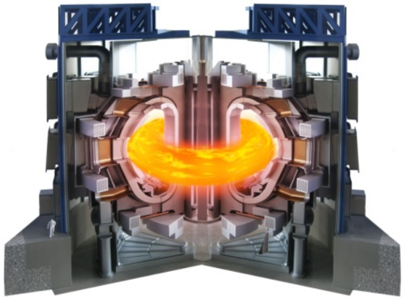

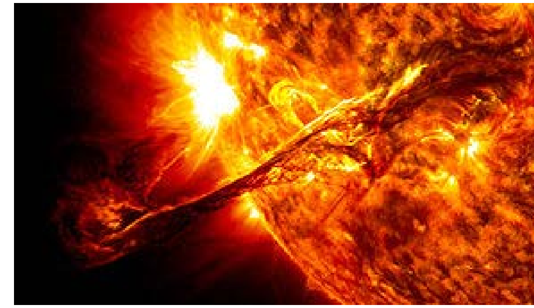

The Mathematics of Fusion Plasmas

M. J. Hole(*), R. L. Dewar, Z. Qu, A. Wright, H. Hezaveh, C. Bowie,
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S. Sharapov[5], J. Svensson[7], Y. Todo[9], G. Hao[10]

Plasma Theory and Modelling

(*) ITER Science Fellow, ITPA Energetic Particles Expert Member;
Australian Delegate, International Fusion Research Council, IAEA;
Chief Division Secretary, Association of Asia Pacific Physical Societies;
Chair, Australian ITER Forum; Board Member, Plasma Physics and Controlled Fusion

MSI Colloquium, 9 August 2018

Acknowledgement: Australian Research Council, ANU

- Supported by ~$3m in funding over last 10 years (ARC, ISL, Simons)

5 CCFE CULHAM CENTRE FOR FUSION ENERGY
3 THE UNIVERSITY OF TEXAS AT AUSTIN
2 PPPL PRINCETON PLASMA PHYSICS LABORATORY
6 NFRI 국가핵융합연구소 National Fusion Research Institute

The Mathematics of Fusion Plasmas

7 IPP Max-Planck-Institut für Plasmaphysik
4 THE UNIVERSITY OF WARWICK
8 iter china eu india japan korea russia usa
9 NIFS
10 SWIP

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Plasma Theory & Modelling - *mathematics*

- Grad-Shafranov
- Beltrami fields
- Sand pile

Reduced Models (fluid)

- Gyrokinetic
- Full orbit particle dynamics

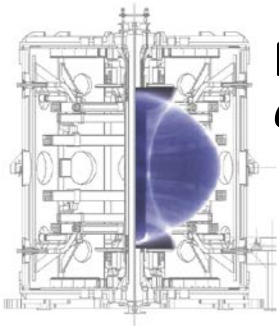
simulation

- Bayesian inference

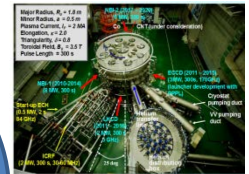
Model/data fusion

Electrical Engineering (Power, rotating machines, signals and systems). *Hole*
Physics (Electrodynamics, Kinetic Theory, Fluid dynamics, Plasmas) *Hole, Dewar, Qu*
Applicable Mathematics (Calculus, PDEs, computation, Classical Mechanics, maths methods) *Hole, Dewar, Qu*

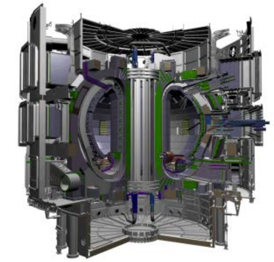
Plasma Theory & Modelling - *physics*



MAST (UK)
compact

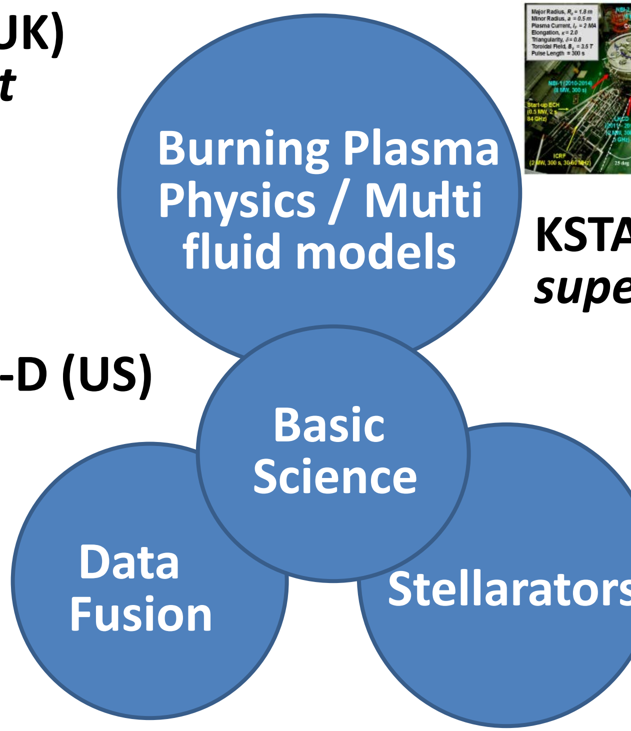


KSTAR (Korea)
superconducting

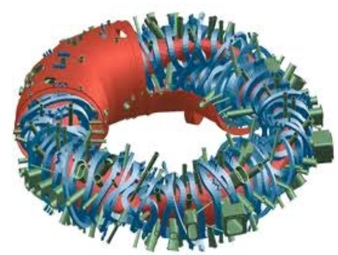


ITER (Earth)
ITPA

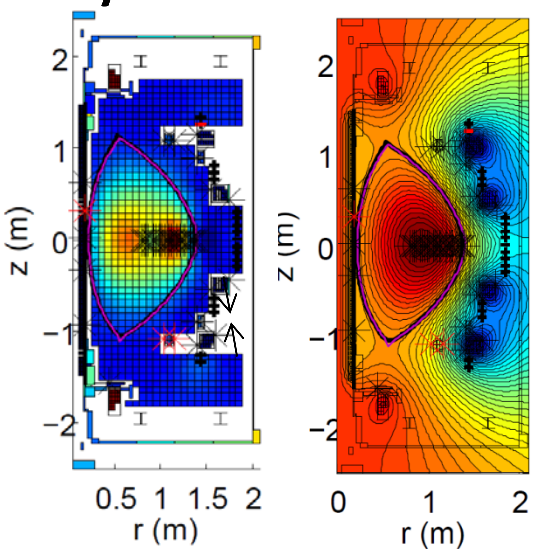
DIII-D (US)



W7-X (Germany)
steady-state, reduced chaos



Bayesian inference



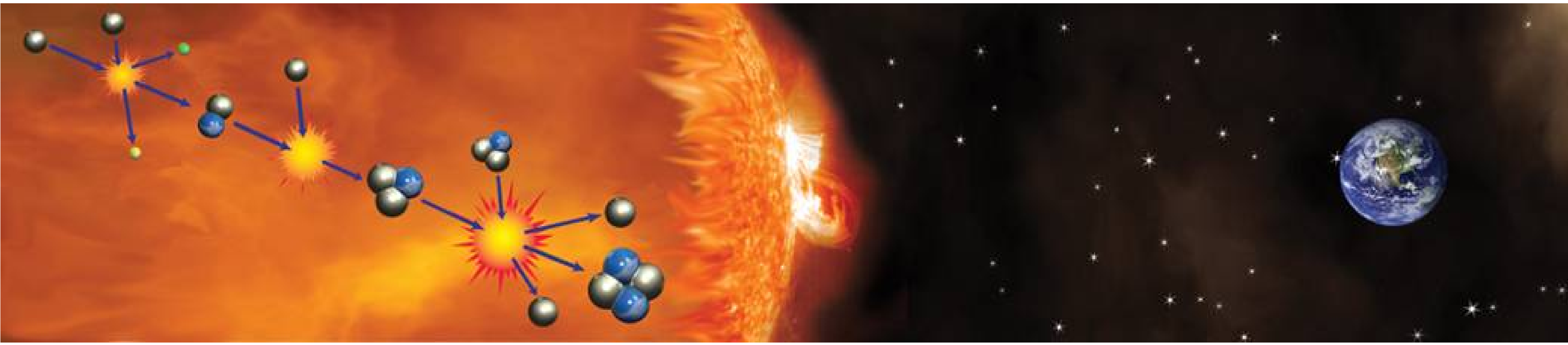
currents flux surfaces

RFX-mod (Itlay)
Self-organising

Stellar dynamics

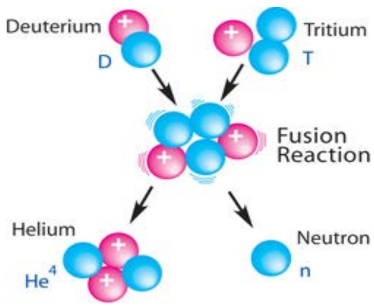
Fusion, the power of the sun and the stars, is one option

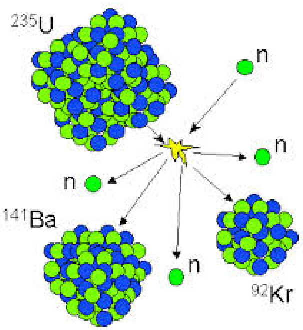
"…Prometheus steals fire from the heaven"



On Earth, fusion could provide:

fusion cf fission





- Large-scale energy production
- Essentially limitless fuel, available all over the world
- No greenhouse gases
- Intrinsic safety
- No long-lived radioactive waste

Conditions for *terrestrial* fusion power

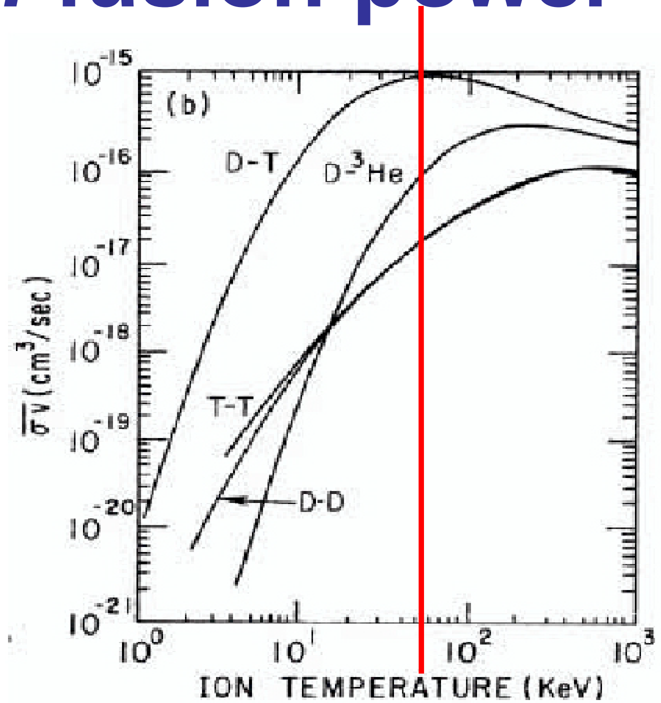
- Achieve sufficiently high

ion temperature T_i

$\Rightarrow$ exceed Coulomb barrier

density $n_D \propto$ energy yield

energy confinement time τ_E



τ_E = insulation parameter: e.g. time taken for a jug of hot water to lose energy to the surroundings

≈600 million °C

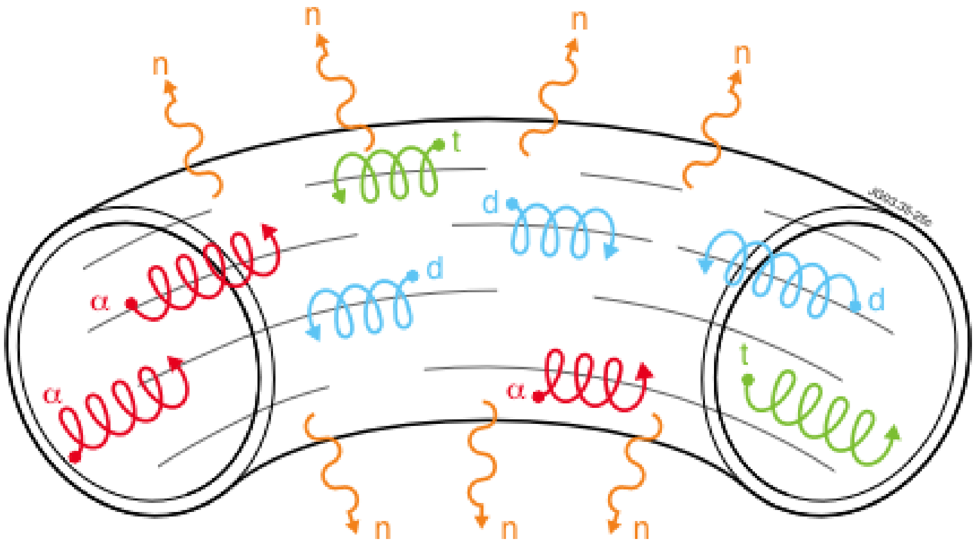
- "Lawson" ignition criteria : Fusion power > heat loss

Fusion triple product $\longrightarrow$ $n_D \tau_E T_i > 3 \times 10^{21}$ m^{-3} keV s

- Steady-state access requires confinement

Toroidal Magnetic Confinement

- Magnetic fields cause charged particles to spiral around field lines. Plasma particles are lost to the vessel walls only by relatively slow diffusion across the field lines



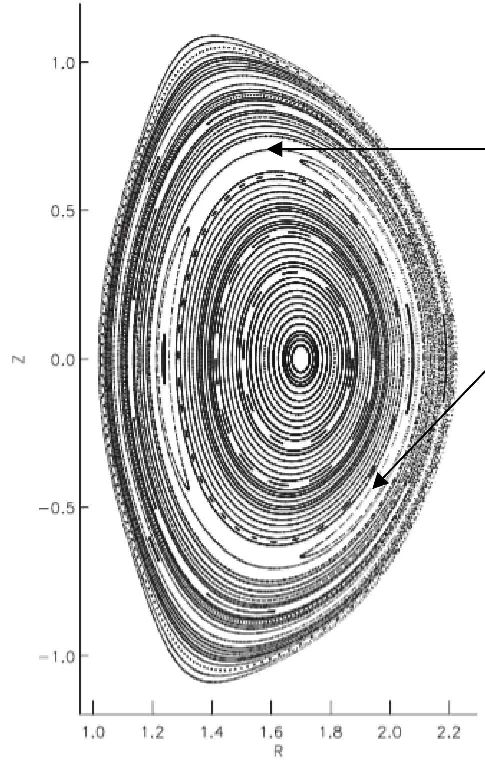
- Only charged particles (D^+, T^+, He^+…) are confined Neutrons escape and release energy

- Toroidal (ring shaped) device: a closed system to avoid end losses

Mathematics of Magnetic Field Lines

- Motion on field lines is dynamical system
  $\dot{\mathbf{r}} \equiv d\mathbf{r}/d\zeta \propto \mathbf{B}$ where ζ is a toroidal angle
- System is Hamiltonian
- Analyze with flux-preserving *return map* to Poincaré section $\zeta = 0$:

*Non*axisymmetric (3-D) fields are not generically *integrable*, i.e. some points do not lie exactly on flux surfaces (e.g. KAM* *invariant tori*). Instead follow *chaotic* orbits in island separatrices.
*Kolmogorov, Arnol'd & Moser

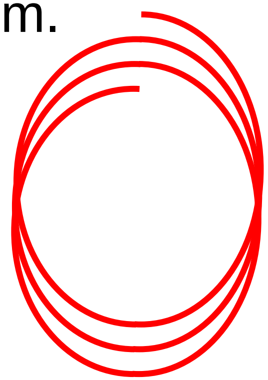


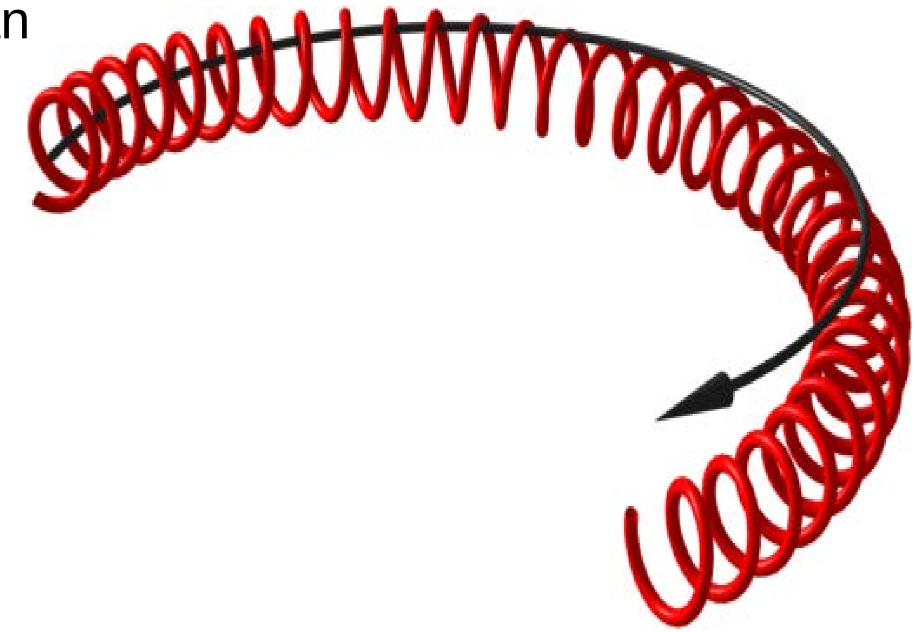
invariant torus:
$\iota\!\!\!- = \mathrm{irrational}$

island:
$\iota\!\!\!- = 1/2$

$\iota\!\!\!-$ = (average rotation angle per return)/2π is *rotation number*

But if field lines are bent as in an axisymmetric torus, particles drift off them.





→ To confine particles, constrain their position with a conservation law.

<u>Noether's theorem:</u>

For each **continuous symmetry** of a system, there is a corresponding **conserved quantity**.

