The Mathematics of Fusion Plasmas


Plasma Theory and Modelling

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MSI Colloquium, 9 August 2018

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The Mathematics of Fusion Plasmas

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Plasma Theory & Modelling - *mathematics*

- Grad-Shafranov
- Beltrami fields
- Sand pile

- Gyrokinetic
- Full orbit particle dynamics
- Bayesian inference

- **Reduced Models (fluid)**
- **simulation**
- **Model/data fusion**

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**Electrical Engineering** (Power, rotating machines, signals and systems). *Hole*

**Physics** (Electrodynamics, Kinetic Theory, Fluid dynamics, Plasmas) *Hole, Dewar, Qu*

**Applicable Mathematics** (Calculus, PDEs, computation, Classical Mechanics, maths methods) *Hole, Dewar, Qu*
Plasma Theory & Modelling - *physics*

- MAST (UK) *compact*
- DIII-D (US)
- ITER (Earth) *ITPA*
- KSTAR (Korea) *superconducting*
- W7-X (Germany) *steady-state, reduced chaos*
- RFX-mod (Italy) *Self-organising*

**Stellarators**

**Basic Science**

**Data Fusion**

**Burning Plasma Physics / Multi-fluid models**

**Stellar dynamics**

**Bayesian inference**

- currents
- flux surfaces
Fusion, the power of the sun and the stars, is one option

“...Prometheus steals fire from the heaven”

**On Earth, fusion could provide:**

- Large-scale energy production
- Essentially limitless fuel, available all over the world
- No greenhouse gases
- Intrinsic safety
- No long-lived radioactive waste
Conditions for *terrestrial* fusion power

- Achieve sufficiently high ion temperature $T_i$
  - $\Rightarrow$ exceed Coulomb barrier
- Density $n_D \propto$ energy yield
- Energy confinement time $\tau_E$

$\tau_E = \text{insulation parameter: e.g. time taken for a jug of hot water to lose energy to the surroundings}$

- "Lawson" ignition criteria: Fusion power $> \text{heat loss}$
  - Fusion triple product $n_D \tau_E T_i > 3 \times 10^{21} \text{ m}^{-3} \text{ keV s}$

- Steady-state access requires confinement

$\approx 600 \text{ million } ^\circ\text{C}$
Toroidal Magnetic Confinement

• Magnetic fields cause charged particles to spiral around field lines. Plasma particles are lost to the vessel walls only by relatively slow diffusion across the field lines.

• Only charged particles ($D^+$, $T^+$, $He^+$...) are confined. Neutrons escape and release energy.

• Toroidal (ring shaped) device: a closed system to avoid end losses.
Mathematics of Magnetic Field Lines

- Motion on field lines is dynamical system
  \[ \dot{r} \equiv \frac{dr}{d\zeta} \propto B \] where \( \zeta \) is a toroidal angle
- System is Hamiltonian
- Analyze with flux-preserving return map to Poincaré section \( \zeta = 0 \):

Nonaxisymmetric (3-D) fields are not generically integrable, i.e. some points do not lie exactly on flux surfaces (e.g. KAM* invariant tori). Instead follow chaotic orbits in island separatrices.
*Kolmogorov, Arnol’d & Moser

\( t = \text{rotation number} \) where \( \zeta \) is a toroidal angle

\( t = \frac{\text{average rotation angle per return}}{2\pi} \)
To confine particles, constrain their position with a conservation law.

→ **Noether’s theorem:**

For each **continuous symmetry** of a system, there is a corresponding **conserved quantity**.

Emmy Noether (1882-1935)
Axisymmetry and Noether’s theorem is one way to achieve confinement

Continuous rotational symmetry \( \Rightarrow \) Canonical angular momentum conserved.

\[
L_\phi = mv_\phi R + qA_\phi R = \text{constant}
\]

Vector potential: \( \mathbf{B} = \nabla \times \mathbf{A} \)

Strong B limit \( \Rightarrow \) \(|mv_\phi| \ll |qA_\phi| \Rightarrow \) particles stuck to \( A_\phi R \) surfaces

If \( A_\phi R \) surfaces are bounded, then particles will be confined.
Complication: Axisymmetric confinement requires an internal current

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \text{nested } A_\phi R \text{ surfaces require a } J_\phi$$

- The most successful Magnetic Confinement device is the TOKAMAK (Russian acronym for ‘Toroidal Magnetic Chamber’).
Energy confinement: **big** is better

