Global stability problems in General Relativity

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Einstein vacuum equations

$$\mathrm{Ric}(g) + \Lambda g = 0.$$

- g: Lorentzian metric $(+---)$ on 4-manifold M
- $\Lambda \in \mathbb{R}$: cosmological constant

Examples.

- Minkowski ($\Lambda = 0$):

$$M = \mathbb{R}^4 = \mathbb{R}_t \times \mathbb{R}^3_x, \quad g = dt^2 - dx^2.$$

- Schwarzschild ($\Lambda = 0$) and Schwarzschild–de Sitter ($\Lambda > 0$):

$$M = \mathbb{R}_t \times I_r \times \mathbb{S}^2, \quad g = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 g_{\mathbb{S}^2},$$

where $f(r) = 1 - \frac{2M_\bullet}{r} - \frac{\Lambda r^2}{3}$, $M_\bullet > 0$ black hole mass.

Initial value problem for the Einstein vacuum equation

Data:

- Σ: 3-manifold
- h: Riemannian metric on Σ
- k: symmetric 2-tensor on Σ

Find spacetime (M, g), $\Sigma \hookrightarrow M$, solving $\mathrm{Ric}(g) + \Lambda g = 0$, with

$$h = -g|_\Sigma, \quad k = II_\Sigma.$$

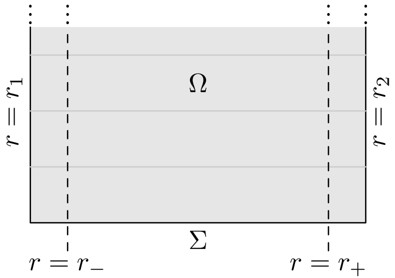
Theorem (Choquet-Bruhat '53)

Necessary and sufficient for local well-posedness: constraint equations for (h, k).

Key difficulty: diffeomorphism invariance $\Rightarrow$ need for gauge fixing

Kerr–de Sitter family: $\mathrm{Ric}(g) + \Lambda g = 0$, $\Lambda > 0$

- manifold: $M = [0, \infty)_t \times [r_1, r_2]_r \times \mathbb{S}^2$
- Cauchy surface: $\Sigma = \{t = 0\}$
- $\mathcal{C}^\infty$ family of stationary metrics g_b, $b = (M_\bullet, \vec{a}) \in \mathbb{R} \times \mathbb{R}^3$
  - $M_\bullet$: mass of the black hole
  - $\vec{a}$: angular momentum



Special case. $b_0 = (M_\bullet, \vec{0})$: Schwarzschild–de Sitter

Black hole stability ($\Lambda > 0$)

Theorem (H.–Vasy '16)

Given $\mathcal{C}^\infty$ *initial data* (h, k) *on* Σ

- *satisfying the constraint equations,*
- *close (in* H^{21}*) to the initial data induced by* g_{b_0}*,*

there exist

- *a* $\mathcal{C}^\infty$ *metric* g *on* M *solving* $\mathrm{Ric}(g) + \Lambda g = 0$ *with initial data* (h, k) *at* Σ*,*
- *parameters* $b \in \mathbb{R}^4$ *close to* b_0 *such that*

$$g = g_b + \widetilde{g}, \quad |\widetilde{g}| = \mathcal{O}(e^{-\alpha t}), \ \alpha > 0.$$

Exponential decay towards a Kerr–de Sitter solution!

Setup for Minkowski stability: $\mathrm{Ric}(g) = 0$ $(\Lambda = 0)$

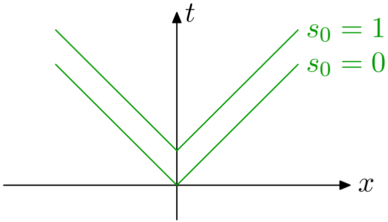
Minkowski spacetime.

$$M = \mathbb{R}^4 = \mathbb{R}_t \times \mathbb{R}^3_x, \quad g = g_{\mathrm{Mink}} = dt^2 - dx^2.$$

Polar coordinates in $\mathbb{R}^3_x$: $g_{\mathrm{Mink}} = dt^2 - dr^2 - r^2\, g_{\mathbb{S}^2}$

Outgoing null direction: $\partial_t + \partial_r$

Light cones: $s_0 := t - r = \mathrm{const}$



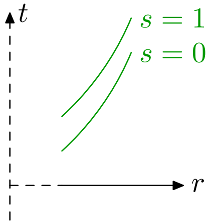
Schwarzschild spacetime: far field of system with mass $M_\bullet > 0$.

$$M = \{(t,x)\colon r = |x| \geq R\}, \quad R \gg 1;$$
$$g = g_{\mathrm{S}}^{M_\bullet} = (1 - \tfrac{2M_\bullet}{r})dt^2 - (1 - \tfrac{2M_\bullet}{r})^{-1}dr^2 - r^2 g_{\mathbb{S}^2}.$$

Perturbation of Minkowski: $g_{\mathrm{Mink}} - g_{\mathrm{S}}^{M_\bullet} = \mathcal{O}(r^{-1})$.