Emmy Noether (1882-1935)

Axisymmetry and Noether's theorem is one way to achieve confinement

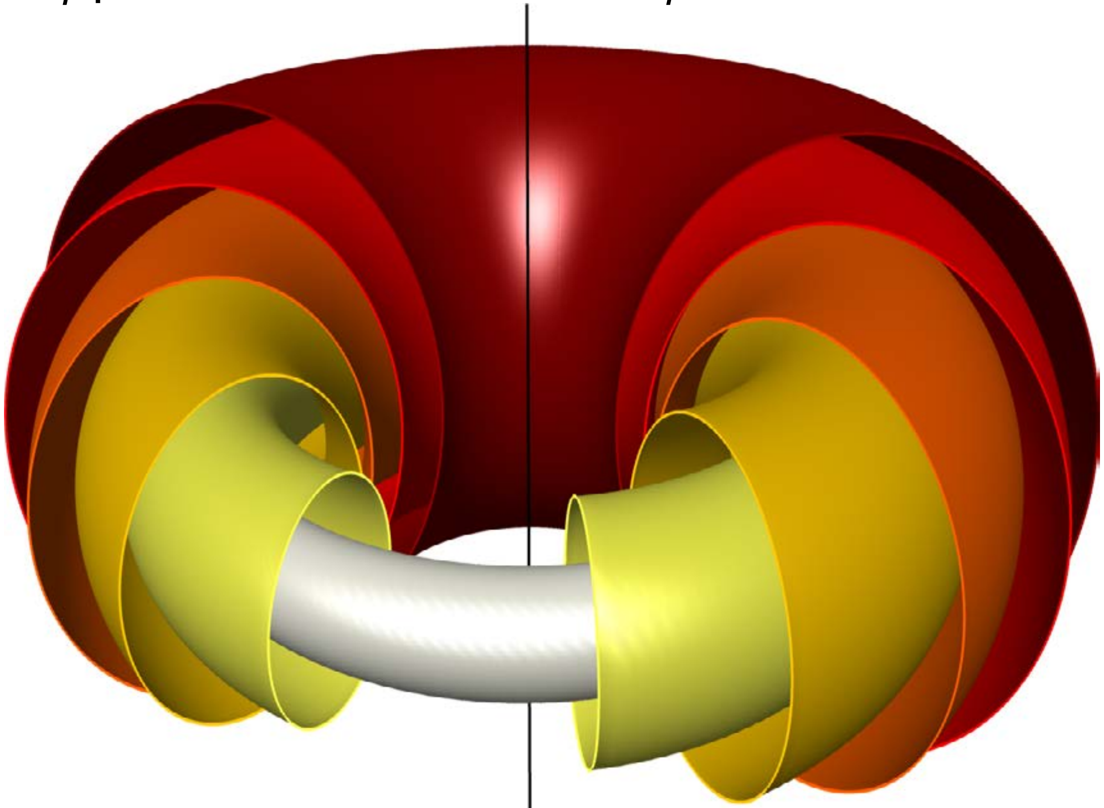
Continuous rotational symmetry ⇒ Canonical angular momentum conserved.

$$L_\phi = m v_\phi R + q A_\phi R = \text{constant}$$

Vector potential: $\mathbf{B}=\nabla\times\mathbf{A}$

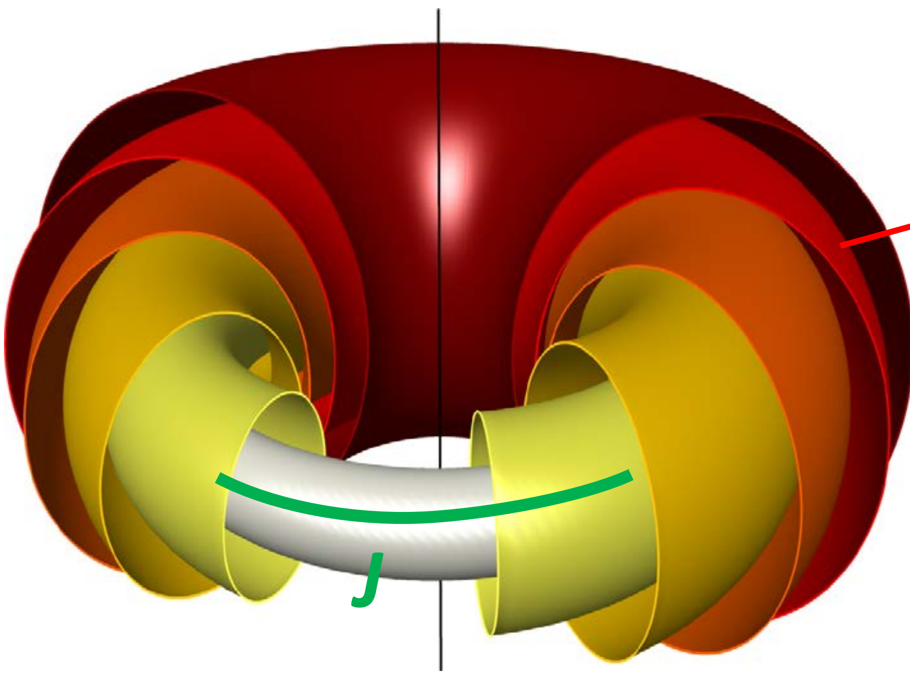
Strong B limit $\Rightarrow |m v_\phi| \ll |q A_\phi| \Rightarrow$ particles stuck to $A_\phi R$ surfaces

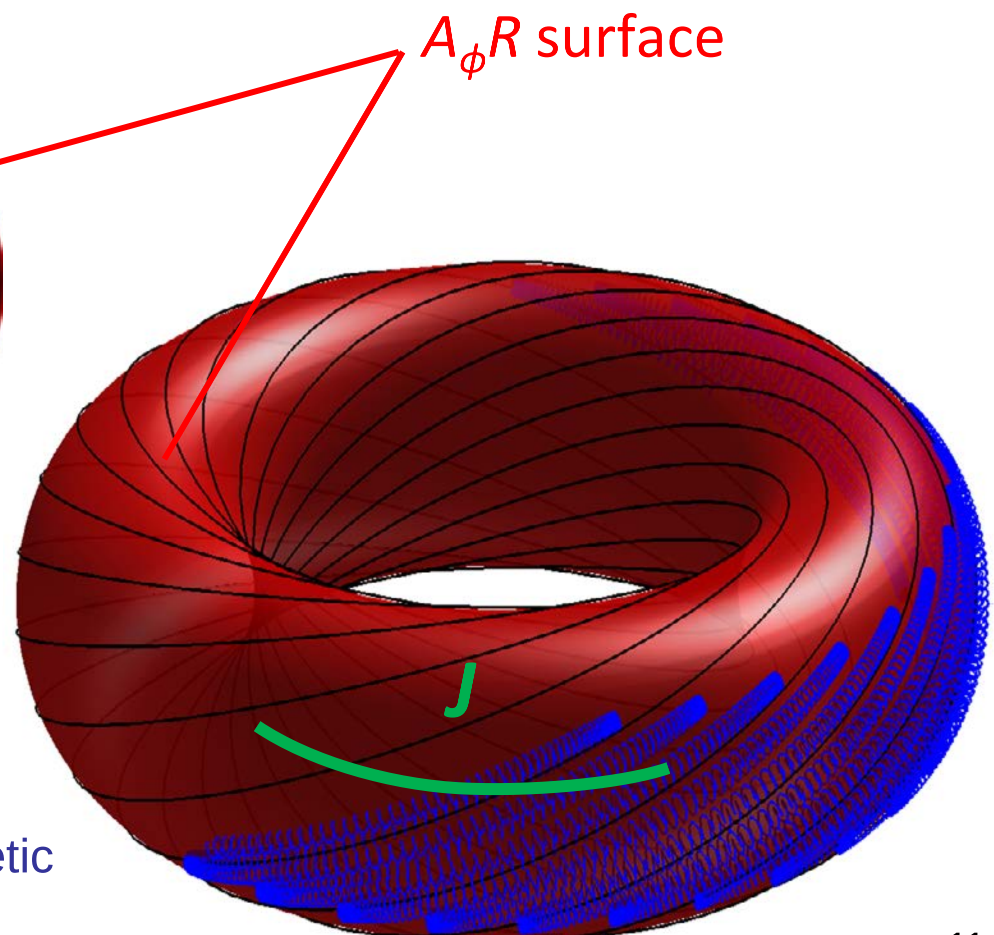
If $A_\phi R$ surfaces are bounded, then particles will be confined.



Complication: Axisymmetric confinement requires an internal current

$\nabla \times (\nabla \times \mathbf{A}) = \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow$ nested $A_\phi R$ surfaces require a J_ϕ





- The most successful Magnetic Confinement device is the TOKAMAK (Russian acronym for 'Toroidal Magnetic Chamber')

Energy confinement : ***big*** is better

- τ_E empirical scaling

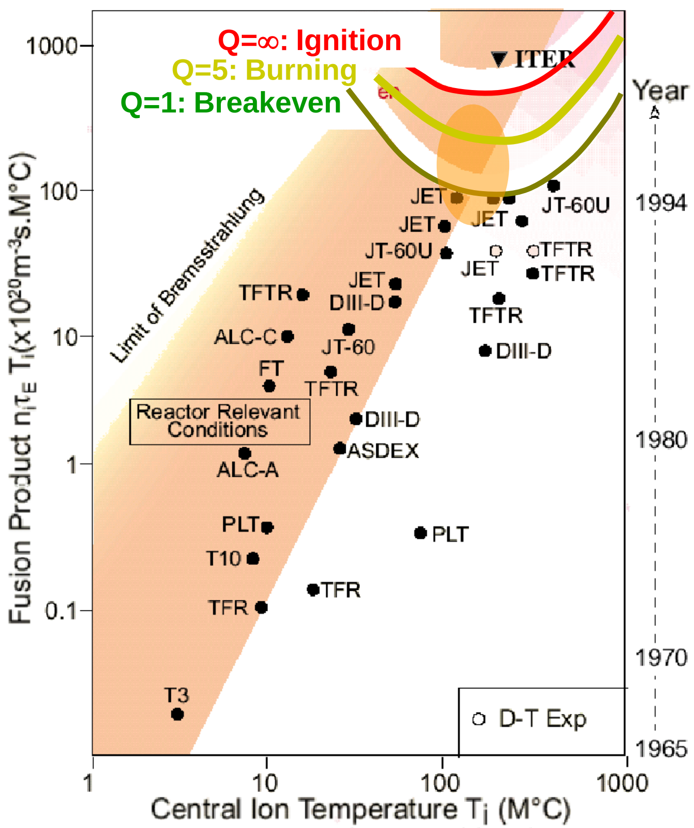
$$\tau_E = f\left(H_H, I_p, B_T, n, P_H, R_m, \kappa, M, a/R\right) \propto I_p B_T^{0.15} R_m^{2} P_H^{-2/3}$$

Plasma current

magnetic field

heating power

major radius

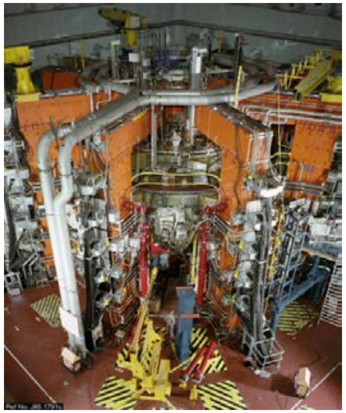


- "Breakeven" regime :

$Q = P_{out} / P_{heat} = 1$

Eg. JET

Q=0.7, 16.1MW fusion



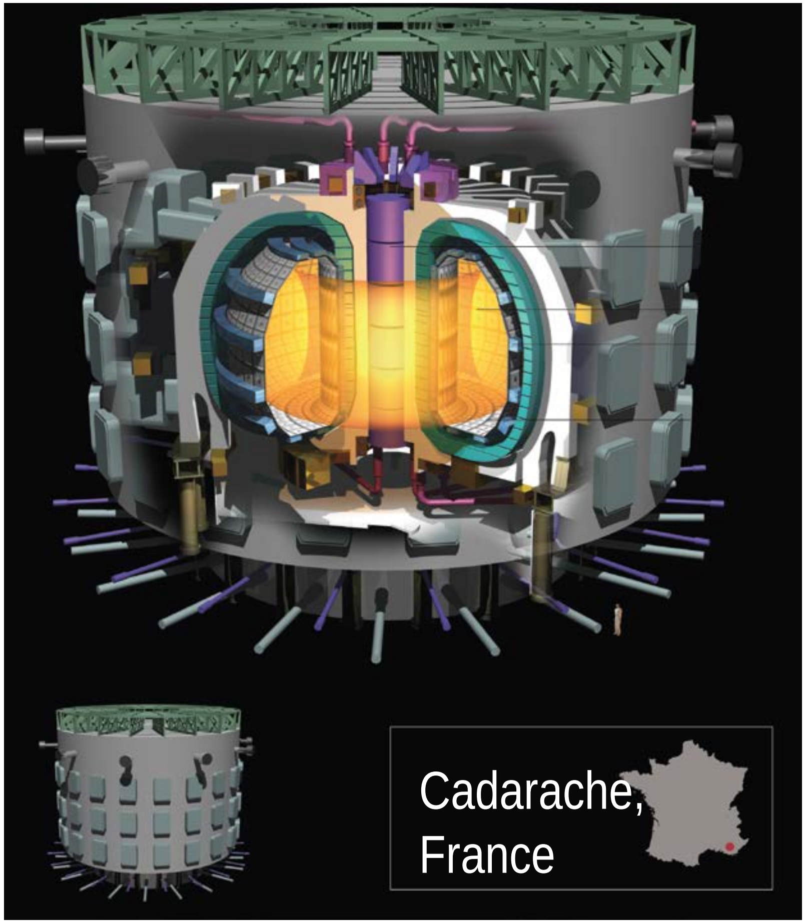
- "Burning" regime : **ITER**

$$D^2 + T^3 \rightarrow \underbrace{He^4\,(3.5\ \text{MeV})}_{\geq P_{heat}} + \underbrace{n^1\,(14.1\ \text{MeV})}_{P_{out}}$$

Q>5 ⇒ITER

- "Ignition" regime, $Q\rightarrow\infty$: **Power Plant.**

International Thermonuclear Experimental Reactor (ITER)



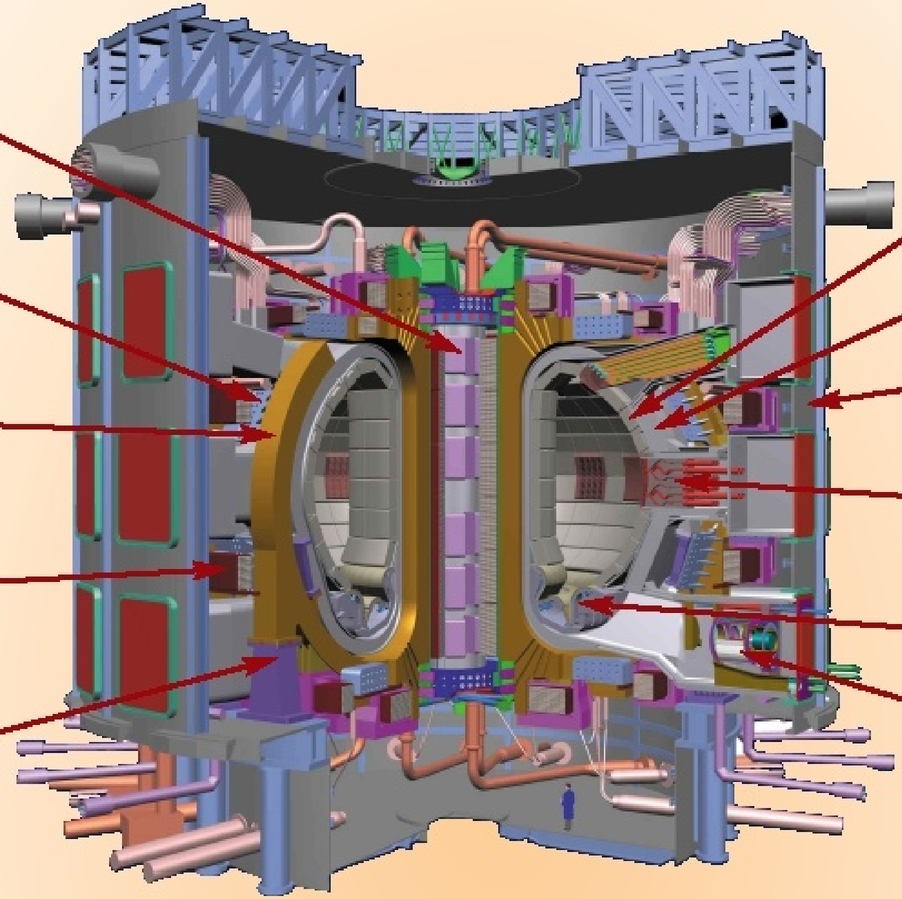
- Fusion power = 500MW
- Power Gain (Q) > 10
- Temperature ~ 100 million °C

- Growing Consortium

- Collaboration agreements with
  - International Atomic Energy Agency
  - CERN – world's largest accelerator
  - Principality of Monaco
  - Australia 30/09/2016
  - *Iran (4 July 2016, High-level Iranian delegation visits ITER worksite)*

Construction +10 year operation cost ~$40 billion ?