- $\tau_E$ empirical scaling

$$\tau_E = f\left(H_H, I_p, B_T, n, P_H, R_m, \kappa, M, a / R \right) \propto I_p B_T^{0.15} R_m^2 P_H^{-2/3}$$

- **"Breakeven" regime:**
  $Q = P_{\text{out}} / P_{\text{heat}} = 1$
  Eg. JET
  $Q=0.7$, 16.1MW fusion

- **"Burning" regime:** 
  ITER
  $Q>5 \Rightarrow$ ITER
  $D^2 + T^3 \rightarrow \text{He}^4 (3.5 \text{ MeV}) + n^1 (14.1 \text{ MeV})$

- **"Ignition" regime, $Q \rightarrow \infty$:** Power Plant.
International Thermonuclear Experimental Reactor (ITER)

- Fusion power = 500MW
- Power Gain (Q) > 10
- Temperature ~ 100 million °C

- Growing Consortium

- Collaboration agreements with
  - International Atomic Energy Agency
  - CERN – world’s largest accelerator
  - Principality of Monaco
  - Australia 30/09/2016
  - Iran (4 July 2016, High-level Iranian delegation visits ITER worksite)

Cadarache, France

Construction +10 year operation cost ~$40 billion?
ITER in detail

<table>
<thead>
<tr>
<th>Total Fusion power</th>
<th>500MW</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minor (a), major (R) radius</td>
<td>2.0m, 6.2m</td>
</tr>
<tr>
<td>Ip, plasma current</td>
<td>15MA</td>
</tr>
<tr>
<td>Toroidal field @6.2m</td>
<td>5.3T</td>
</tr>
<tr>
<td>Plasma Volume</td>
<td>837 m³</td>
</tr>
<tr>
<td>Auxillary heating, current drive</td>
<td>73MW</td>
</tr>
</tbody>
</table>
However ... poloidal field can be also be outcome of geometry

e.g. Figure-8 Stellarator

Some advantages
• Very low (no) $J_\phi$: eliminates some plasma instabilities and disruption
• Intrinsically steady-state
Averaging over fast gyration, dynamics depend on $B$ through $|B|$

Lagrangian for particle in magnetic field: 

$$L = \frac{m}{2} \dot{x}^2 + qA \cdot \dot{x}$$

(neglect $E$)

Average over fast gyration, use angle coordinates:

$$L = \frac{mG^2\dot{\phi}^2}{2B^2} - \mu B + q\psi \dot{\theta} - q\chi \dot{\phi}$$

Only depends on $\theta$ and $\phi$ through $B=|B|$

If $\frac{\partial |B|}{\partial \phi} = 0$, then canonical angular momentum $\frac{\partial L}{\partial \dot{\phi}}$ would be conserved.
Quasi-symmetry: $|B|$ is symmetric

• Can you actually make a non-symmetric $B$ with symmetric $|B|$?

• Can you start with a vacuum field with $\nabla \times B = 0$ to eliminate most of the internal current in a plasma with finite pressure?

Yes!

….but the cost is (to date) engineering complexity
Magnetic field magnitude $|B|$ in Tesla

$\theta$ (short way around)

$\varphi$ (long way around)

HSX:
Helically Symmetric eXperiment
(University of Wisconsin)
Example of very nonaxisymmetric magnetic confinement: Wendelstein 7-X (Germany)

Oct 21, 2015
30 minute plasmas eventually

Electromagnetic coils

Magnetic field lines

Magnetic surfaces, plasma
Common: heterogeneous diagnostics

- Large number of *heterogeneous* diagnostics sampling *overlapping* parts of configuration and phase-space
- Diagnostic forward model is inter-dependent on physics model and other parameters.
  e.g. interferometer signal \( \Delta \phi \propto \int n_e dl \)

Some ITER diagnostics within the bioshield
Model/Data Fusion: Bayesian Inference of B, $\delta B$

Aim: Develop a probabilistic framework for validating equilibrium (magnetic force balance models) and mode structure

Motivation: data from multiple diagnostics with strong model dependency

E.g. Current tomography on MAST

\[
P(J|D) = P(D|J)P(J)/P(D)
\]

vector of beam currents

Data vector of Flux loops, pickup coils, MSI


ISL CG130047 $395k$ (2008-2012)
Virtual Control / Remote Participation

- Our group has virtual control infrastructure, access to data from MAST and JET tokamaks (UK, exps. worth ~ $1 billion), KSTAR tokamak (South Korea, ~$350m), NSTX-U (~US$300m)

- ITER scenarios (Integrated Modelling Analysis Suite)
- ITER is an Exascale data-class experiment: data acquisition systems are 50GB/s, with plasma durations targeting 400s and ~20 discharges per day
Characteristics of fusion physics

• Fusion plasma physics is now big science: the leading-edge fusion experiments are billion dollar class machines.