Outgoing null direction: $\partial_t + (1 - \frac{2M_\bullet}{r})\partial_r$

Light cones: $s := t - r_* = \text{const}$, $r_* \approx r - 2M_\bullet \log r$



Stability of Minkowski space ($\Lambda = 0$)

Theorem (H.–Vasy '17; Christodoulou–Klainerman '93, Klainerman–Nicolò '03, Lindblad–Rodnianski '05, '10, Bieri '09, Lindblad '17, ...)

Given: $M_\bullet \in \mathbb{R}$ *and smooth data* (γ, k) *on* $\Sigma = \mathbb{R}^3$ *such that:*

$$\begin{aligned} |\gamma - \gamma_{\mathrm{S}}^{M_\bullet}| &\le \delta(1+r)^{-1-b}, \quad r \ge 1, \\ |k - k_{\mathrm{S}}^{M_\bullet}| &\le \delta(1+r)^{-2-b}, \\ |M_\bullet| &\le \delta, \end{aligned}$$

and $|\gamma - \gamma_{\mathrm{Mink}}| \le \delta$, $r \le 1$, *where* $\delta > 0$ *small,* $b > 0$ *fixed.*

Then: $\exists$ *geodesically complete solution* g *of the IVP for* $\mathrm{Ric}(g) = 0$ *on* $M = \mathbb{R}^4$,

$$|g - g_{\mathrm{Mink}}| \lesssim \delta(1 + |t| + r)^{-1+\epsilon} \quad \forall\, \epsilon > 0,$$

with a precise asymptotic description on a compactification of $\mathbb{R}^4$.

Related work

Non-linear stability:

- de Sitter: Friedrich ('80s), Ringström ('08), ...
- Hyperbolic space: Graham–Lee ('91), ...
- Schwarzschild+symmetry: Klainerman–Szeftel (in progress)

Linear (mode) stability of black holes:

- $\Lambda > 0$: Kodama–Ishibashi ('04)
- $\Lambda = 0$: Regge–Wheeler ('57), Whiting ('89), Shlapentokh-Rothman ('14), Dafermos–Holzegel–Rodnianski ('16)

Compactification and precise asymptotics on Minkowski:

- Wang '14
- Baskin–Vasy–Wunsch '15, '16

Gauge fixing: generalities

Goal: Solve $\mathrm{Ric}(g) + \Lambda g = 0$.

DeTurck: Demand that $\mathbf{1} : (M, g) \to (M, g_{b_0})$ be a wave map

$$\iff W(g) = \begin{pmatrix} \text{1-form only} \\ \text{involving } g, \partial g \end{pmatrix} = 0.$$

Reduced Einstein equation:

$$(\mathrm{Ric} + \Lambda)(g) - \delta_g^* W(g) = 0. \qquad (*)$$

- $(h, k) \mapsto$ Cauchy data for g in (*) with $W(g) = 0$ at Σ.
- Solve (*) (quasilinear wave equation) $\implies \mathcal{L}_{\partial_t} W(g) = 0$ at Σ.
- Second Bianchi: $\delta_g G_g \delta_g^* W(g) = 0$. Wave equation!

$\implies W(g) \equiv 0$, and $(\mathrm{Ric} + \Lambda)(g) = 0$.

Gauge fixing: generalities

Goal: Solve $\mathrm{Ric}(g) + \Lambda g = 0$.

Friedrich: Demand that

$$\iff W(g) = \begin{pmatrix} \text{1-form only} \\ \text{involving } g, \partial g \end{pmatrix} = -\theta.$$

Reduced Einstein equation:

$$(\mathrm{Ric} + \Lambda)(g) - \delta_g^*(W(g) + \theta) = 0. \qquad (*)$$

- $(h, k) \mapsto$ Cauchy data for g in (*) with $W(g) + \theta = 0$ at Σ.
- Solve (*) (quasilinear wave equation) $\implies \mathcal{L}_{\partial_t} W(g) = 0$ at Σ.
- Second Bianchi: $\delta_g G_g \delta_g^*(W(g) + \theta) = 0$. Wave equation!