ITER in detail

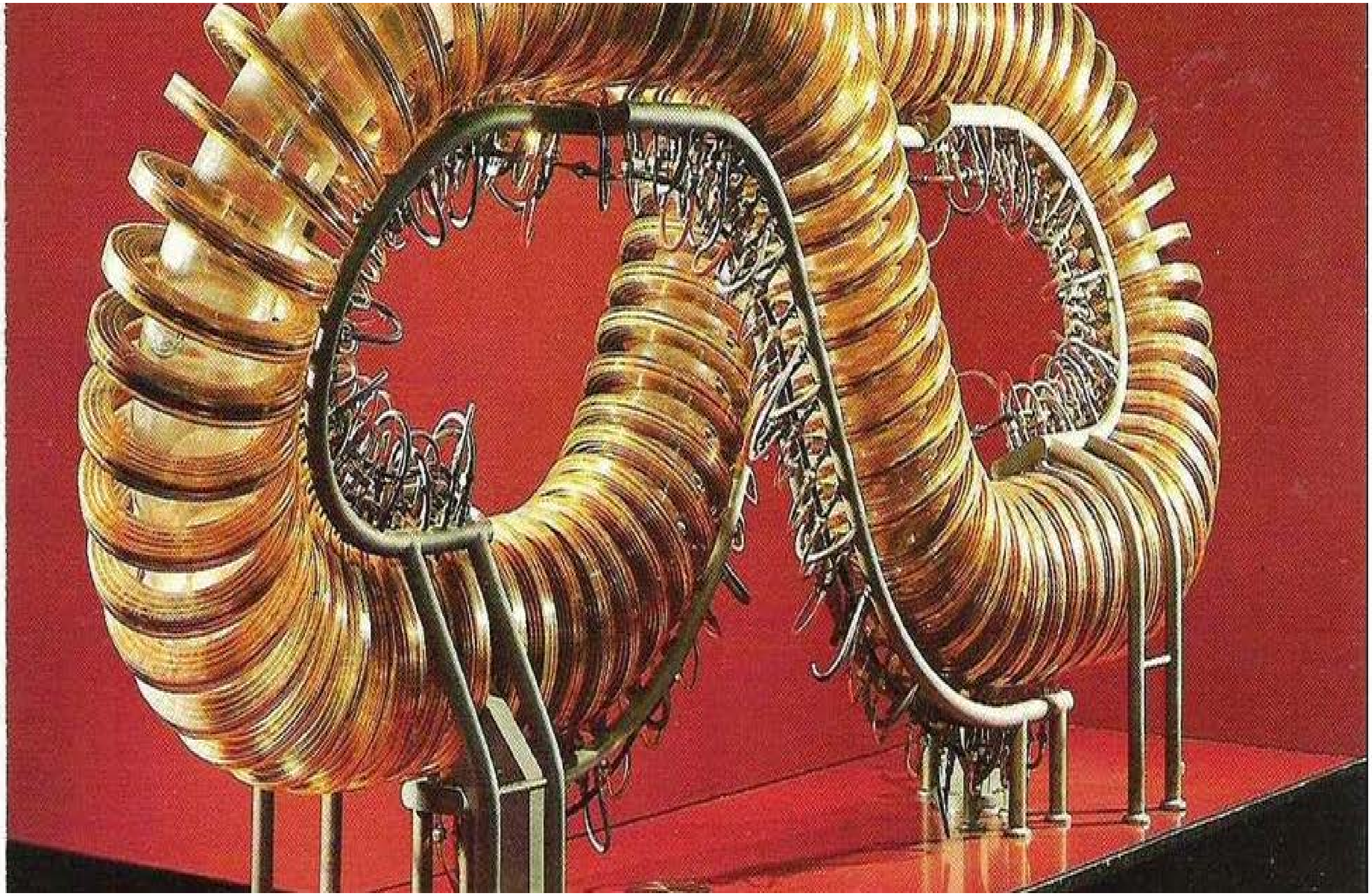


Total Fusion power	500MW
Minor (a), major (R) radius	2.0m, 6.2m
Ip, plasma current	15MA

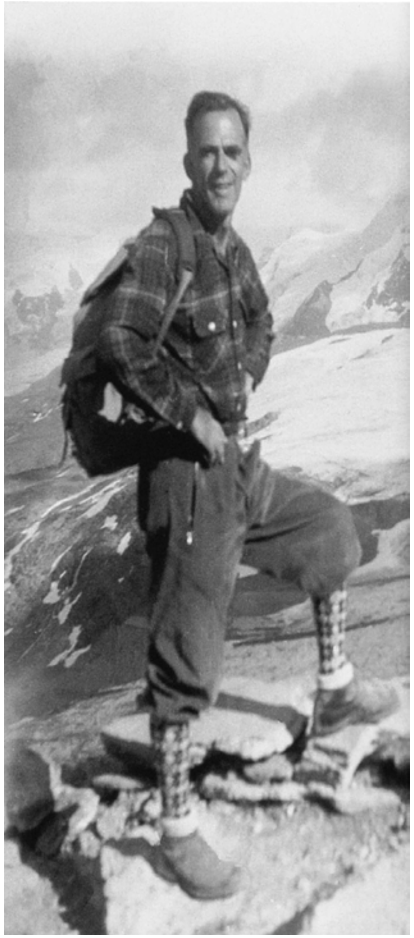
Toroidal field @6.2m	5.3T
Plasma Volume	837 m^3
Auxillary heating, current drive	73MW

However … poloidal field can be also be outcome of geometry

e.g. Figure-8 Stellarator



Lyman Spitzer (1914-1997)



Some advantages

- Very low (no) J_ϕ: eliminates some plasma instabilities and disruption
- Intrinsically steady-state

Averaging over fast gyration, dynamics depend on B through |B|

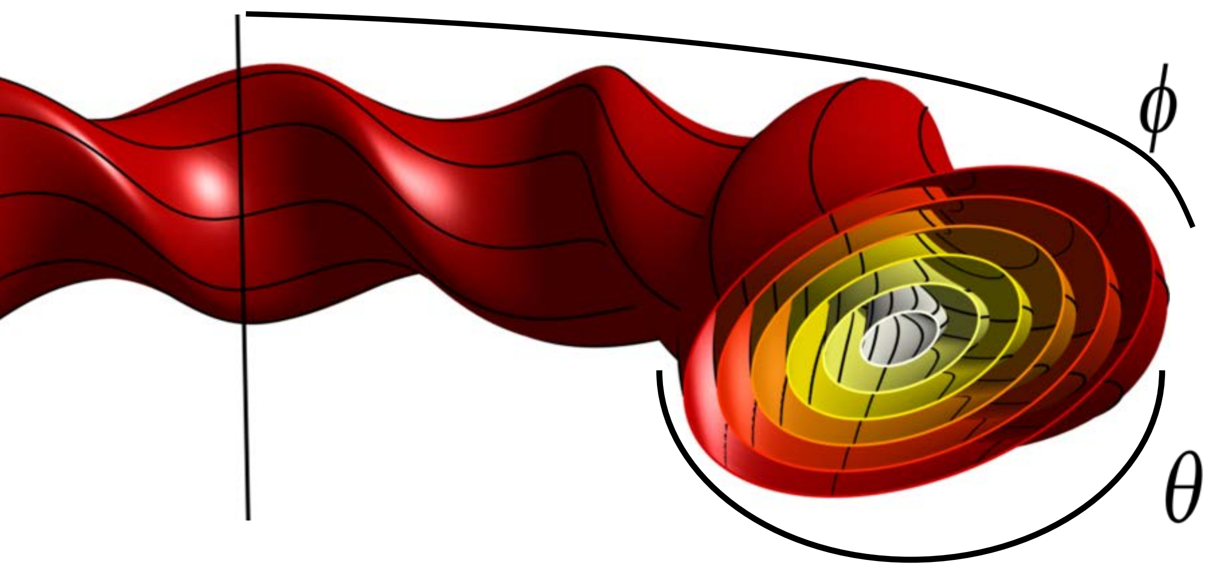
Lagrangian for particle in magnetic field: $\mathcal{L} = \frac{m}{2}\left|\dot{\mathbf{x}}\right|^2 + q\mathbf{A}\cdot\dot{\mathbf{x}}$
(neglect **E**)

Independent of ϑ and ϕ

Average over fast gyration,
use angle coordinates:

$$\mathcal{L} = \frac{mG^2\dot{\phi}^2}{2B^2} - \mu B + q\psi\dot{\theta} - q\chi\dot{\phi}$$

Only depends on θ and ϕ through B=|**B**|



If $\frac{\partial|\boldsymbol{B}|}{\partial\phi}$ =0, then canonical angular momentum $\frac{\partial\mathcal{L}}{\partial\dot{\phi}}$ would be conserved

Quasi-symmetry: |B| is symmetric

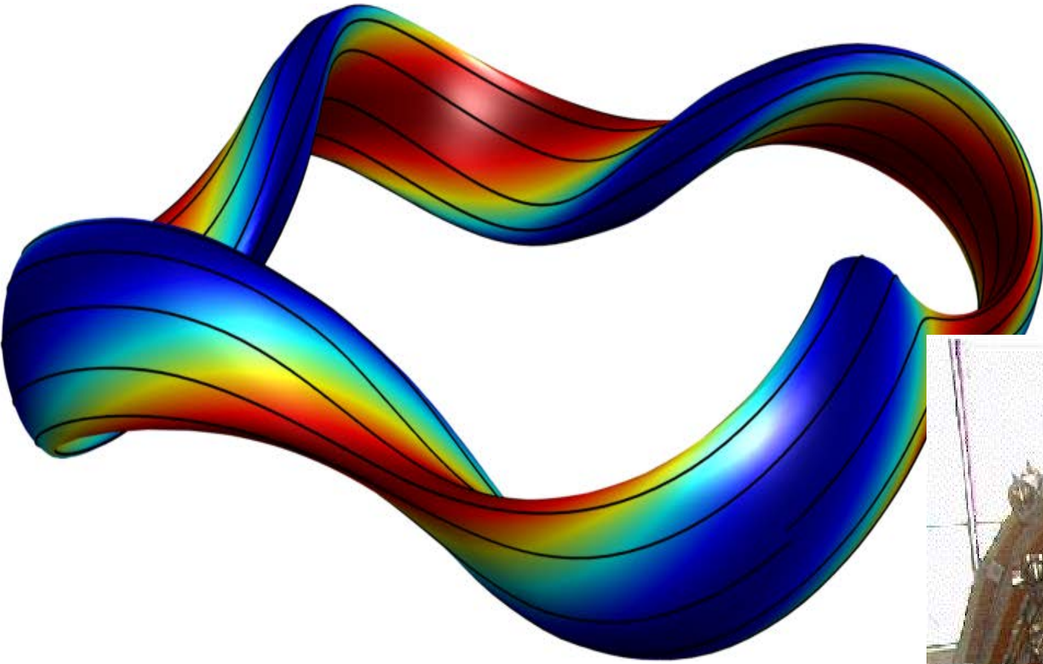
- Can you actually make a non-symmetric $\mathbf{B}$ with symmetric $|\mathbf{B}|$?

- Can you start with a vacuum field with $\nabla\times\mathbf{B} = 0$ to eliminate most of the internal current in a plasma with finite pressure?

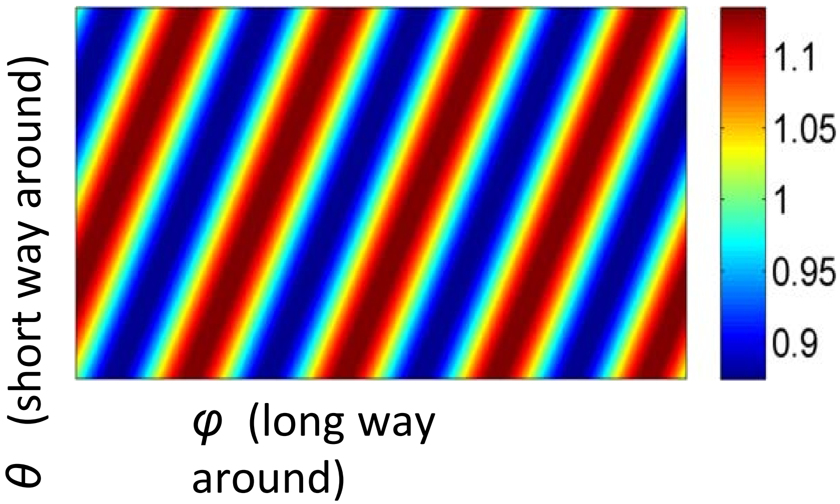
Yes!

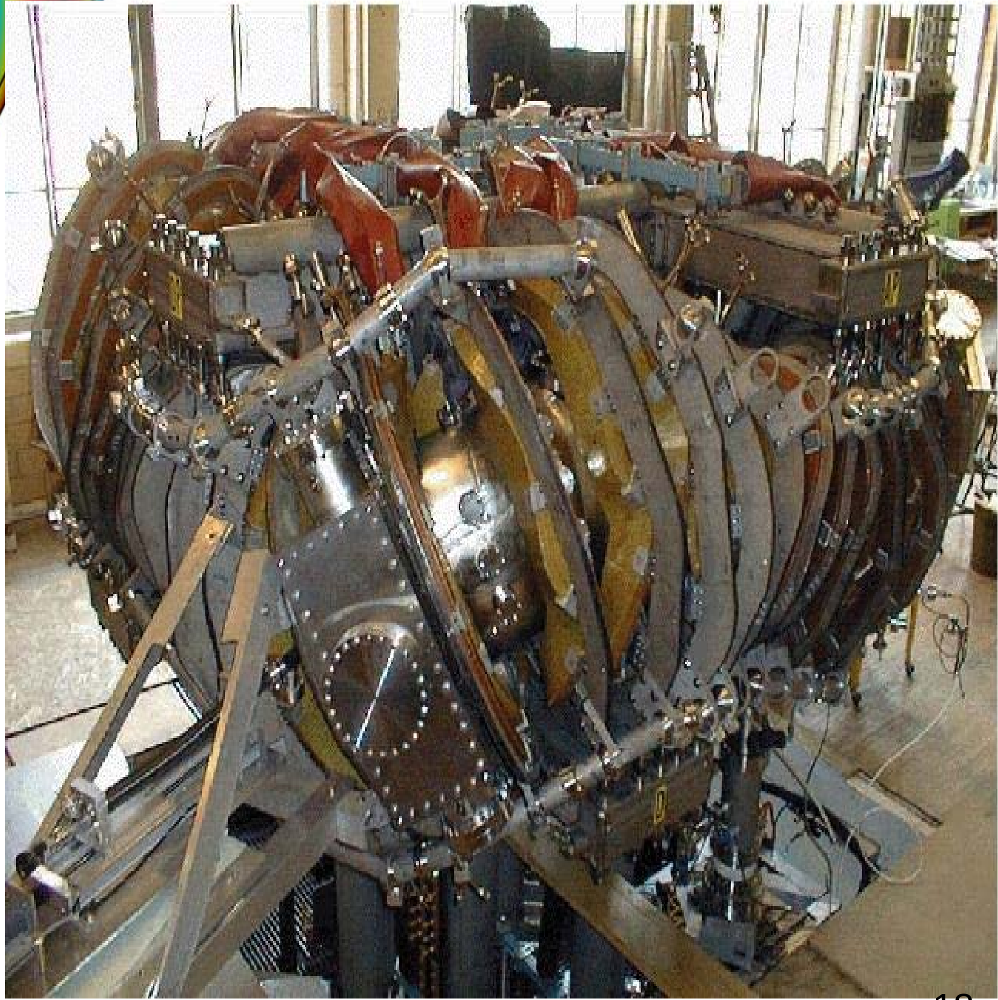
….but the cost is (to date) engineering complexity

HSX:
Helically Symmetric eXperiment
(University of Wisconsin)



Magnetic field magnitude |B| in Tesla

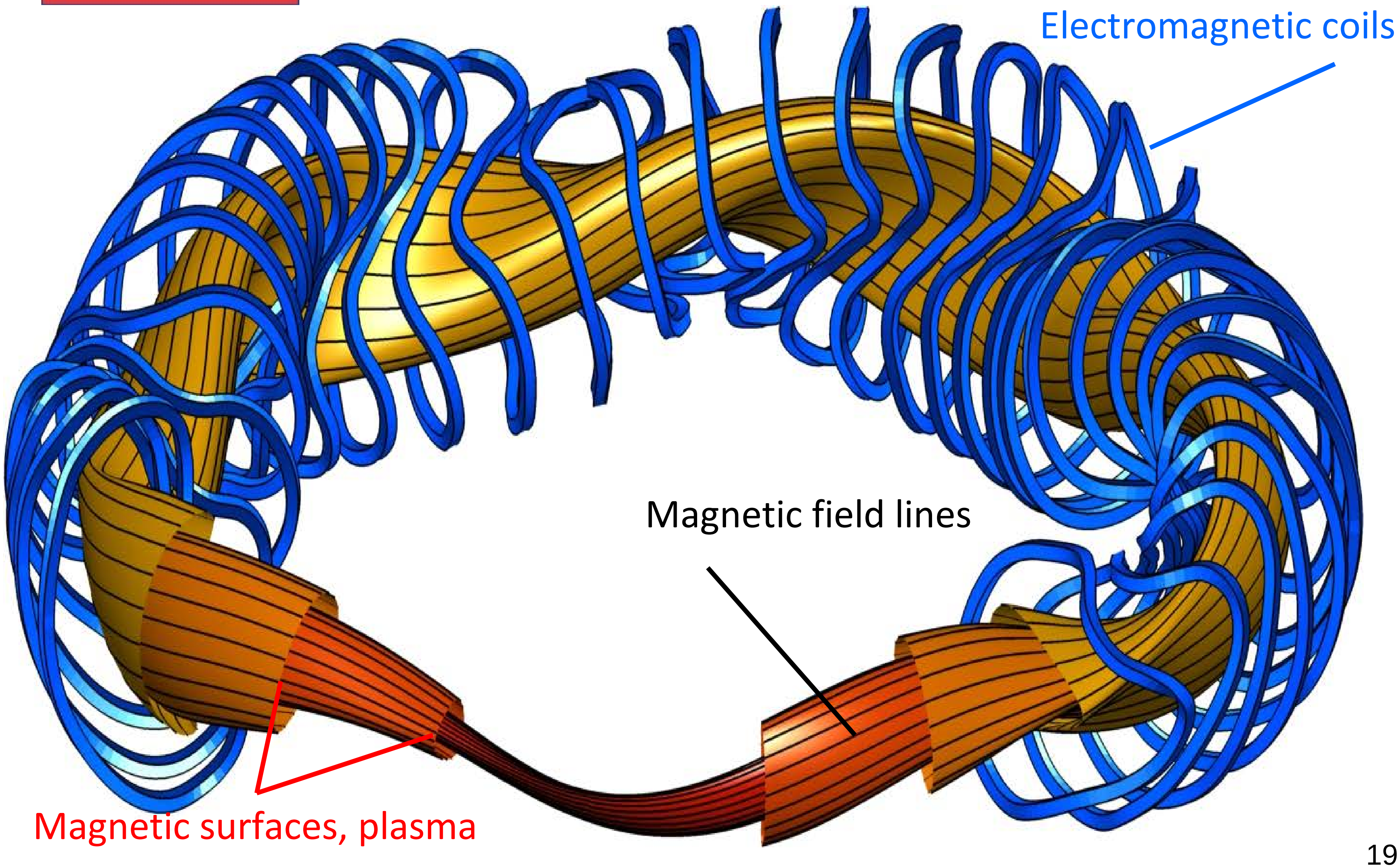




Wendelstein 7-X

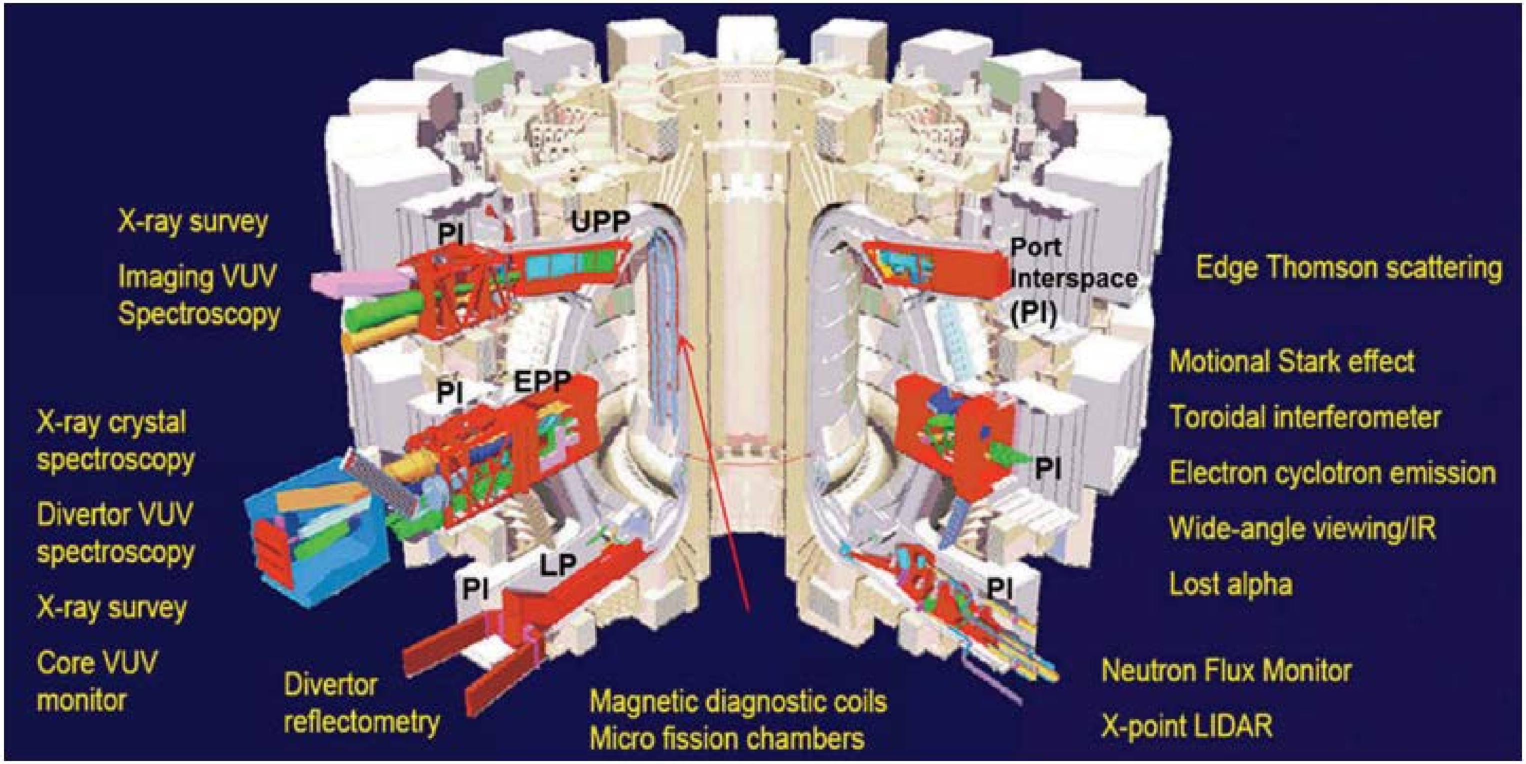
Science Oct 21, 2015

30 minute plasmas eventually



Common: heterogeneous diagnostics

- Large number of *heterogeneous* diagnostics sampling *overlapping* parts of configuration and phase-space
- Diagnostic forward model is inter-dependent on physics model and other parameters.
  e.g. interferometer signal $\Delta\phi \propto \int n_e dl$