• Multi-scale:
  - Spatial: electron gyro-radius $5 \times 10^{-6}$ m $\rightarrow$ 10m (device scale) anisotropic
  - Temporal: electron gyration $4 \times 10^{-10}$s $\rightarrow$ $\tau_E \sim$ 3 seconds
  - Generates many expansion parameters for asymptotic analysis

• High dimensional phase-space: gyro-kinetic is 6D (3 spatial, 2 velocity, time).
• $n_i = 10^{20}$ m$^{-3}$ (ideal gas at 1atm, 0C is $2.7 \times 10^{25}$ m$^{-3}$)
• Strong nonlinearity
• Constraint and diagnostic model interdependency
• Machine design is a complicated optimisation problem (especially for stellarators)
Topical Research Fields

- Example of reduced modelling: magnetohydrodynamics
- 2D: Tokamak equilibrium model
- ND: Spectral theory
- ND: Computation and simulation
- 3D: Describing and optimising stellarator fields
Magnetohydrodynamics (MHD)

- Single conducting fluid:  \( \mathbf{J} = n_i Z e \mathbf{v}_i - n_e e \mathbf{v}_e \) \( \rho \approx m_i n_i \)
  \[ \mathbf{v} = \left( m_i \mathbf{v}_i + m_e \mathbf{v}_e \right) / \rho \approx \mathbf{v}_i \]
  \( p = p_i + p_e \)

- Continuity:
  \( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \)

- Momentum:
  \( \rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{J} \times \mathbf{B} \)

If \( d\mathbf{v}/dt = 0 \) \( \Rightarrow \) \( \mathbf{J} \times \mathbf{B} = \nabla p \)

- Generalised Ohm’s law:  \( \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} \)

- Maxwells equations, Adiabatic equation:
  \( \frac{p}{\rho^\gamma} = \text{const.} \),
What is $B$ for a stationary tokamak plasma?

(1) $\mathbf{J} \times \mathbf{B} = \nabla p$ ⇒\[ \begin{cases} \mathbf{B} \cdot \nabla p = 0 \Rightarrow \text{No pressure gradient along } \mathbf{B} \\ \mathbf{J} \cdot \nabla p = 0 \Rightarrow \text{Current flows within surfaces.} \end{cases} \]

(2) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

(3) $\nabla \cdot \mathbf{B} = 0$

Introduce poloidal magnetic flux function $\psi(R, Z)$ and co-ord. system $(R, \phi, z)$. In axisymmetry $z$ Eq. (1), (2) become Grad-Shafranov equation:

$$\nabla \cdot \frac{1}{R^2} \nabla \psi = -\frac{\mu_0 J_\phi}{R} = -\mu_0 p'(\psi) - \frac{\mu_0^2}{R^2} f(\psi)f'(\psi)$$

second order PDE for field and currents.

With $f(\psi)$ a toroidal flux function:

$$f(\psi) = RB_\phi(\psi, R)/\mu_0$$

• To solve: prescribe $p'(\psi), f(\psi)f'(\psi)$ and boundary

• Solve numerically by current-field iteration: compute $J_\phi$ → solve for $\psi(R, Z)$

separatrix
“MHD with anisotropy in velocity, pressure”

- Pressure different parallel and perpendicular to field due mainly to directed neutral beam injection

Illustrative 1D slice

\[ \lambda = \frac{v_\parallel}{v} \]

\[
\begin{align*}
n &= \int_{0}^{\infty} \int_{-1}^{1} \hat{f}(E, \lambda) \, d\lambda \, dE \\
n u_\parallel &= \int_{0}^{\infty} \int_{-1}^{1} v_\parallel \hat{f}(E, \lambda) \, d\lambda \, dE \\
p_\parallel &= m \int_{0}^{\infty} \int_{-1}^{1} (v_\parallel - u_\parallel)^2 \hat{f}(E, \lambda) \, d\lambda \, dE \\
p_\perp &= \frac{m}{2} \int_{0}^{\infty} \int_{-1}^{1} v_\perp^2 \hat{f}(E, \lambda) \, d\lambda \, dE.
\end{align*}
\]
MHD with rotation & anisotropy

