(t,\cdot)||_{L^2(\mathbb{R}^d)} \leq C\varepsilon.$

- There exist a subsequence ε_k such that, for every $a<b$, $\phi\in\mathcal{C}^\infty_0(\mathbb{R}^d)$,

$$\lim_{k\to\infty}\int_a^b\int_{\mathbb{R}^d}\varphi(x)|\psi^\varepsilon(t,x)|^2 dxdt = \sum_{n\in I,\, n\leq N}\int_a^b\int_{\mathbb{R}^{2d}}\varphi(x)\mu^n_t(dx,d\xi)dt,$$

where, for each $n\in\mathbb{N}$, the measures $\mu^n_t\in\mathcal{M}_+(\mathbb{R}^d\times\mathbb{R}^d)$ are Wigner measures of (ψ^ε_n).

Generalized effective mass Theorem

Theorem

Under the assumptions described above, $\exists (\varepsilon_k)_{k\in\mathbb{N}}$, $\nu_n \in \mathcal{M}_+(T^*\Lambda_n)$ *and a meas. fami. of self-adjoint, positive, trace-class operators*

$$M_{0,n} : T^*_\xi \Lambda_n \ni (z,\xi) \longmapsto M_{0,n}(z,\xi) \in \mathcal{L}^1_+(L^2(N_\xi\Lambda_n)), \quad \mathrm{Tr}_{L^2(N_\xi\Lambda_n)} M_{0,n}(z,\xi) = 1,$$

such that for every for every $a<b$ *and every* $\phi \in \mathcal{C}_c(\mathbb{R}^d)$ *one has:*

$$\lim_{k\to\infty} \int_a^b \int_{\mathbb{R}^d} \phi(x)|\psi^{\varepsilon_k}(t,x)|^2 dxdt$$

$$= \sum_{n\in I} \int_a^b \int_{T^*\Lambda_n} \mathrm{Tr}_{L^2(N_\xi\Lambda_n} \left[m_\phi(z,\xi) M_n(t,z,\xi) \right] \nu_n(dz,d\xi)dt,$$

with
$$\begin{cases} i\partial_t M_n(t,z,\xi) + \left[\frac{1}{2}\Delta_{N_\xi\Lambda_n} + m_V(z,\xi), M_n(t,z,\xi) \right] = 0, \\ M_n|_{t=0} = M_{0,n}. \end{cases}$$

Conclusion

- Second microlocalisation along Λ has led to a complete description of the mechanism for any (ε-oscillating) initial data.
- The analysis is not restricted to too "special" initial data (strong ε-oscillation).
- We have introduced a generalized effective mass equation with an operator-valued macroscopic item satisfying a Heisenberg equation (instead of a function satisfying a Schrödinger equation).
- Question : what to do for a Bloch band containing two eigenvalues which cross?
- Question : how to ameliorate the reconstruction?

Thank you for your attention !

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