$\implies W(g) + \theta \equiv 0$, and $(\mathrm{Ric} + \Lambda)(g) = 0$.

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- Solve (*) (quasilinear wave equation) $\Longrightarrow \mathcal{L}_{\partial_t} W(g) = 0$ at Σ.
- Second Bianchi: $\delta_g G_g \delta^* W(g) = 0$. Wave equation!

$\Longrightarrow W(g) \equiv 0$, and $(\mathrm{Ric} + \Lambda)(g) = 0$.

Constraint damping

Why choose $\delta^* := \delta^*_g + l.o.t.$ carefully in

$$P(g) := (\mathrm{Ric} + \Lambda)(g) - \delta^* W(g) = 0?$$

Numerical relativity.

- want: solution W of wave-type equation

$$\delta_g G_g \delta^* W = 0$$

decays when W is initially small. (Gundlach et al '05)

Constraint damping, $P(g) = (\mathrm{Ric} + \Lambda)(g) - \delta^* W(g)$

Show that one can choose δ^* so that:

$$\boxed{\text{Solutions } W \text{ of } \delta_g G_g \delta^* W = 0 \text{ decay (fast).}}$$

Analysis. Use $\approx$ Newton iteration, $D_g P(h) = -P(g)$, $g \to g + h$.

- Example 1 (Minkowski). If $(\mathrm{Ric} + \Lambda)(g) \approx 0$, then $\delta_g G_g \delta^*(D_g W(h)) \approx 0 \Rightarrow$ parts of h have better decay.
- Example 2 (black holes). Control of mode solutions: $(\mathrm{Ric} + \Lambda)(g) = 0$ and $D_g P(e^{ct} h(x)) = 0$, $\operatorname{Re} c \geq 0$

$$\Rightarrow \;\; D_g W(e^{ct} h(x)) \equiv 0 \Rightarrow D_g(\mathrm{Ric} + \Lambda)(e^{ct} h(x)) \equiv 0,$$

so $e^{ct} h(x)$ satisfies a geometric equation!

Black hole stability ($\Lambda > 0$)

Theorem

Given $\mathcal{C}^\infty$ initial data (h, k) on Σ close to the data of the Schwarzschild–de Sitter metric g_{b_0}, there exist

- *a $\mathcal{C}^\infty$ metric g on M solving $\mathrm{Ric}(g) + \Lambda g = 0$ with initial data (h, k) at Σ,*
- *parameters $b \in \mathbb{R}^4$ close to b_0 such that*

$$g = g_b + \widetilde{g}, \quad |\widetilde{g}| = \mathcal{O}(e^{-\alpha t}), \ \alpha > 0.$$

Use Newton iteration for IVP for

$$P(g) = (\mathrm{Ric} + \Lambda)(g) - \delta^* W(G) = 0.$$

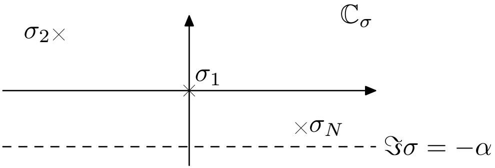
Solve linear equation globally at each step!

Linearization around $g = g_{b_0}$

Asymptotics for equation $Lh = D_g(\mathrm{Ric} + \Lambda)(h) - \delta^* W'(h) = 0$:

$$h = \sum_{j=1}^{N} h_j a_j(x) e^{-i\sigma_j t} + \widetilde{h}(t,x).$$

- $\sigma_j \in \mathbb{C}$ resonances (quasinormal modes)
- $a_j(x)e^{-i\sigma_j t}$ resonant states, $L(a_j(x)e^{-i\sigma_j t}) = 0$
- $h_j \in \mathbb{C}$, $\widetilde{h} = \mathcal{O}(e^{-\alpha t})$, $\alpha > 0$ fixed, small

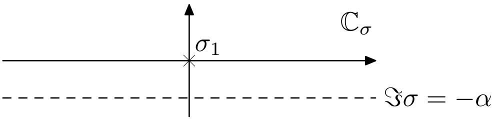


(Wunsch–Zworski '11, Vasy '13, Dyatlov '15, H. '15)

In an ideal world...

($Lh = 0$, $L = D_g(\mathrm{Ric} + \Lambda) - \delta^* W'$.)