- Inclusion of anisotropy and flow in equilibrium MHD equations

\[ \nabla \cdot (\rho \mathbf{v}) = 0, \quad \rho \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{P}, \quad \nabla \cdot \mathbf{B} = 0 \]

\[ \mu_0 \mathbf{J} = \nabla \times \mathbf{B}, \quad \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \]

\[ \bar{\mathbf{P}} = p_\perp \mathbf{I} + \Delta \mathbf{B} \mathbf{B} / \mu_0 , \quad \Delta = \frac{\mu_0 (p_\parallel - p_\perp)}{B^2} \]

- Frozen flux condition + axis-symmetry:

\[ \mathbf{v} = \frac{\psi'_M (\psi)}{\rho} \mathbf{B} - R\phi'_E (\psi) \mathbf{e}_\phi . \]

Generalised Grad-Shafranov:

\[ \nabla \cdot \left[ \tau \left( \frac{\nabla \psi}{R^2} \right) \right] = - \frac{\partial p_\parallel}{\partial \psi} - \rho H'_M (\psi) + \rho \frac{\partial W}{\partial \psi} - I'_M (\psi) \frac{I}{R^2} - \psi''_M (\psi) \mathbf{v} \cdot \mathbf{B} + R \rho \nu \phi'_E (\psi) \]

\[ I = RB_\varphi \]

\[ I_M (\psi) = \tau I - \mu_0 R^2 \psi'_M (\psi) \phi'_E (\psi) \]

\[ H_M (\psi) = W_M (\rho, B, \psi) - \frac{1}{2} \left[ R \phi'_E (\psi) \right]^2 + \frac{1}{2} \left[ \frac{\psi'_M (\psi) B}{\rho} \right]^2 , \quad \tau = 1 - \Delta - \mu_0 (\psi'_M)^2 / \rho , \]

- Implemented into EFIT TENSOR and HELENA+ATF

Set of 6 profile constraints

\[ \left\{ I_M (\psi), \psi_M (\psi), \phi_E (\psi), H_M (\psi), \frac{\partial p_\parallel}{\partial \psi}, \frac{\partial W}{\partial \psi} \right\} \]
PDE can be elliptic, hyperbolic or parabolic

- Expand the highest order derivative. If $\Delta = 0$, this is

$$\left( A_{RR} \frac{\partial^2 \psi}{\partial R^2} + A_{RZ} \frac{\partial^2 \psi}{\partial R \partial Z} + A_{ZZ} \frac{\partial^2 \psi}{\partial Z^2} \right)$$

$$A_{RR} = 1 - \frac{v_\theta^2 v_Z^2}{c_s^2 c_A \theta^2 - (c_s^2 + c_A^2) v_\theta^2 + v_\theta^4}, \quad A_{RZ} = \frac{2v_\theta^2 v_R v_Z}{c_s^2 c_A \theta^2 - (c_s^2 + c_A^2) v_\theta^2 + v_\theta^4}, \quad A_{ZZ} = 1 - \frac{v_\theta^2 v_R^2}{c_s^2 c_A \theta^2 - (c_s^2 + c_A^2) v_\theta^2 + v_\theta^4}$$

$$v_\theta^2 = v_R^2 + v_Z^2, \quad c_s = (\gamma p/\rho)^{1/2}, \quad c_A = B/(\mu_0 \rho)^{1/2}, \quad c_A \theta = B_\theta/(\mu_0 \rho)^{1/2}$$

Properties depend on discriminant, $D$

- $D<0$ : elliptic
- $D=0$ : parabolic
- $D>0$: hyperbolic

E.g. McClements and Hole, Phys. Plasmas 17, 082509 2010

Possible transonic (jump) equilibria

- Elliptic PDE : no characteristic curves $\rightarrow$ no information propagation
- Hyperbolic PDE : characteristic curves $\rightarrow$ admit wave like disturbance
Impact of anisotropy & flow on equilibrium

- **Plasma Configuration** (magnetic structure)
  - **EFIT TENSOR** reconstruction code: Adds physics of flow/ anisotropy and kinetic constraints
  - **HELENA+ATF** enables stability studies
  - What is the impact?
    - $p_{||}$, $p_{\perp}$, $\rho$ not a flux function
    - can modify rotational transform
    - if $p_{||} > p_{\perp}$, $p_{||}$ surfaces distorted and displaced inward relative to flux surfaces
    - if $p_{\perp} > p_{||}$, an increase will occur in centrifugal shift.
  - Impact on ITER scenarios: made progress in May 2018 visit (5 weeks) as part of ITER Science Fellowship
Tokamak Stability Zoo