Some ITER diagnostics within the bioshield

Model/Data Fusion: Bayesian Inference of B, δB

Aim: Develop a probabilistic framework for validating equilibrium (magnetic force balance models) and mode structure

Motivation: data from multiple diagnostics with strong model dependency

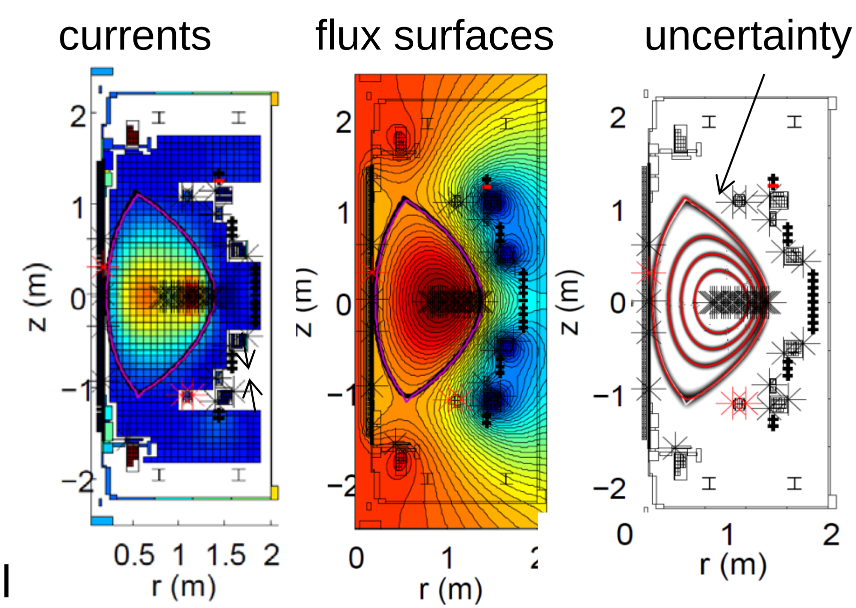
E.g. Current tomography on MAST

Posterior *Likelihood* *Prior* *Evidence*

$$P(\mathbf{J}|\mathbf{D}) = P(\mathbf{D}|\mathbf{J})P(\mathbf{J})/P(\mathbf{D})$$

vector of beam currents

Data vector of Flux loops, pickup coils, MSE



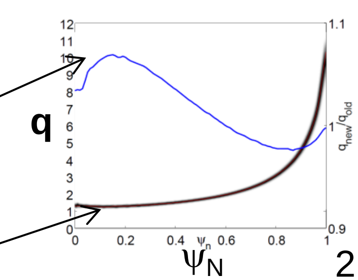
[M.J. Hole, G. von Nessi, J. Svensson, L.C. Appel, Nucl. Fusion 51 (2011) 103005]

[G. von Nessi, M. J. Hole, J. Phys. A: Math. Theor. **46** (2013) 185501]

ISL CG130047 $395k (2008-2012)

change of q if J_θ included

q profile and uncertainty



Virtual Control / Remote Participation

- Our group has virtual control infrastructure, access to data from MAST and JET tokamaks (UK, exps. worth ~ $ 1 billion), KSTAR tokamak (South Korea, ~$350m), NSTX-U (~US$300m)

- ITER scenarios (Integrated Modelling Analysis Suite)
- ITER is an Exascale data-class experiment:
  data acquisition systems are 50GB/s, with plasma durations targeting 400s and ~20 discharges per day

Characteristics of fusion physics

- Fusion plasma physics is now big science : the leading-edge fusion experiments are billion dollar class machines.
- Multi-scale:
  - Spatial: electron gyro-radius 5 x 10^{-6} m $\rightarrow$ 10m (device scale) anisotropic
  - Temporal: electron gyration 4 x 10^{-10}s $\rightarrow \tau_E \sim$ 3 seconds
  - Generates many expansion parameters for asymptotic analysis
- High dimensional phase-space: gyro-kinetic is 6D (3 spatial, 2 velocity, time).
- $n_i = 10^{20}$ m^{-3} (ideal gas at 1atm, 0C is 2.7 x 10^{25} m^{-3})
- Strong nonlinearity
- Constraint and diagnostic model interdependency
- Machine design is a complicated optimisation problem (especially for stellarators)

Topical Research Fields

- Example of reduced modelling: magnetohydrodynamics
- 2D: Tokamak equilibrium model
- ND: Spectral theory
- ND: Computation and simulation
- 3D: Describing and optimising stellarator fields

Magnetohydrodynamics (MHD)

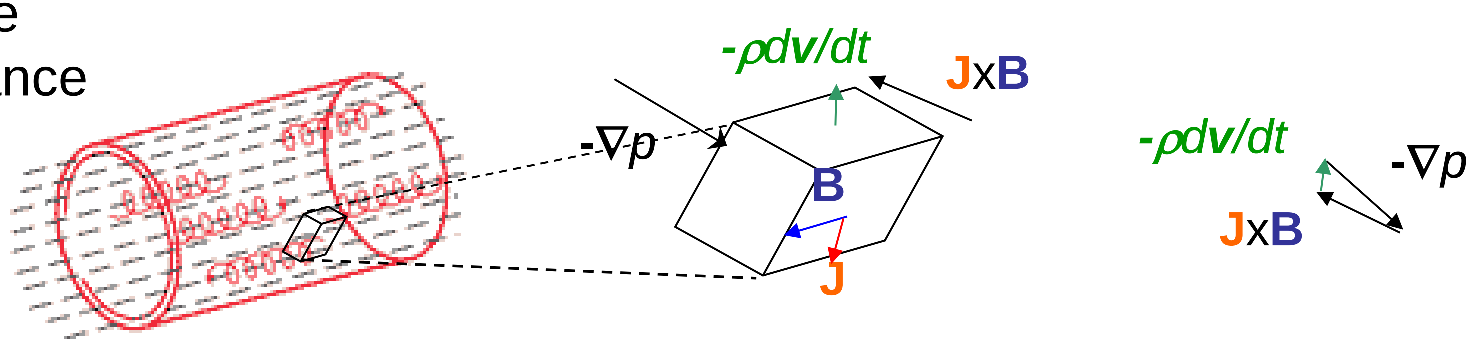
- Single conducting fluid: $\mathbf{J} = n_i Ze\mathbf{v_i} - n_e e\mathbf{v_e}$ $\qquad \rho \approx m_i n_i$

$$\mathbf{v} = \left(m_i \mathbf{v_i} + m_e \mathbf{v}_e\right)/\rho \approx \mathbf{v_i} \qquad p = p_i + p_e$$

- Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v}\right) = 0$$

- Momentum: $$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{J} \times \mathbf{B}$$

force balance



If dv/dt = 0 $\Rightarrow$ $\mathbf{J} \times \mathbf{B} = \nabla p$

- Generalised Ohm's law: $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$

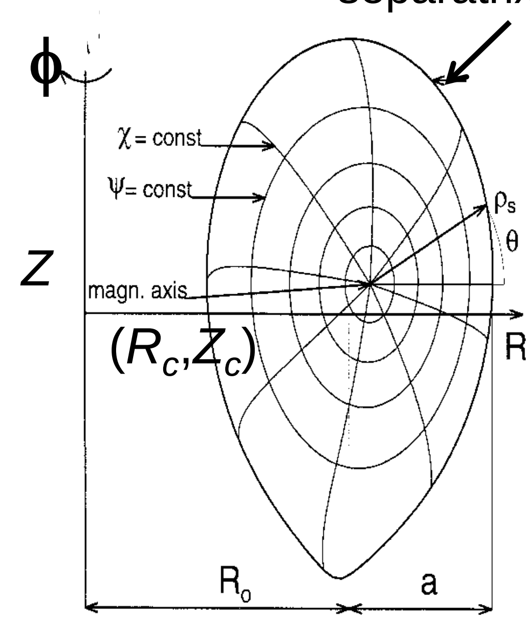
- Maxwells equations, Adiabatic equation: $\dfrac{p}{\rho^\gamma} = \text{const.},$

What is *B* for a stationary tokamak plasma?

(1) $\mathbf{J}\times\mathbf{B}=\nabla p \Rightarrow \begin{cases} \mathbf{B}\cdot\nabla p=0 \Rightarrow \text{No pressure gradient along } \mathbf{B} \\ \mathbf{J}\cdot\nabla p=0 \Rightarrow \text{Current flows within surfaces.} \end{cases}$

(2) $\nabla\times\mathbf{B}=\mu_0\,\mathbf{J}$

(3) $\nabla\cdot\mathbf{B}=0$



Introduce poloidal magnetic flux function ψ(R,Z) and co-ord. system (R, ϕ ,z). In axisymmetry Eq. (1), (2) become **Grad-Shafranov equation:**

$$\boxed{\nabla\cdot\frac{1}{R^2}\nabla\psi=-\frac{\mu_0 J_\phi}{R}=-\mu_0 p'(\psi)-\frac{\mu_0^2}{R^2}f(\psi)f'(\psi)}$$

second order PDE for field and currents.

With *f*(ψ) a toroidal flux function

$$f(\psi)=RB_\phi(\psi,R)/\mu_0$$

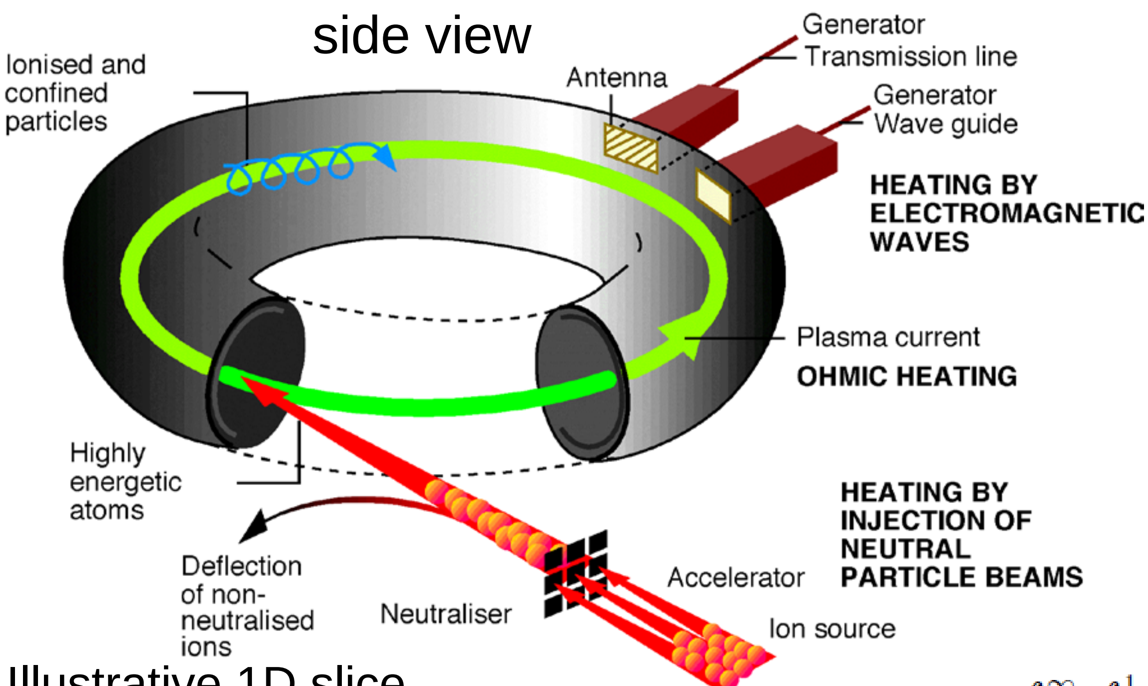
- To solve: prescribe $p'(\psi)$, $f(\psi)\,f'(\psi)$ and boundary
- Solve numerically by current-field iteration:

compute J_ϕ $\rightarrow$ solve for ψ(R,Z)

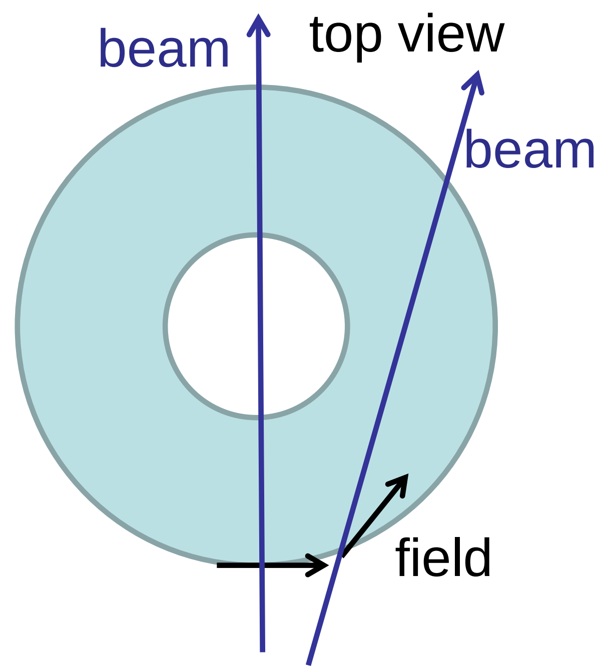
"MHD with anisotropy in velocity, pressure"

- Pressure different parallel and perpendicular to field due mainly to *directed* neutral beam injection

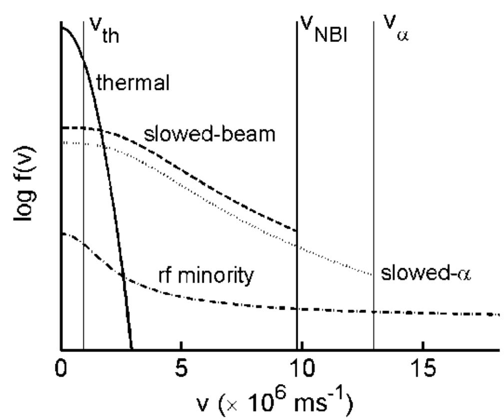
side view



top view



Illustrative 1D slice



$$n = \int_0^\infty \int_{-1}^1 \hat{f}(E,\lambda)\, d\lambda\, dE$$

$$nu_\parallel = \int_0^\infty \int_{-1}^1 v_\parallel \hat{f}(E,\lambda)\, d\lambda\, dE$$

$$p_\parallel = m \int_0^\infty \int_{-1}^1 (v_\parallel - u_\parallel)^2 \hat{f}(E,\lambda)\, d\lambda\, dE$$

$$p_\perp = \frac{m}{2} \int_0^\infty \int_{-1}^1 v_\perp^2 \hat{f}(E,\lambda)\, d\lambda\, dE.$$

$\lambda = v_\parallel / v$

MHD with rotation & anisotropy

- Inclusion of anisotropy and flow in equilibrium MHD equations

[R. Iacono, et al Phys. Fluids B 2 (8). 1990]

$$\nabla\cdot(\rho\mathbf{v})=0,\qquad \rho\mathbf{v}\cdot\nabla\mathbf{v}=\mathbf{J}\times\mathbf{B}-\nabla\cdot\overline{\mathbf{P}},\qquad \nabla\cdot\mathbf{B}=0$$

$$\mu_0\mathbf{J}=\nabla\times\mathbf{B},\qquad \nabla\times(\mathbf{v}\times\mathbf{B})=0,$$

$$\overline{\mathbf{P}}=p_\perp\overline{\mathbf{I}}+\Delta\mathbf{B}\mathbf{B}/\mu_0\,,\qquad \Delta=\frac{\mu_0\left(p_{\|}-p_\perp\right)}{B^2}$$

- Frozen flux condition + axis-symmetry: $\mathbf{v}=\dfrac{\psi_M'(\psi)}{\rho}\mathbf{B}-R\phi_E'(\psi)\mathbf{e}_\varphi.$

Generalised Grad-Shafranov:

$$\boxed{\nabla\cdot\left[\tau\left(\frac{\nabla\psi}{R^2}\right)\right]=-\frac{\partial p_{\|}}{\partial\psi}-\rho H_M'(\psi)+\rho\frac{\partial W}{\partial\psi}-I_M'(\psi)\frac{I}{R^2}-\psi_M''(\psi)\mathbf{v}\cdot\mathbf{B}+R\rho v_\phi\phi_E''(\psi)}$$

$$I=RB_\varphi$$

$$I_M(\psi)=\tau\, I-\mu_0R^2\psi_M'(\psi)\phi_E'(\psi)$$

$$H_M(\psi)=W_M(\rho,B,\psi)-\frac{1}{2}\left[R\phi_E'(\psi)\right]^2+\frac{1}{2}\left[\frac{\psi_M'(\psi)B}{\rho}\right]^2,\qquad \tau=1-\Delta-\mu_0\left(\psi_M'\right)^2/\rho,$$

$$\left\{I_M(\psi),\psi_M(\psi),\phi_E(\psi),H_M(\psi),\frac{\partial p_{\|}}{\partial\psi},\frac{\partial W}{\partial\psi}\right\}$$