Hope: $N = 1$, $\sigma_1 = 0$.



$$h = \frac{d}{ds} g_{b_0 + sb}\Big|_{s=0} + \widetilde{h}.$$

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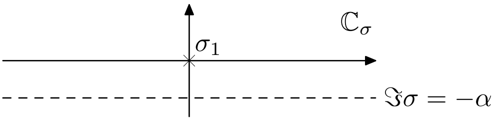

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Then: Could solve

$$(\mathrm{Ric} + \Lambda)(g_b + \widetilde{g}) - \delta^* W(g_b + \widetilde{g}) = 0$$

for $\widetilde{g} = \mathcal{O}(e^{-\alpha t})$, $b \in \mathbb{R}^4$. (H.–Vasy '16)

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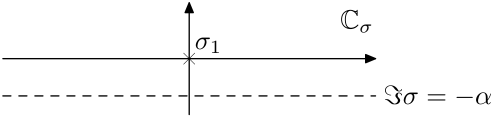

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Newton iteration; solve linearized equation globally at each step.

Automatically find final black hole parameters b and tail $\widetilde{g}$.

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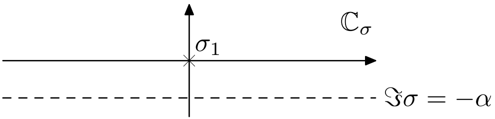

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Difficult to verify!

Dealing with reality...

$(Lh = 0,\ L = D_g(\mathrm{Ric} + \Lambda) - \delta^* W'.)$

Say $N = 2$, $\operatorname{Im} \sigma_2 > 0$, and (ignoring linearized KdS family)

$$h = a_2(x) e^{-i\sigma_2 t} + \widetilde{h}(t, x).$$

Recall: $a_2(x) e^{-i\sigma_2 t} \in \ker D_g(\mathrm{Ric} + \Lambda)$ (constraint damping!)

Kodama–Ishibashi '04 $\Rightarrow a_2(x) e^{-i\sigma_2 t} = \mathcal{L}_V g$ is pure gauge

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Kodama–Ishibashi '04 $\Rightarrow$ $a_2(x)e^{-i\sigma_2 t} = \mathcal{L}_V g$ is pure gauge, so $h = \mathcal{L}_{\chi V} g + \widetilde{h}$. Find equation for interesting bit $\widetilde{h}$:

$$L\widetilde{h} = -L(\mathcal{L}_{\chi V} g) = \delta^* \underbrace{W'(\mathcal{L}_{\chi V} g)}_{\theta}$$

Thus:

$$D_g(\mathrm{Ric} + \Lambda)(\widetilde{h}) - \delta^*\big(W'(\widetilde{h}) + \theta\big) = 0.$$

Improved setup

Then: Can solve

$$(\mathrm{Ric}+\Lambda)(g_b+\widetilde{g})-\delta^*\big(W(g_b+\widetilde{g})+\theta\big)=0$$

for $\widetilde{g}$ and (b,θ) (finite-dimensional parameters).

Nash–Moser iteration scheme:

- Solve linearized equation globally at each step.
- Use linear solutions to update $\widetilde{g}$, b and θ.

Automatically find

- final black hole parameters b,
- finite-dimensional modification θ of the gauge.

How to arrange constraint damping?

Theorem (H.–Vasy '16)

For g the Schwarzschild–de Sitter metric, one can choose δ^* such that $\delta_g G_g \delta^*$ has no resonances with $\operatorname{Im}\sigma \geq 0$. Take

$$\delta^* w = \delta_g^* w + \hbar^{-1}\, dt \otimes_s w - \tfrac{1}{2}\hbar^{-1} w(\nabla t) g, \quad 0 < \hbar \ll 1.$$

Idea of proof.

$$2\hbar^2 \delta_g G_g \delta^* \approx \hbar^2 \Box_g - i L_\hbar, \quad L_\hbar \approx \tfrac{\hbar}{i} \nabla_{\mathrm{grad}\ t}.$$

- High frequencies: propagation along null-geodesic flow
- Low frequencies: $L_\hbar - i\hbar^2 \Box_g$, transport along $\mathrm{grad}\ t$; exponential decay due to sign of subprincipal symbol of $L_\hbar$

Further applications of the global point of view

Non-linear stability of:

- slowly rotating Kerr–Newman–de Sitter black holes (H., '16)
- Minkowski space (H.–Vasy, '17)

Thank you!