A whole zoo of modes. Can divide them as:

- **Most-serious (disruptive):** e.g. external modes such as the \((n, m) = (1, 1)\) *external kink*, driven by gradients in pressure and current density

- **Serious but tolerable (performance-limiting):**
  - *Sawteeth*, internal kink, \((n, m) = (1, 1)\) – reconnection of core. Periodic collapse of central temperature
  - *Alfven eigenmodes*, wave-particle resonance driven. Loss of fast particle confinement
  - *Edge-Localised Modes (ELMs)*, which occur for moderately high \(m\) and \(n\).

  ELM mitigation / suppression demonstrated by application of resonant magnetic perturbation coils, that deliberately perturb edge
Linearisation - Spectral theory

• Formally, stability theory often conducted as a perturbation treatment:

\[
\begin{align*}
\mathbf{v}(r, t) &= \mathbf{v}_0(r) + \varepsilon \mathbf{v}_1(r, t) \\
\mathbf{B}(r, t) &= \mathbf{B}_0(r) + \varepsilon \mathbf{B}_1(r, t) \\
\rho(r, t) &= \rho_0(r) + \varepsilon \rho_1(r, t) \\
p(r, t) &= p_0(r) + \varepsilon p_1(r, t)
\end{align*}
\]

\( \varepsilon = \text{linear expansion parameter} \)

• Introduce a Lagrangian displacement vector field \( \xi \) of a plasma away from an equilibrium state

\[
\mathbf{v} = \frac{\text{D}\xi}{\text{D}t} = \frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \nabla \xi
\]

• Substitute into continuity, momentum, Faraday’s law (with ideal Ohm’s law), and equation of state

\[
\rho_0 \frac{\partial^2 \xi}{\partial t^2} = F(p_1(\xi), B_1(\xi), \rho_1(\xi))
\]

where \( F \) is the ideal MHD Force operator

\[
F(\xi) = -\nabla p_1 - \mathbf{B} \times (\nabla \times \mathbf{B}_1) + (\nabla \times \mathbf{B}) \times \mathbf{B}_1 + (\nabla \Phi)\nabla \cdot (\rho \xi)
\]

• It can be shown that \( F \) is self-adjoint, and so if \( \xi \) has a finite norm, solutions of \( \xi \) lie in a Hilbert space.
Spectral theory

- Equation of motion becomes: \( \rho^{-1} F(\xi) = \frac{\partial^2 \xi}{\partial t^2} \)
- Normal modes with exponential-time leads to spectral equation: \( \rho^{-1} F(\xi) = -\omega^2 \xi \)
- Stability problem reduces to a study of the marginal equation: \( \rho^{-1} F(\xi) = 0 \)
- Discretisation of equation of motion leads to: \( L \cdot x = \lambda x \)
- The spectrum of \( L \) obtained by study of the inhomogeneous equation
  \( (L - \lambda I) \cdot x = a \)

where \( a \) is a given vector in Hilbert space

For complex \( \lambda \), three possibilities exist

(\( L - \lambda I \))\(^{-1} \) does not exist as \( (L - \lambda I) \cdot x = 0 \) has a solution \( \Rightarrow \lambda \in \) discrete spectrum of \( L \)

(\( L - \lambda I \))\(^{-1} \) exists but is unbounded \( \Rightarrow \lambda \in \) continuous spectrum of \( L \)

(\( L - \lambda I \))\(^{-1} \) exists but is bounded \( \Rightarrow \lambda \in \) resolvent set of \( L \)

Schematic spectrum of MHD waves for a static 1D equilibrium, where \( \lambda = -\omega^2 \),

unstable
Continuum of a periodic cylinder (1.5D)

- Continuum resonance frequency versus radial position for a periodic cylinder.

$n = 1 \quad m = \text{azimuthal mode number}$

$m = 1 \quad m = 2 \quad m = 2 \quad m = 3 \quad m = 3$

Radial coordinate $(r)$
Continuum of a tokamak (2D)

- Continuum resonance frequency versus radial position for a large-aspect ratio elliptical cross-section tokamak.

\[ m = \text{poloidal mode number} \]
\[ n = 1 \quad n = \text{toroidal mode number} \]

Gaps due to poloidal variation in \( B \) arising due to ellipticity of cross-section


Ellipticity of cross-section

Toroidicity of plasma
• A zoo of gaps and gap modes (n=3)

Beta induced Alfven eigenmode (BAE): Low frequency mode that exists due to finite beta (pressure)

**Burning plasma physics**

**Motivation:**
1. Fast ions from auxiliary heating and fusion $\alpha$'s make the plasma non-Maxwellian (e.g. **anisotropy** and **plasma rotation**).
2. As the ions collisionally slow, they drive modes of the plasma.
3. At large amplitude, modes can eject energetic ions from plasma, short-circuit heating mechanism → prevent burn.