Set of 6 profile constraints

- Implemented into EFIT TENSOR and HELENA+ATF

PDE can be elliptic, hyperbolic or parabolic

- Expand the highest order derivative. If $\Delta = 0$, this is

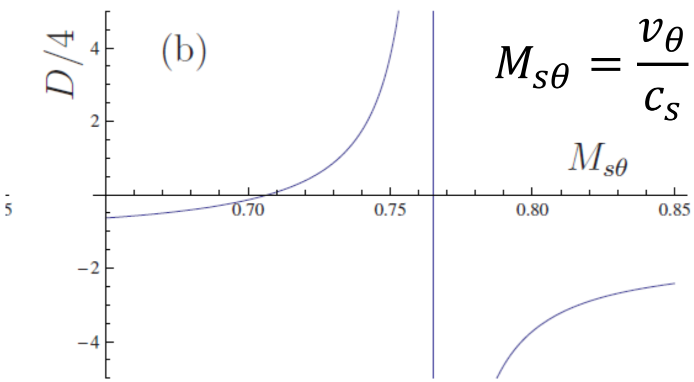
$$\left(A_{RR} \frac{\partial^2 \psi}{\partial R^2} + A_{RZ} \frac{\partial^2 \psi}{\partial R \partial Z} + A_{ZZ} \frac{\partial^2 \psi}{\partial Z^2} \right)$$

$$A_{RR} = 1 - \frac{{v_\theta}^2 {v_Z}^2}{{c_s}^2 {c_{A\theta}}^2 - ({c_s}^2 + {c_A}^2){v_\theta}^2 + {v_\theta}^4},\ A_{rz} = \frac{2{v_\theta}^2 v_R v_Z}{{c_s}^2 {c_{A\theta}}^2 - ({c_s}^2 + {c_A}^2){v_\theta}^2 + {v_\theta}^4},\ A_{zz} = 1 - \frac{{v_\theta}^2 {v_R}^2}{{c_s}^2 {c_{A\theta}}^2 - ({c_s}^2 + {c_A}^2){v_\theta}^2 + {v_\theta}^4}$$

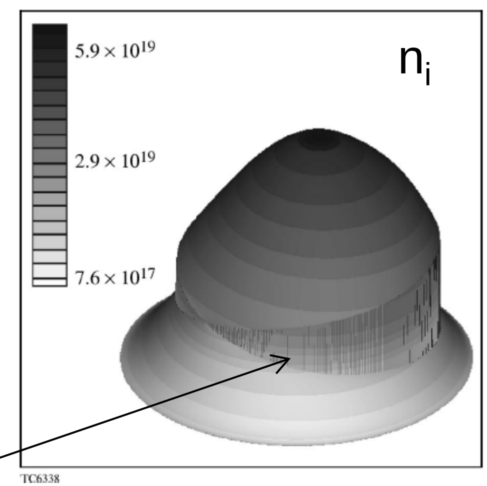
$${v_\theta}^2 = {v_R}^2 + {v_Z}^2, c_s = (\gamma p/\rho)^{1/2}, c_A = B/(\mu_0 \rho)^{\frac{1}{2}}, c_{A\theta} = B_\theta/(\mu_0 \rho)^{1/2}$$

Properties depend on discriminant, D

D<0 : elliptic
D=0 : parabolic
D>0: hyperbolic



$$M_{s\theta} = \frac{v_\theta}{c_s} \frac{B}{B_\theta}$$



e.g. McClements and Hole, Phys. Plasmas **17**, 082509 2010

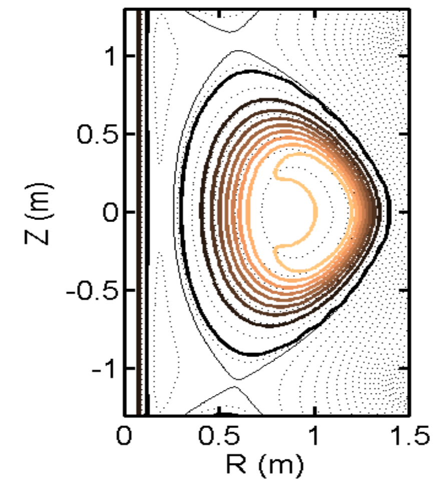
Possible transonic (jump) equilibria

- Elliptic PDE : no characteristic curves → no information propagation
- Hyperbolic PDE : characteristic curves → admit wave like disturbance

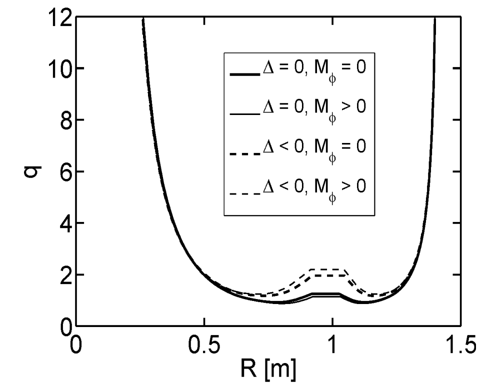
Impact of anisotropy & flow on equilibrium

➢ **Plasma Configuration** (magnetic structure)

- **EFIT TENSOR** reconstruction code: Adds physics of flow/ anisotropy and kinetic constraints
  [Fitzgerald, Appel, Hole, Nucl. Fusion **53** (2013) 113040]
- **HELENA+ATF** enables stability studies
  [Qu, Fitzgerald, Hole, Plas. Phys. Con. Fus. **56** (2014) 075007]



- What is the impact?
  - $p_{||}$, $p_{\perp}$, ρ not a flux function
  - can modify rotational transform
  - if $p_{||} > p_{\perp}$, $p_{||}$ surfaces distorted and displaced inward relative to flux surfaces
  - if $p_{\perp} > p_{||}$, an increase will occur in centrifugal shift.



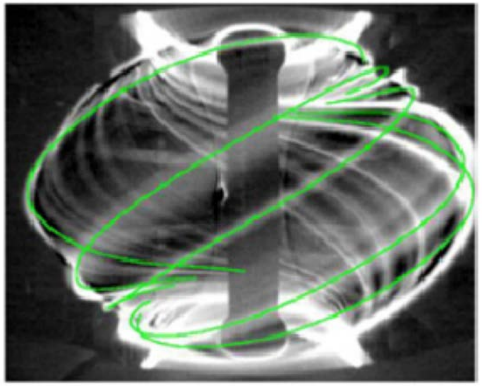
- Impact on ITER scenarios: made progress in May 2018 visit (5 weeks) as part of ITER Science Fellowship

Tokamak Stability Zoo

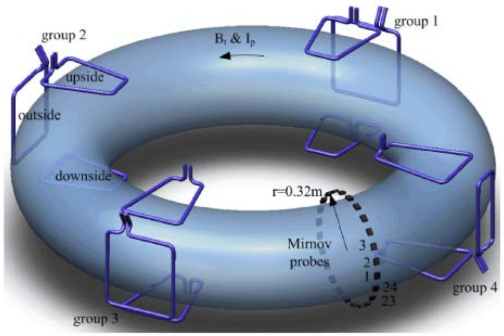
A whole zoo of modes. Can divide them as:

- Most-serious (disruptive): e.g. external modes such as the (n, m) = (1,1) *external kink*, driven by gradients in pressure and current density

- Serious but tolerable (performance-limiting):
  - *Sawteeth*, internal kink, (n, m) = (1,1) – reconnection of core. Periodic collapse of central temperature
  - **Alfven eigenmodes**, wave-particle resonance driven. Loss of fast particle confinement
  - *Edge-Localised Modes* (ELMs), which occur for moderately high *m* and *n*.



ELM mitigation / suppression demonstrated by application of resonant magnetic perturbation coils, that deliberately perturb edge



Linearisation - Spectral theory

- Formally, stability theory often conducted as a perturbation treatment:

$$\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v_0}(\boldsymbol{r}) + \boldsymbol{\varepsilon}\,\boldsymbol{v_1}(\boldsymbol{r},t)$$
$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{B_0}(\boldsymbol{r}) + \boldsymbol{\varepsilon}\,\boldsymbol{B_1}(\boldsymbol{r},t)$$
$$\rho(\boldsymbol{r},t) = \rho_0(\boldsymbol{r}) + \boldsymbol{\varepsilon}\,\rho_1(\boldsymbol{r},t)$$
$$p(\boldsymbol{r},t) = p_0(\boldsymbol{r}) + \boldsymbol{\varepsilon}\,p_1(\boldsymbol{r},t)$$

ε = linear expansion parameter

- Introduce a Lagrangian displacement vector field ξ of a plasma away from an equilibrium state

$$\boldsymbol{v} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} = \frac{\partial\boldsymbol{\xi}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{\xi}$$

- Substitute into continuity, momentum, Faraday's law (with ideal Ohm's law), and equation of state

$$\rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = \mathbf{F}(p_1(\boldsymbol{\xi}), B_1(\boldsymbol{\xi}), \rho_1(\boldsymbol{\xi}))$$

where **F** is the ideal MHD Force operator

$$\mathbf{F}(\boldsymbol{\xi}) = -\nabla p_1 - \mathbf{B}\times(\nabla\times\mathbf{B}_1) + (\nabla\times\mathbf{B})\times\mathbf{B}_1 + (\nabla\Phi)\nabla\cdot(\rho\boldsymbol{\xi})$$

- It can be shown that **F** is self-adjoint, and so if ξ has a finite norm, solutions of ξ lie in a Hilbert space.

Spectral theory

- Equation of motion becomes: $\rho^{-1}\mathbf{F}(\boldsymbol{\xi}) = \dfrac{\partial^2 \boldsymbol{\xi}}{\partial t^2}$
- Normal modes with exponential-time leads to *spectral equation:*

$$\rho^{-1}\mathbf{F}(\boldsymbol{\xi}) = -\omega^2 \boldsymbol{\xi}$$

- Stability problem reduces to a study of the marginal equation: $\rho^{-1}\mathbf{F}(\boldsymbol{\xi}) = 0$
- Discretisation of equation of motion leads to: $\mathbf{L} \cdot \mathbf{x} = \lambda \mathbf{x}$
- The spectrum of **L** obtained by study of the inhomogeneous equation

$$(\mathbf{L} - \lambda \mathbf{I}) \cdot \mathbf{x} = \boldsymbol{a}$$

where a is a given vector in Hilbert space

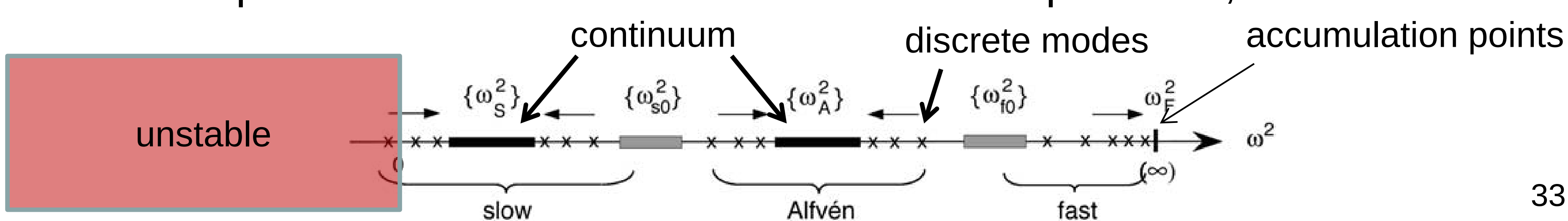
For complex λ, three possibilities exist

$(\mathbf{L} - \lambda \mathbf{I})^{-1}$ does not exist as $(\mathbf{L} - \lambda \mathbf{I}) \cdot \mathbf{x} = \mathbf{0}$ has a solution $\Rightarrow \lambda \in$ discrete spectrum of L

$(\mathbf{L} - \lambda \mathbf{I})^{-1}$ exists but is unbounded $\Rightarrow \lambda \in$ continuous spectrum of L

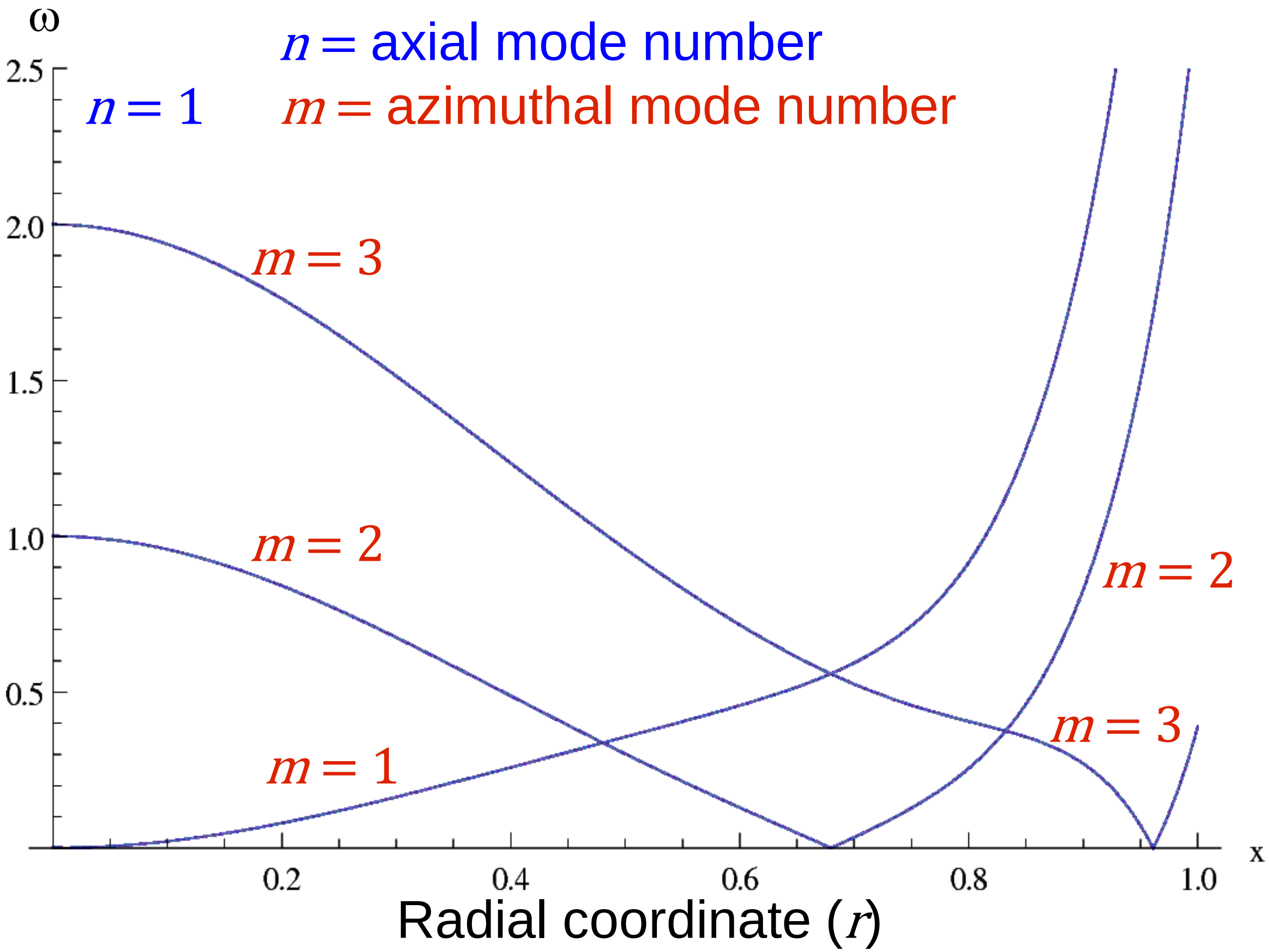
$(\mathbf{L} - \lambda \mathbf{I})^{-1}$ exists but is bounded $\Rightarrow \lambda \in$ resolvent set of L

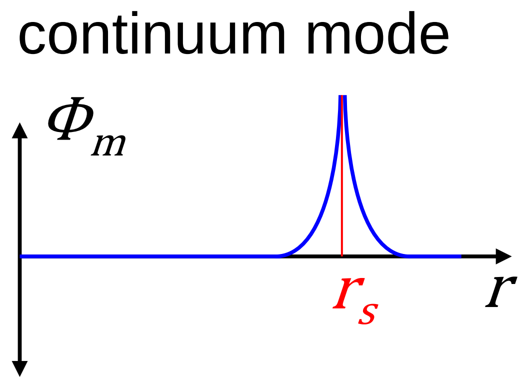
Schematic spectrum of MHD waves for a static 1D equilibrium, where $\lambda = -\omega^2$,



Continuum of a periodic cylinder (1.5D)

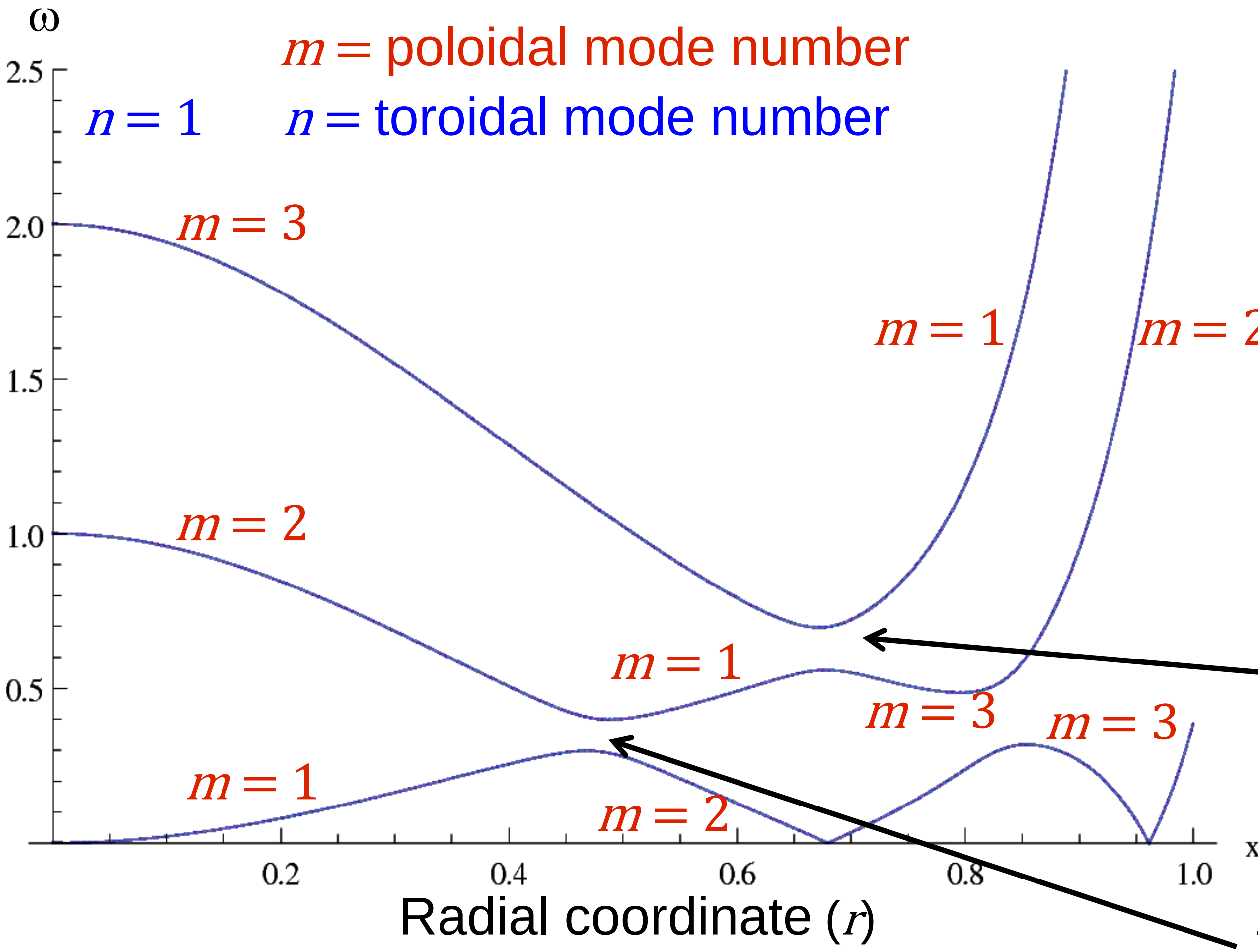
- Continuum resonance frequency versus radial position for a periodic cylinder.