![Graph showing plasma physics](image)
Example: Bursty TAE modes in KSTAR

- Staggered early NBI heating
- Application of RMP coil (delayed onset of H mode, reduced TAE mode intensity)
• Mode chirps $\Delta \omega/\omega_0 \approx 25/133 = 0.19$ in $\Delta t \omega_A = 5500$ wave periods.

• Mode frequency scales with middle of $(m,n) = (4,2)$ TAE gap

• 1D analytic theory can not capture complicated 3D behaviour [Breizman. Nuc. Fus., 50:084014, 2010.]

\[ \delta \omega = \frac{16}{3\pi^2} \sqrt{\frac{2}{3}} \gamma_l \sqrt{\gamma_d} \delta t \]

• Mode at marginal stability ($\gamma_l = \gamma_d$) Calculation gives $\gamma_l/\omega = \gamma_d/\omega_0 = 3\%$
  Fit gives $\gamma_l/\omega = \gamma_d/\omega_0 = 5\%$
Detailed Numerical Modelling

- \( q \) profile at 640ms available from coherence imaging spectroscopy. [Howard et al, JI. Instrumentation, 10, 2015]
- Remapped to correct \( I_p \) at 400ms
- Continuum (CSCAS) and Global Modes (MISHKA)
Wave-particle drive: (non)linear dynamics

Evolution of unstable modes in realistic geometry and with simulated distribution functions uses drift-kinetic wave-particle interaction codes, such as HAGIS.

NCI installed simulation codes:
HAGIS: > 64 cores, ~2 hours :
MEGA: > 512 cores, ~6 hours
GENE: >2k nodes (~48k cores)
Simulation: HAGIS Wave Drive

- Lowest frequency mode unstable
- Wave saturates at $\delta B/B_0 \approx 2.5 \times 10^{-3}$. Coils measure $\delta B/B_0 \approx 10^{-4}$
- Mode growth phase = $\Delta t \omega_A / (2\pi) \sim 100 << \Delta t \omega_A = 5550$ (measured)
- Linear growth rate $\gamma/\omega \sim 3\%$
3D equilibria in toroidal plasmas

- Simplest model to approximate global, macroscopic force-balance is magnetohydrodynamics (MHD).

\[ \nabla p = J \times B, \quad \nabla \times B = J, \quad \nabla \cdot B = 0 \]

- Non-axisymmetric \( \Rightarrow \) field does not lie in nested flux surfaces \textbf{unless} surface currents allowed.
- Existing 3D solvers (e.g. VMEC) assume nested flux surfaces.

[CTH stellarator, Hanson et al, IAEA 2012]
Taylor Relaxed States: Relaxed MHD

- Zero pressure gradient regions are force-free magnetic fields:
- In 1974, Taylor argued that turbulent plasmas with small resistivity, and viscosity relax to a Beltrami field

Internal energy: \[ W = \int_{P \cup V} \left( \frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1} \right) d\tau^3 \]

Total Helicity: \[ H = \int_P (A \cdot B) d\tau^3 \]

Taylor solved for minimum \( W \) subject to fixed \( H \)

i.e. solutions to \( \delta F = 0 \) of functional \( F = W - \mu H / 2 \)

\( P : \) \[ \nabla \times B = \mu B \]

\( I : \) \[ \left[ \frac{B^2}{2\mu_0} + p \right] = 0 \]

\( V : \) \[ \nabla \times B = 0 \]

Model very successful for toroidal pinches, multipinch, and spheromaks

\[ \text{discretised } J \times B = \nabla p \]
Generalised Taylor Relaxation: MRxMHD

• Assume each invariant tori \( I_i \) act as ideal MHD barriers to relaxation, so that Taylor constraints are localized to subregions.

New system comprises:

- \( N \) plasma regions \( P_i \) in relaxed states.
- Regions separated by ideal MHD barrier \( I_i \).
- Enclosed by a vacuum \( V \),
- Encased in a perfectly conducting wall \( W \).

\[
W_i = \int_{R_i} \left( \frac{B_i^2}{2\mu_0} + \frac{P_i}{\gamma - 1} \right) d\tau^3
\]

\[
H_i = \int_V (A_l \cdot B_l) d\tau^3
\]

Seek minimum energy state:

\[
F = \sum_{l=1}^{N} \left( W_i - \mu_l H_i / 2 \right)
\]

\[
P_i : \quad \nabla \times \mathbf{B} = \mu_l \mathbf{B}
\]

\[
P_i = \text{constant}
\]

\[
I_i : \quad \mathbf{B} \cdot \mathbf{n} = 0
\]

\[
[[P_i + B^2 / (2\mu_0)]] = 0
\]

\[
V : \quad \nabla \times \mathbf{B} = 0
\]

\[
\nabla \cdot \mathbf{B} = 0
\]

\[
W : \quad \mathbf{B} \cdot \mathbf{n} = 0
\]
MRXMHD approaches ideal MHD as $N \to \infty$.

MRxMHD implemented in SPEC code

[Hudson et al Phys. Plasmas 19, 112502 (2012)]

- 3D boundary of a DIII-D plasma with $n=3$ applied error field.
- Irrational interfaces chosen to coincide with pressure gradients.
- Island formation is permitted
- No rational “shielding currents” included in calculation.

STELLOPT

$P(\psi)$

SPEC

formation of magnetic islands at rational surfaces
Spontaneously formed helical states

• The quasi-single helicity state is a stable helical state in RFP: becomes purer as current is increase

“Experimental” Poincaré plot

[Fig. 6 of P. Martin et al., Nuclear Fusion 49, 104019 (2009)]

• State can be described by a sequence of SPEC solutions, which are in a minimum energy state

Some recent ANU activity in MRxMHD

- Generalized straight field line coordinates concept to fully 3D plasmas
- Developed theory of resonant current sheet formation and reconnection.
- Developed techniques to establish pressure jump a surface can support.
  [M. McGann, ANU PhD thesis, 2013]
- Extended MRxMHD to include non-zero plasma flow, and anisotropy
- Generalize Taylor *minimum energy* equilibrium by *extremizing MHD action* (Hamilton’s Principle)

**Important for study of some modes (e.g. ELMs)**

**Addition of shear-flow to SPEC to more realistically model fusion plasmas.**

- Stability of a two-volume MRxMHD model in slab geometry
- Examined gap mode stability (including continuum damping) in fully 3D plasmas.
  [G. W. Bowden and M. J. Hole, Phys. Plasmas, 22, 022116, 2015]
- A general formulation to compute the spectrum of stable and unstable eigenmodes of MRxMHD

**Study of stability and dynamics of MRxMHD**

*First study of stability of plasmas with chaotic fields*
MRxMHD *equilibria* of stellar plasmas

- Most coronal solar flare models assume the field is force-free, and adjust the field pitch to match local measurements of the photosphere field footprint. (nonlinear force free fields)
- boundary conditions are line-tied (solar) cf toroidal (fusion)

Solar magnetogram

![Solar magnetogram](image)

e.g. Polarization, intensity of radio thermal free-free emission

*Iwai et al Earth, Planets and Space, Volume 66:149, 10 pp.*
Multi-Region Relaxation Dynamics

DP170102606  Multi-Region Relaxation Dynamics — a new paradigm for fusion and stellar plasma physics, R. L. Dewar, M. J. Hole, S. R. Hudson; A. Bhattacharjee

Q1) What is the MRxMHD spectrum of normal modes, and what are the effects of field-line curvature and mass flow on their stability?
Q2) When are the MRxMHD current sheets topologically stable towards internal plasmoid formation?
Q3) When do unstable modes saturate at a low level or develop nonlinearly into explosive events?

• compare the effect of toroidal (fusion) and line-tied (solar) boundary conditions on wave spectrum and stability
• Develop codes to treat dynamics in full 3D:
  - time-domain evolution code SPDC
  - frequency-domain code SPECN to calculate linear normal modes
• Comparison of results with experimental and observational data
Hidden Symmetries and Fusion Energy

Simons Foundation Collaboration on Hidden Symmetries and Fusion Energy, directed by Amitava Bhattacharjee of Princeton University.