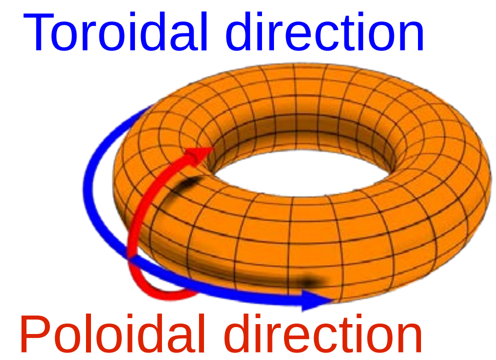




Continuum of a tokamak (2D)

- Continuum resonance frequency versus radial position for a large-aspect ratio elliptical cross-section tokamak.





Gaps due to poloidal variation in *B* arising due to

Ellipticity of cross-section

R. L. Dewar *et al.*, Phys. Fluids, **17,** 930, 1974.

Toroidicity of plasma

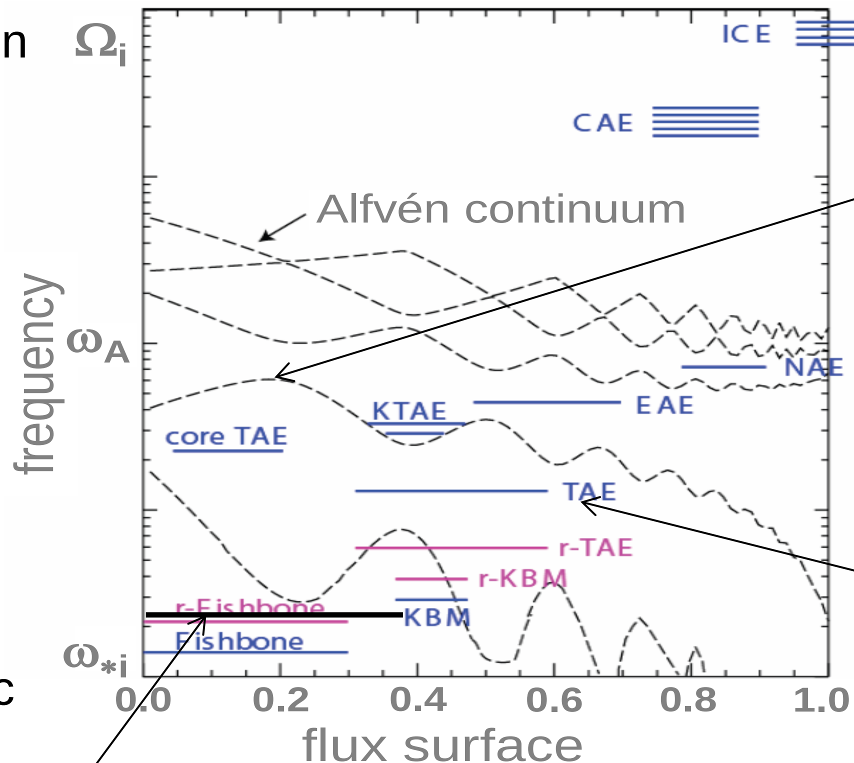
Illustrative tokamak spectrum

- A zoo of gaps and gap modes (n=3)

Ion cyclotron frequency

Alfven frequency

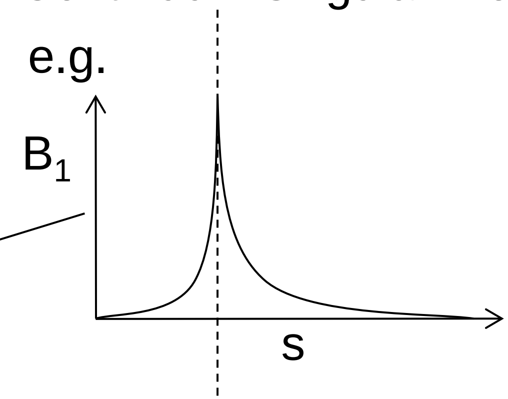
Thermal diamagnetic frequency



(radial coordinate)

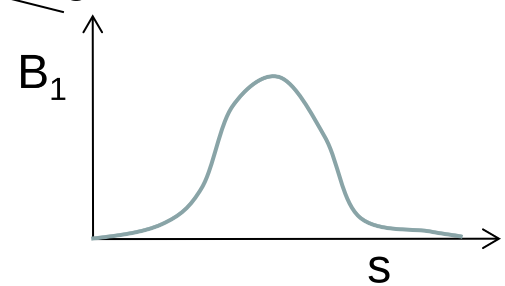
Continuum singular modes

e.g.



Global Alfven eigenmodes

e.g.



Beta induced Alfven eigenmode (BAE): Low frequency mode that exists due to finite beta (pressure)

[Heidbrink Phys. Plasmas, Vol. 9, No. 5, May 2002]

Burning plasma physics

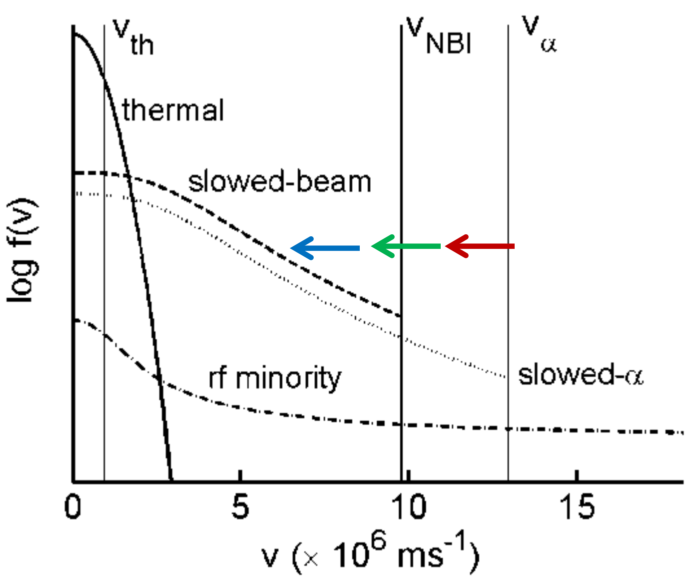
Motivation:

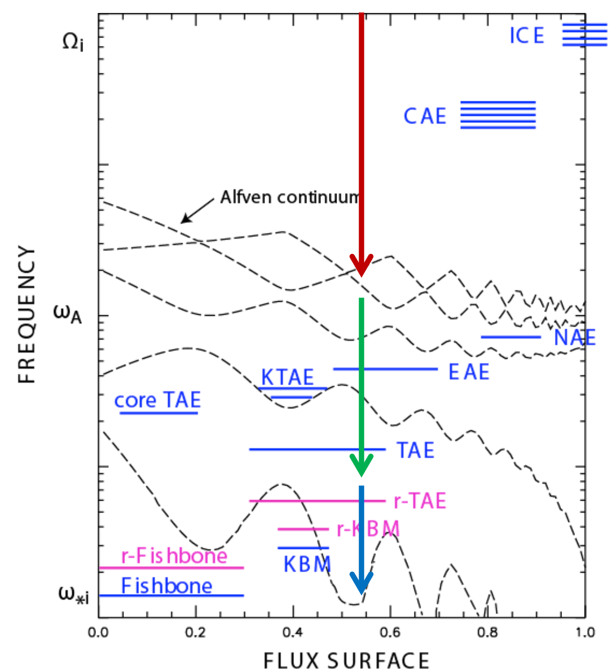
(1) fast ions from auxiliary heating and fusion α's make the plasma non-Maxwellian (e.g. **anisotropy** and **plasma rotation**)

(2) As the ions collisionally slow, they drive modes of the plasma.

(3) At large amplitude, modes can eject energetic ions from plasma, short-circuit heating mechanism → prevent burn

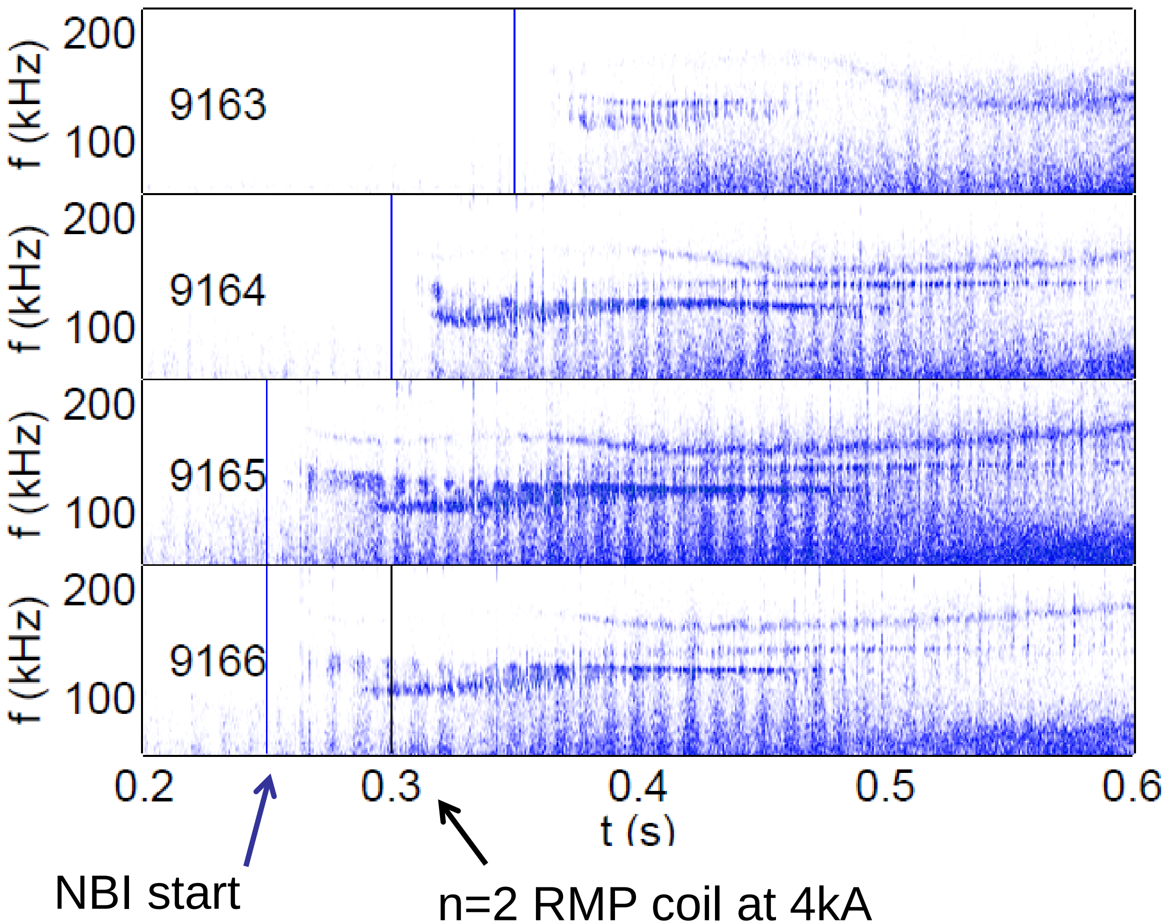
neutral beam injection (NBI)



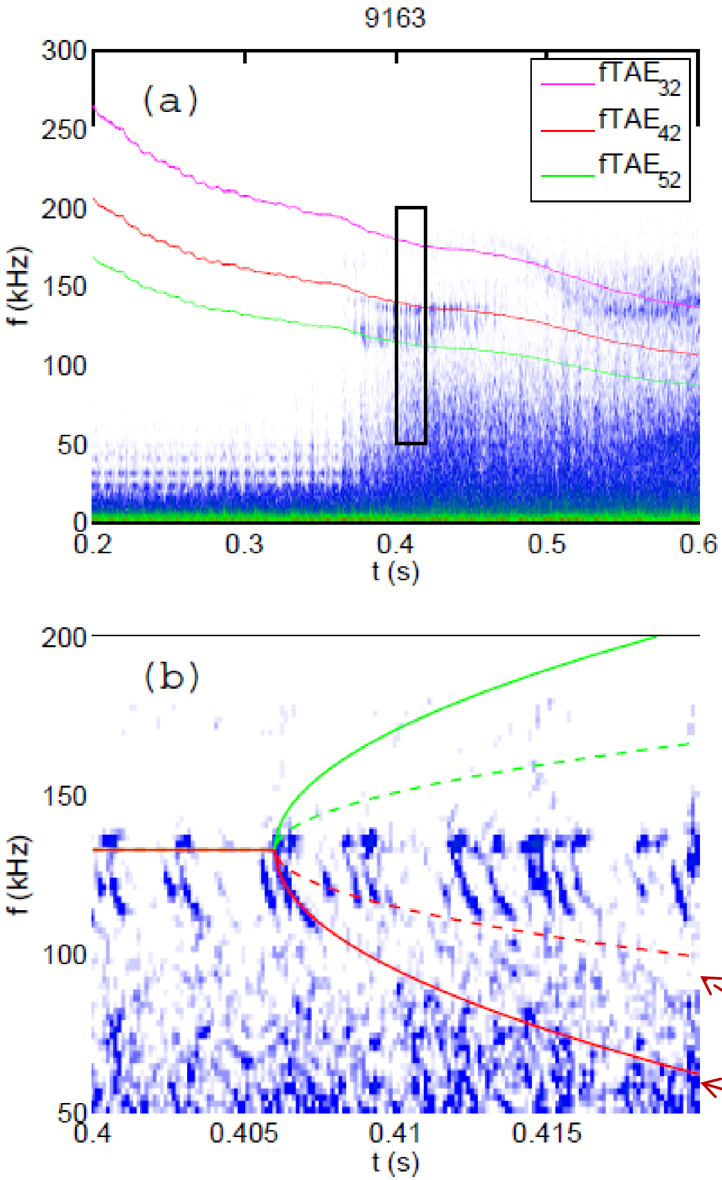


Example: Bursty TAE modes in KSTAR

- Staggered early NBI heating
- Application of RMP coil (delayed onset of H mode, reduced TAE mode intensity)



Mode evolution highly nonlinear

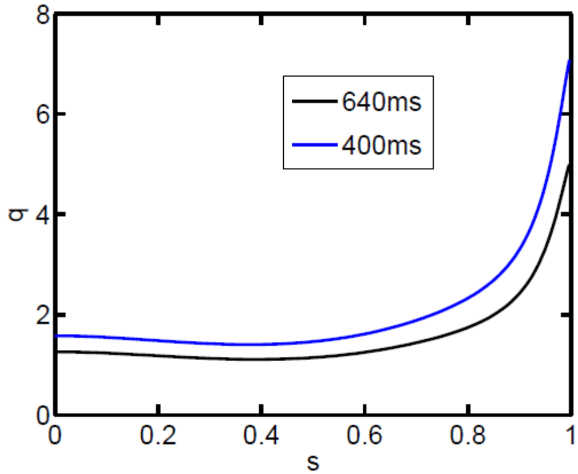


- Mode frequency scales with middle of (m,n)= (4,2) TAE gap
  - Mode chirps $\Delta\omega/\omega_0 \approx 25/133 = 0.19$ in $\Delta t\ \omega_A = 5500$ wave periods.
  - 1D analytic theory can not capture complicated 3D behaviour [Breizman. Nuc. Fus., 50:084014, 2010.]

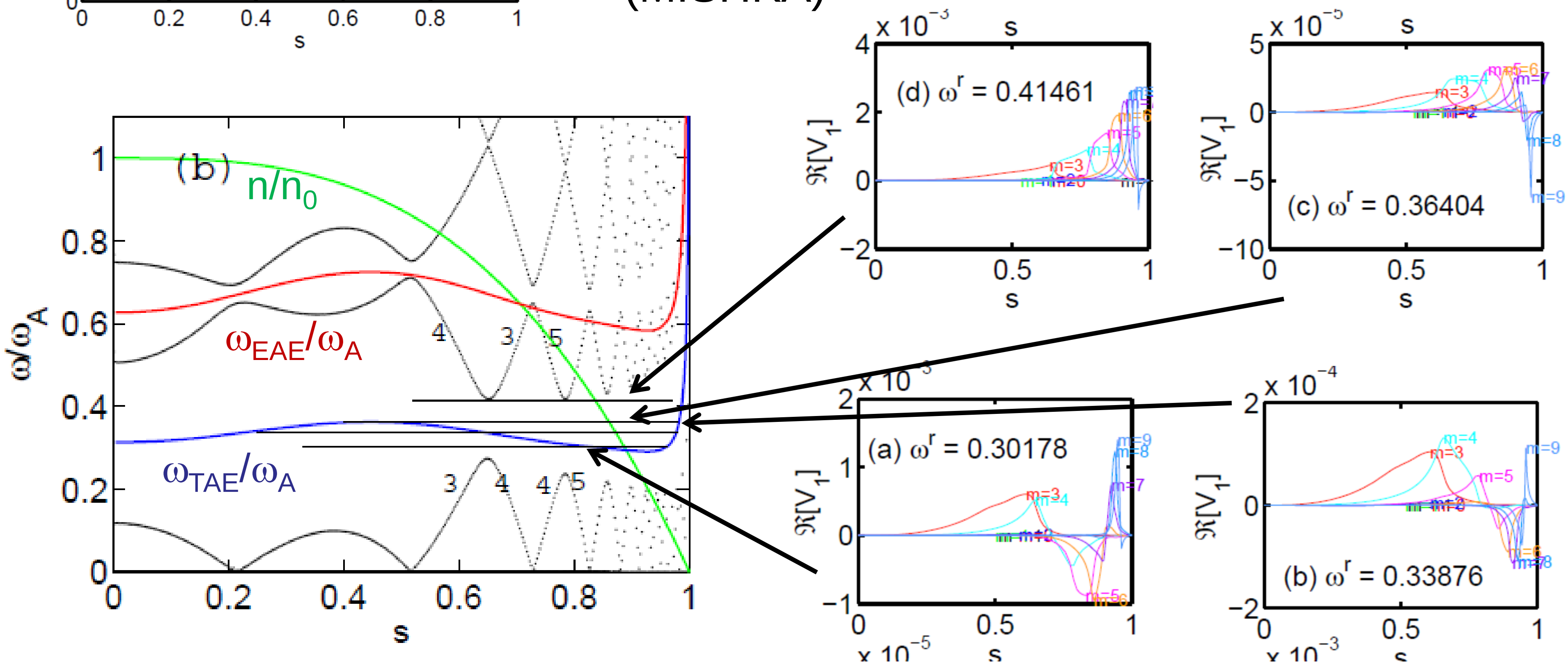
$$\delta\omega = \frac{16}{3\pi^2}\sqrt{\frac{2}{3}\gamma_l\sqrt{\gamma_d\delta t}}$$

  - Mode at marginal stability ($\gamma_l = \gamma_d$)
    Calculation gives $\gamma_l/\omega = \gamma_d/\omega_0 = 3\%$
    Fit gives $\gamma_l/\omega = \gamma_d/\omega_0 = 5\%$

Detailed Numerical Modelling

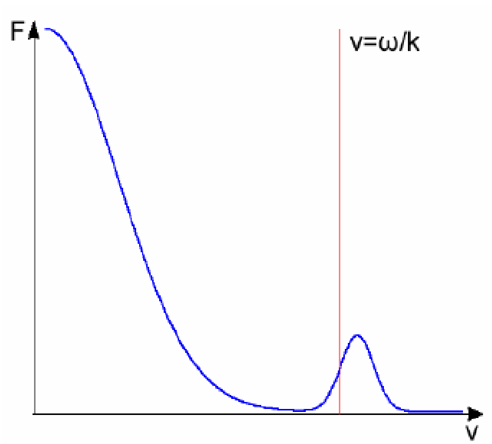


- *q* profile at 640ms available from coherence imaging spectroscopy. [Howard *et al*, Jl. Instrumentation, 10, 2015]
- Remapped to correct I_p at 400ms
- Continuum (CSCAS) and Global Modes (MISHKA)

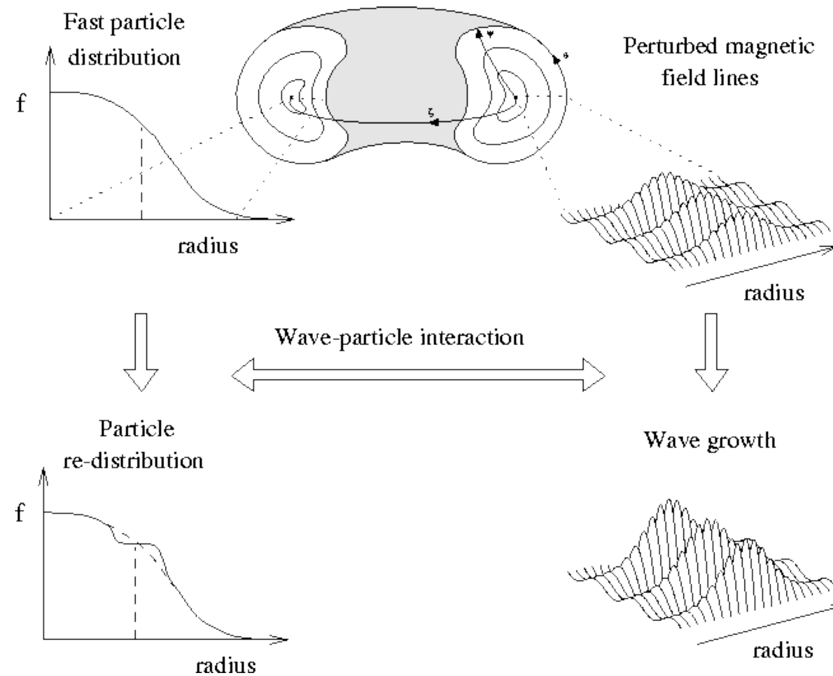


Wave-particle drive: (non)linear dynamics

Evolution of unstable modes in realistic geometry and with simulated distribution functions uses drift-kinetic wave-particle interaction codes, such as HAGIS.
[Pinches *et al,* Comp. Phys. Comm. 111 (1998) 133-149]



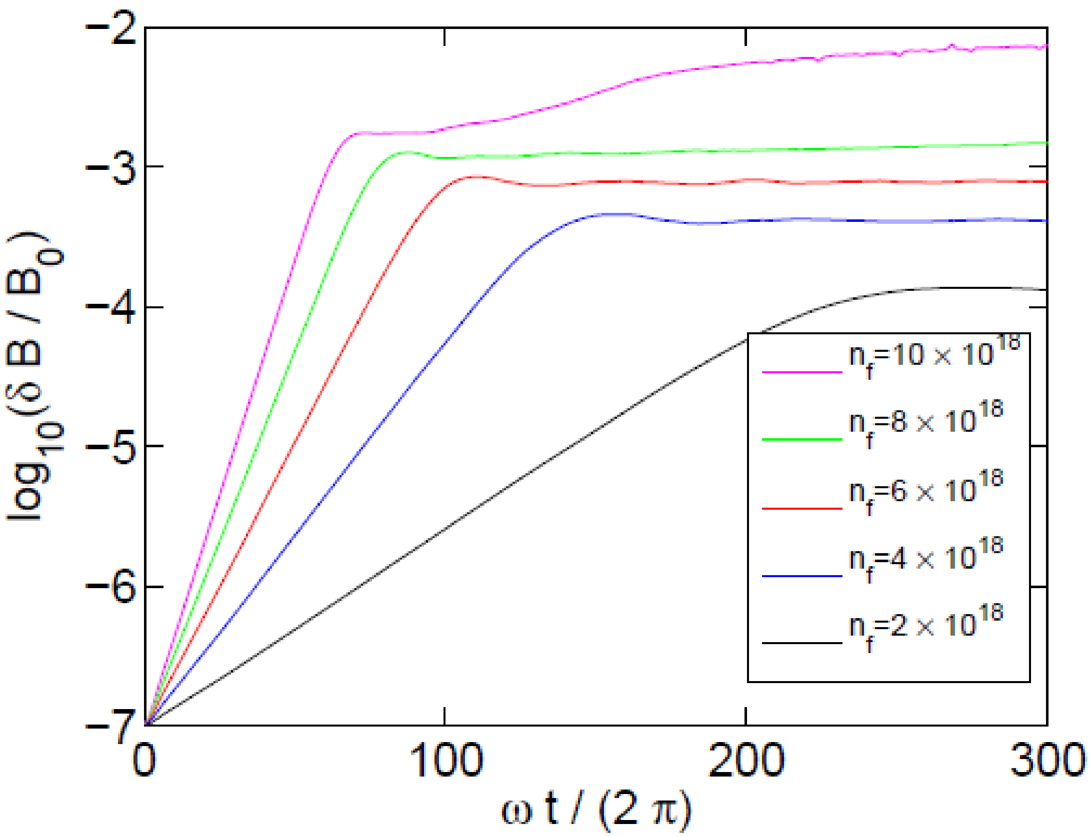
$$\left.\frac{\partial F_0}{\partial v}\right|_{\frac{\omega}{k}} > 0$$



NCI installed simulation codes:
HAGIS: > 64 cores, ~2 hours :
MEGA: > 512 cores, ~6 hours
GENE: >2k nodes (~48k cores)

Simulation: HAGIS Wave Drive

- Lowest frequency mode unstable
- Wave saturates at $\delta B/B_0 \approx 2.5 \times 10^{-3}$. Coils measure $\delta B/B_0 \approx 10^{-4}$
- mode growth phase = $\Delta t\ \omega_A /(2\pi)$~100 << $\Delta t\ \omega_A$ = 5550 (measured)
- Linear growth rate γ_l/ω ~3%

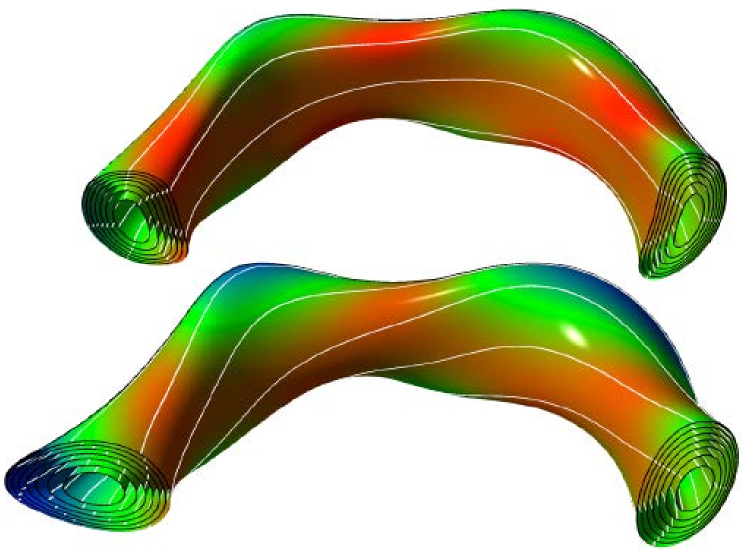


3D equilibria in toroidal plasmas

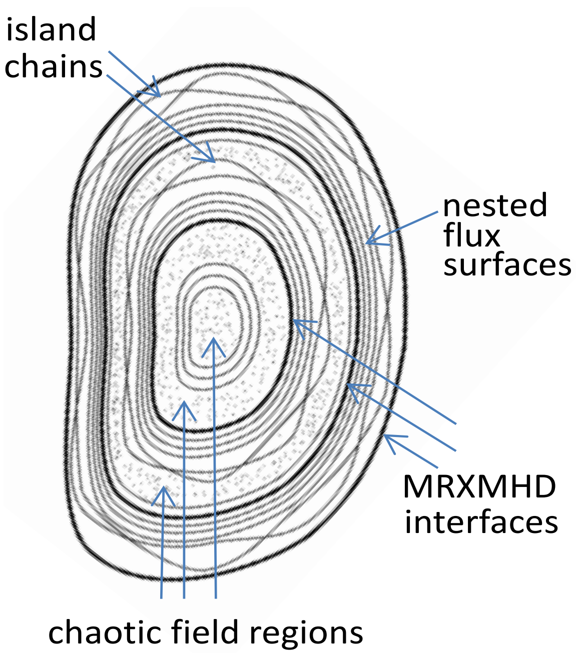
- Simplest model to approximate global, macroscopic force-balance is magnetohydrodynamics (MHD).

$$\nabla p = \mathbf{J} \times \mathbf{B}, \qquad \nabla \times \mathbf{B} = \mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0$$

- Non-axisymmetric $\Rightarrow$ field does **not** lie in nested flux surfaces **unless** surface currents allowed.
- Existing 3D solvers (e.g. VMEC) assume nested flux surfaces.

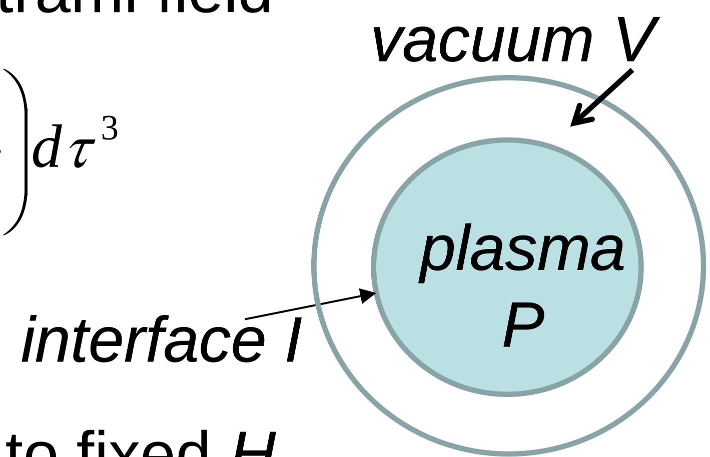


[CTH stellarator, Hanson et al, IAEA 2012]



Taylor Relaxed States: Relaxed MHD

- Zero pressure gradient regions are **force-free** magnetic fields:
- In 1974, Taylor argued that turbulent plasmas with small resistivity, and viscosity relax to a Beltrami field



Internal energy: $$W = \int_{P \cup V} \left(\frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1} \right) d\tau^3$$

Total Helicity : $$H = \int_P (\mathbf{A} \cdot \mathbf{B}) d\tau^3$$

Taylor solved for minimum *W* subject to fixed *H*

i.e. solutions to δF=0 of functional $F = W - \mu H \;\; / 2$

$$P: \qquad \nabla \times \mathbf{B} = \mu \mathbf{B}$$

$$I: \qquad \left[\left[\frac{B^2}{2\mu_0} + p \right]\right] = 0$$

$$V: \qquad \nabla \times \mathbf{B} = 0$$

Model very successful for toroidal pinches, multipinch, and spheromaks

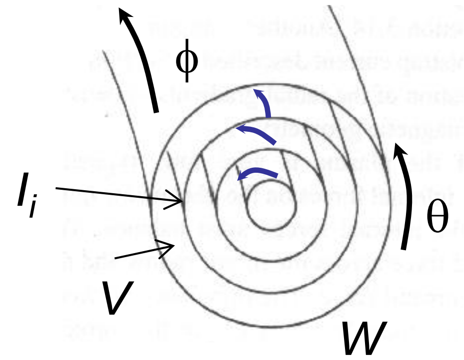
discretised $\mathbf{J} \times \mathbf{B} = \nabla p$

Generalised Taylor Relaxation: MRxMHD

- Assume each invariant tori I_i act as ideal MHD barriers to relaxation, so that Taylor constraints are localized to subregions.

New system comprises:

- ➢ N plasma regions P_i in relaxed states.
- ➢ Regions separated by ideal MHD barrier I_i.
- ➢ Enclosed by a vacuum V,
- ➢ Encased in a perfectly conducting wall W



$$W_l = \int_{R_i} \left(\frac{B_l^2}{2\mu_0} + \frac{P_l}{\gamma - 1} \right) d\tau^3$$

$$H_l = \int_V (\mathbf{A}_l \cdot \mathbf{B}_l) d\tau^3$$

Seek minimum energy state:

$$F = \sum_{l=1}^{N} \left(W_l - \mu_l H_l / 2 \right)$$

$$P_l : \quad \nabla \times \mathbf{B} = \mu_l \mathbf{B}$$

$$P_l = \text{constant}$$

$$I_l : \quad \mathbf{B} \cdot \mathbf{n} = 0$$

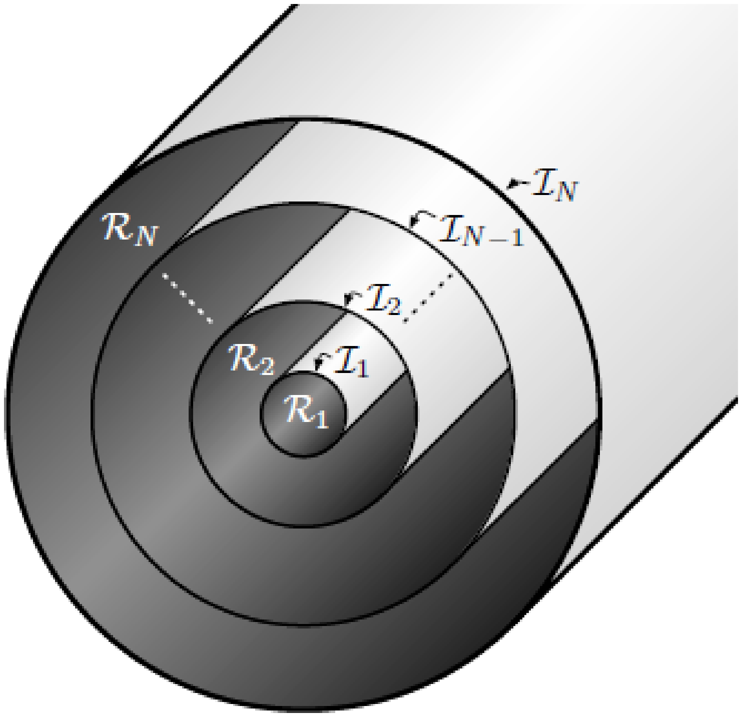
$$[[P_l + B^2/(2\mu_0)]] = 0$$

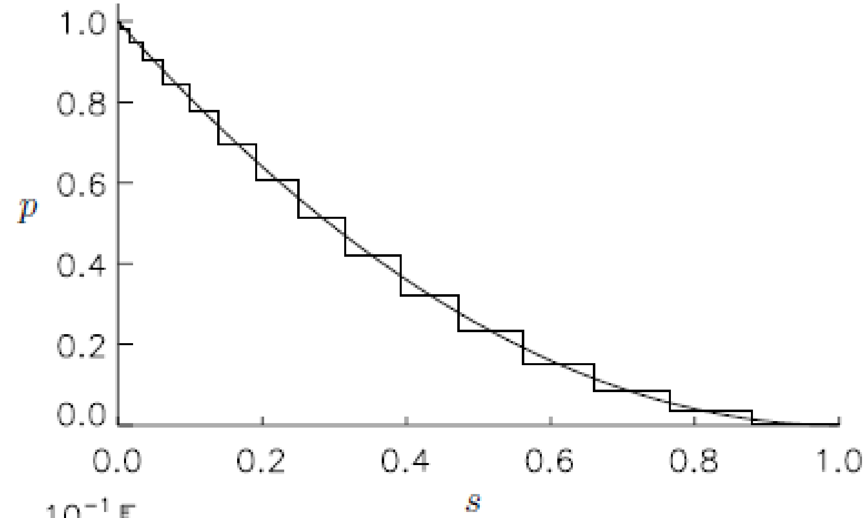
$$V : \quad \nabla \times \mathbf{B} = 0$$