July 24, 2018  https://hiddensymmetries.princeton.edu/

$2m USD per year over 9 institutions: Princeton University, Cornell University, Max Planck Institute for Plasma Physics Greifswald, New York University, Columbia University, University of Maryland, University of Texas (Austin), Warwick University and the University of Colorado, Boulder

Collaborating institutions: Princeton Plasma Physics Laboratory, Oak Ridge National Laboratory, EPFL Lausanne, University of Wisconsin
Hidden Symmetries and Fusion Energy

Multi-disciplinary team: Mathematics + Physics

- Princeton University: A. Bhattacharjee (Collaboration Director), P. Constantin (Founding PI)
- Australian National University: R. Dewar (Founding PI), M. Hole
- Columbia University: A. Boozer (Founding PI)
- Cornell University: D. Bindel (Founding PI)
- EPFL, Lausanne: J. Loizu
- New York University: G. Stadler (Founding PI), A. Cerfon, M. O’Neil, H. Weitzner
- Princeton Plasma Physics Laboratory: S. Hudson, H. Mynick and A. Reiman
- University of Colorado-Boulder: J. Meiss (Founding PI)
- University of Maryland: L.-M. Imbert-Gerard (Founding PI), M. Landreman
- University of Texas at Austin: O. Ghattas (Founding PI)
- Max Planck Institute for Plasma Physics: P. Helander (Founding PI), T. Pedersen, Frank Jenko
- Oak Ridge National Laboratory: D. Spong
- University of Warwick: R. MacKay (Founding PI), J. Robinson, J. Rodrigo
- University of Wisconsin: C. Hegna

Advisory Board: C. Fefferman (Chair), J. Cary, D. Keyes, B. Khesin, B. Wohlmuth
Hidden Symmetries and Fusion Energy

**Objective:** To create and exploit an effective mathematical and computational framework for the design of stellarators with hidden symmetries.

**Research challenges:** Finding optimum magnetic fields with hidden symmetries is an interdisciplinary challenge straddling optimization theory, plasma physics, dynamical systems and the analysis of partial differential equations (PDEs).

**Deliverables:** Optimum design principles of a stellarator, a modern optimization code (SIMSOPT, SIMonS OPTimization code) that can exploit the full power of petascale and exascale computers, and designs of next-generation stellarator experiments

**Features:** Support of the Exascale Computing Project of the US Dep. Of Energy. Aims to harness Aurora, first exascale computer in the US
Some mathematical challenges

- Magnetic field line flow produce a sea of good surfaces and chaos. *How do we quantify non-integrability of field lines and control it? How well do particles track field lines? How are particles transported across “cantori”?*

- MHD describes the magnetized plasma as a fluid. The 3D MHD problem is not known to be globally well-posed even with dissipation added. *What are the special solutions enabled by hidden symmetries?*

- *Formulate the design problem as a constrained, risk-averse, multi-objective stochastic optimization problem. Seek Pareto-optimal solutions. This cannot be done in a physics-agnostic way. Existing stochastic optimization methods are not able to address high-dimensional problems.*
Collaboration Opportunities within MSI

Research – drawing on MSI skill set
• Computational mathematics: sparse grids, optimisation
• Astrophysics: MHD, shocks,
• PDEs: properties and existence
• Inverse problems

Teaching
• M. J. Hole: Physics Honours Advanced Electromagnetic Theory (Jackson) and Statistical Mechanics; Wave Theory (transmission line theory)
• R. L. Dewar: Dynamical Systems, Fluid Dynamics, Classical Mechanics
ANU Grand Challenge(s)

1. Clean Energy from Fusion
   • For over a decade, M. J. Hole has promoted Australian participation in ITER (Australian ITER Forum)
   • Dewar and Hole have significant international profile
   • ANU also has world-recognised profile in advanced diagnostics and materials
   • Wider potential to further engage CECS, CAPS

2. Model / Data Fusion (not nuclear fusion)
   • Develop algorithms / framework to constrain the information-explosion with high-fidelity simulation
   • Referred to as “integrated modelling” in fusion science
   • Multiple fields of application: fusion science, climate science, astronomy, computer-network security?
Summary

• Plasma Theory and Modelling: a vibrant ANU pursuit.
• Very strong international collaboration (people & experiments)
• Important student / post-doc research training dimension
• Current Research areas
  ➢ Burning plasma physics: anisotropy and flow, energetic particle driven modes
  ➢ Fully 3D toroidal physics and MRxMHD. Impacts of 3D structure on plasma. Dynamics of MRxMHD plasmas,
  ➢ Bayesian inference of configurations
     *Bayesian inference (inversion) for particle velocity distribution
  ➢ wakefield accelerator physics, ELM sandpile models