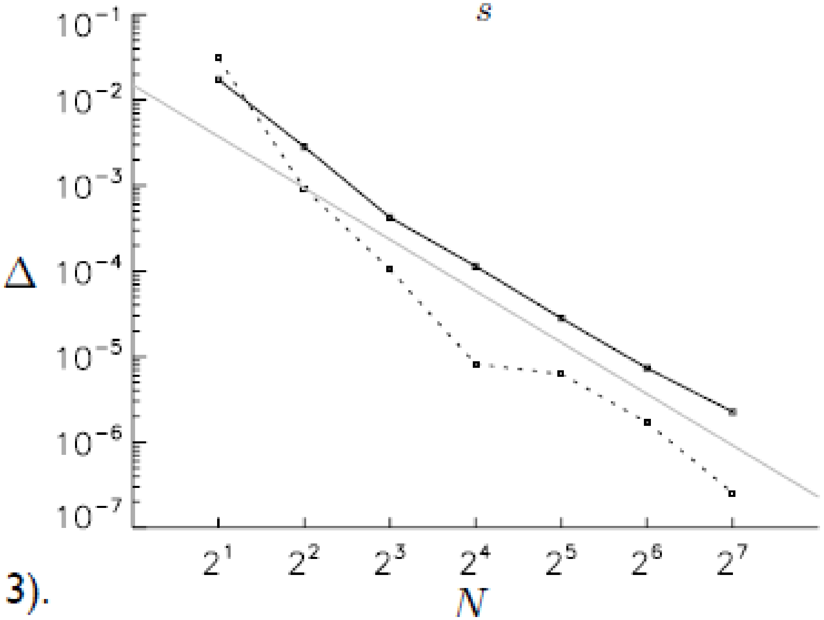
$$\nabla \cdot \mathbf{B} = 0$$

$$W : \quad \mathbf{B} \cdot \mathbf{n} = 0$$

MRXMHD approaches ideal MHD as N→∞





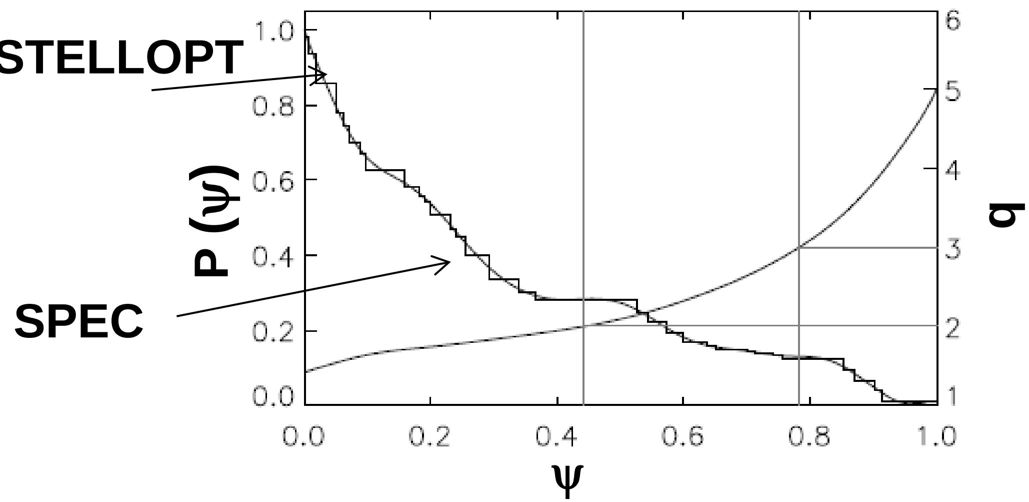


[1] G. Dennis *et al.*, *Phys. Plasmas* **20**, 032509 (2013).

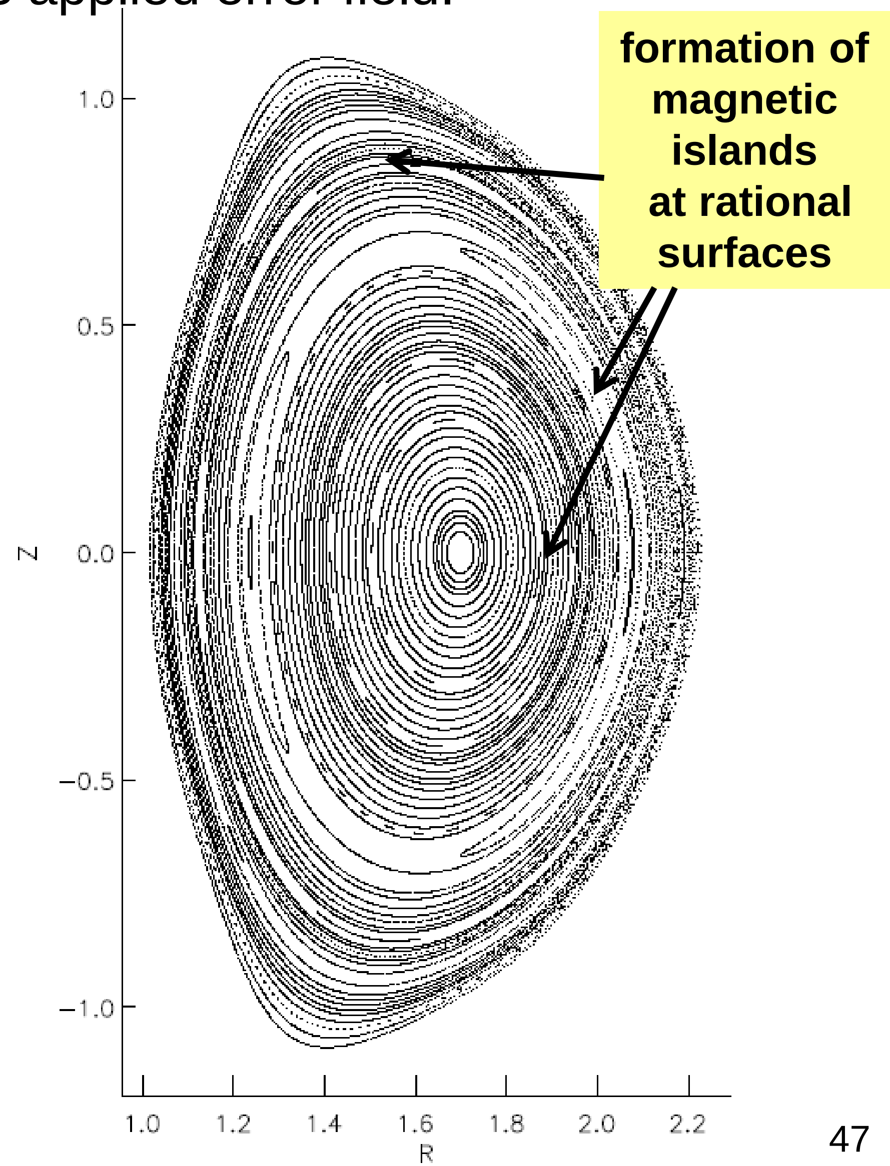
MRxMHD implemented in SPEC code

[Hudson et al Phys. Plasmas 19, 112502 (2012)]

- 3D boundary of a DIII-D plasma with n=3 applied error field.
- Irrational interfaces chosen to coincide with pressure gradients.

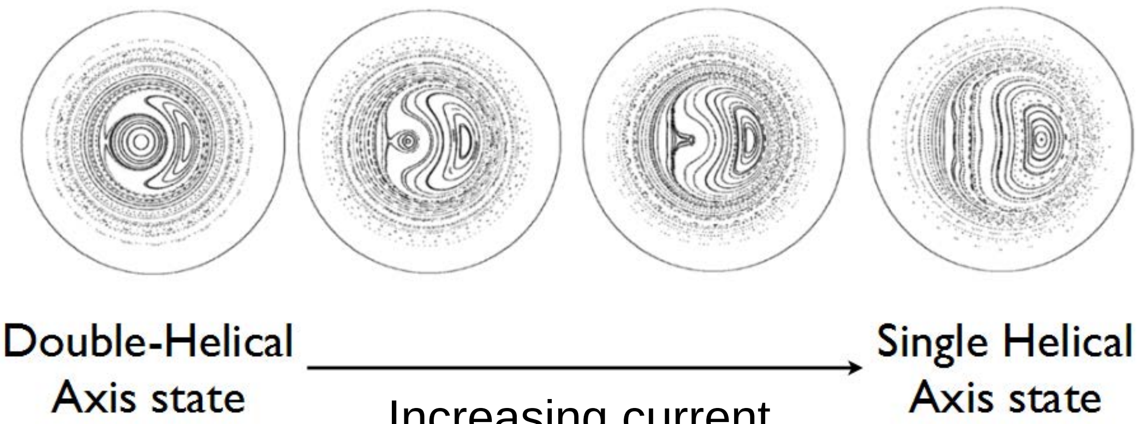


- Island formation is permitted
- No rational "shielding currents" included in calculation.



Spontaneously formed helical states

- The quasi-single helicity state is a stable helical state in RFP: becomes purer as current is increase

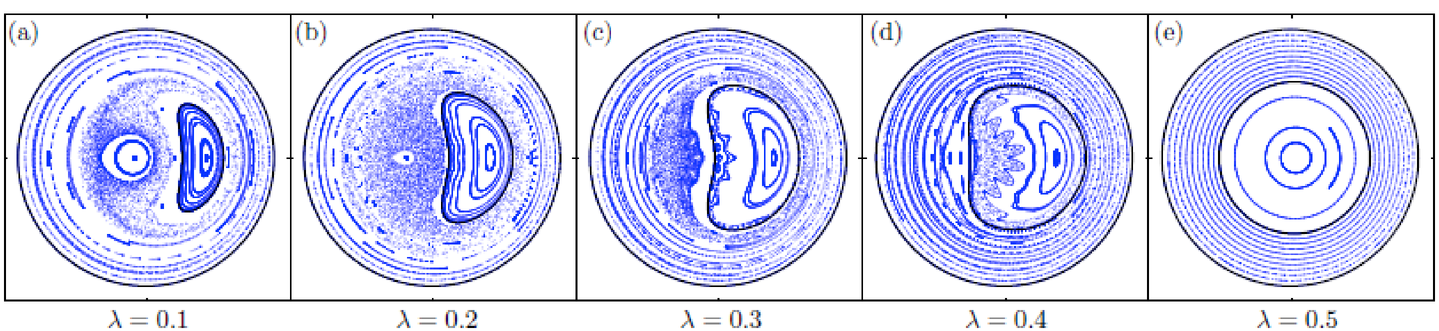


"Experimental" Poincaré plot

[Fig. 6 of P. Martin et al., Nuclear Fusion 49, 104019 (2009)]

- State can be described by a sequence of SPEC solutions, which are in a minimum energy state

[G. R. Dennis *et al* , Phys. Rev. Lett. **111**, 055003, 2013]



Some recent ANU activity in MRxMHD

- Generalized straight field line coordinates concept to fully 3D plasmas
  [R. L. Dewar, S. R. Hudson, A. Gibson, *Plasma Phys. Control. Fusion,* **55,** 014004, 2013]
- Developed theory of resonant current sheet formation and reconnection.
  [R. L. Dewar *et al*, Phys. Plas. **20**, 0832901, 2013.]
- Developed techniques to establish pressure jump a surface can support.
  [M. McGann, ANU PhD thesis, 2013]
- Extended MRxMHD to include non-zero plasma flow, and anisotropy
  [G.R. Dennis, S.R. Hudson, R.L. Dewar, M.J. Hole, PHYSICS OF PLASMAS 21, 042501 (2014)]
  [G.R. Dennis, S.R. Hudson, R.L. Dewar, M.J. Hole, PHYSICS OF PLASMAS 21, 072512 (2014)]
- Generalize Taylor *minimum energy* equilibrium by *extremizing MHD action* (Hamilton's Principle)
  [R. L. Dewar, *et al* J. Plasma Phys. (2015), 515810604]

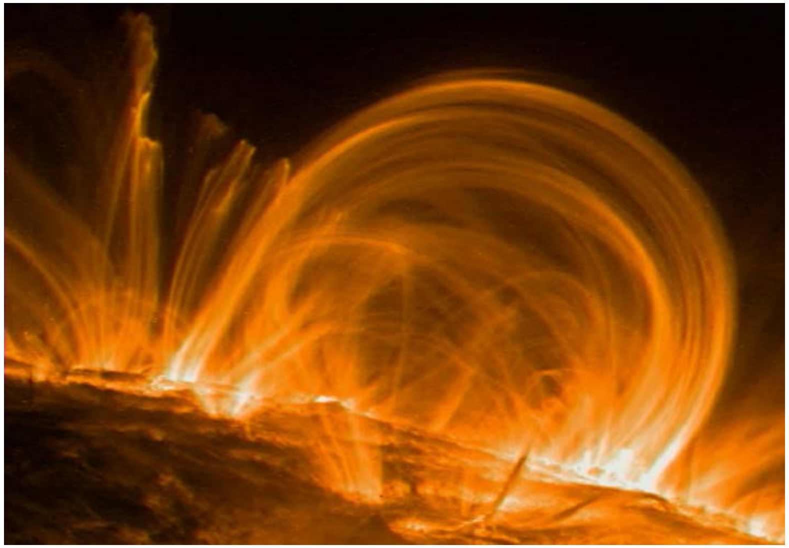
Addition of shear-flow to SPEC to more realistically model fusion plasmas.
Important for study of some modes (e.g. ELMs)

- Stability of a two-volume MRxMHD model in slab geometry
  [L. H. Tuen, ANU Masters thesis, 2015]
- Examined gap mode stability (including continiuum damping) in fully 3D plasmas.
  [G. W. Bowden and M. J. Hole, Phys. Plasmas, **22**, 022116, 2015]
  [G. W. Bowden, M. J. Hole and A. Könies, Phys. Plasmas, **22**, 092114, 2015]
  [M. J. Hole et al Plasma Phys. Control. Fusion 59 (2017) 125007]
- A general formulation to compute the spectrum of stable and unstable eigenmodes of MRxMHD
  [R. L. Dewar, Li Huey Tuen, M. J. Hole, Plasma Phys. Control. Fusion 59 (2017) 044009]

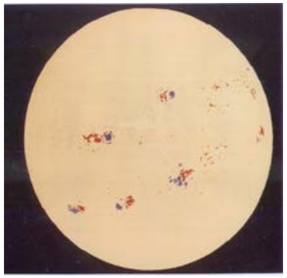
Study of stability and dynamics of MRxMHD
First study of stability of plasmas with chaotic fields

MRxMHD *equilibria* of stellar plasmas

- Most coronal solar flare models assume the field is force-free, and adjust the field pitch to match local measurements of the photosphere field footprint. (nonlinear force free fields)
- boundary conditions are line-tied (solar) cf toroidal (fusion)



Solar magnetogram



e.g. Polarization, intensity of radio thermal free-free emission
[*Iwai et al* Earth, Planets and Space, Volume 66:149, 10 pp.]

intensity

circular polarisation

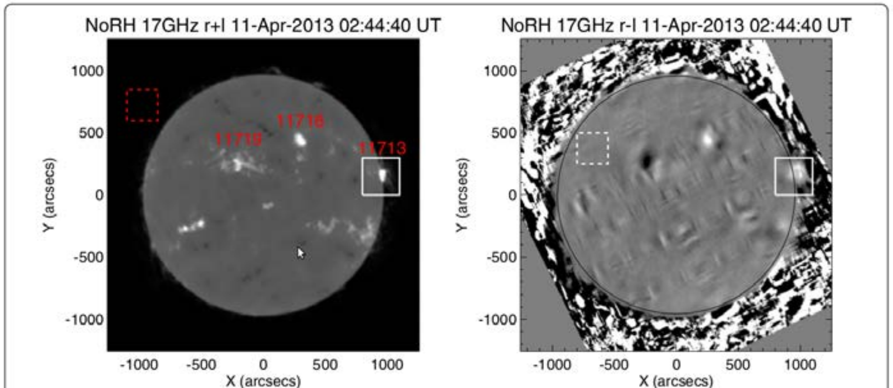


Figure 1 Radio intensity and circularly polarized component observed with NoRH. (Left) Intensity and (right) circularly polarized component of the radio emission at 17 GHz observed with NoRH on 11 April 2013 02:44:40 UT.

Multi-Region Relaxation *Dynamics*

DP170102606 Multi-Region Relaxation Dynamics — a new paradigm for fusion and stellar plasma physics, R. L . Dewar, M. J. Hole, S. R. Hudson; A. Bhattacharjee

Q1) What is the MRxMHD spectrum of normal modes, and what are the effects of field-line curvature and mass flow on their stability?
Q2) When are the MRxMHD current sheets topologically stable towards internal plasmoid formation?
Q3) When do unstable modes saturate at a low level or develop nonlinearly into explosive events?

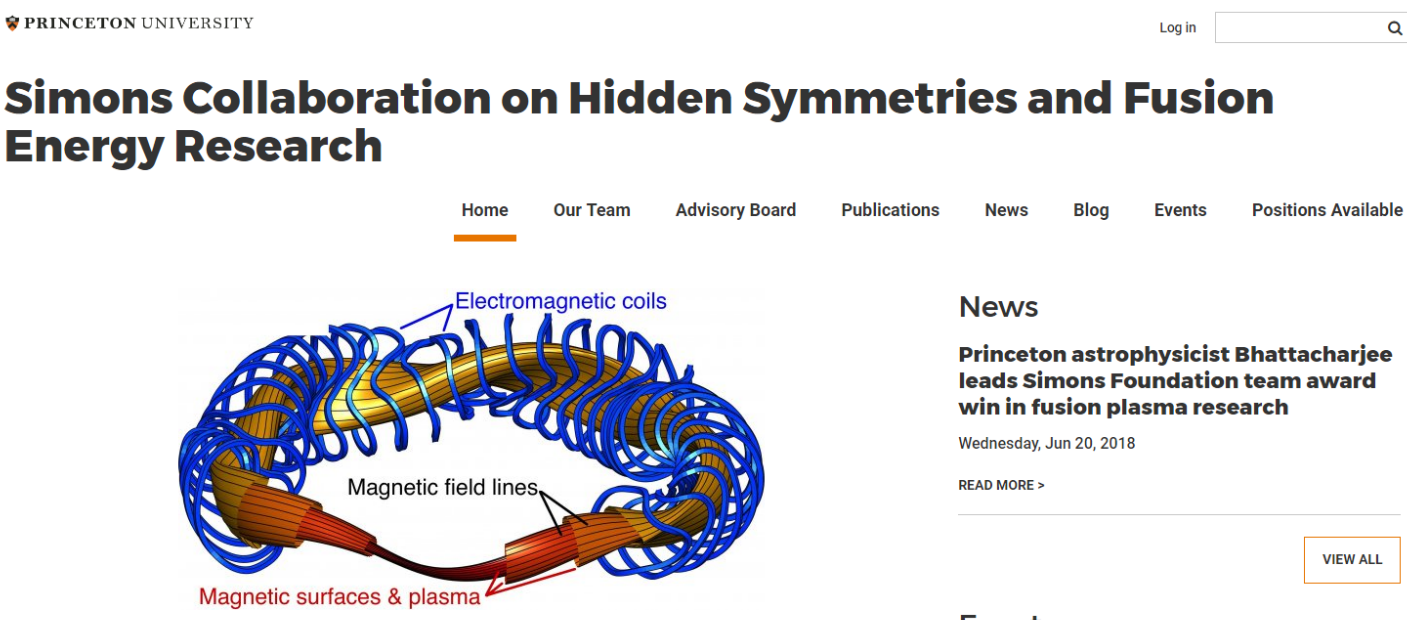
- compare the effect of toroidal (fusion) and line-tied (solar) boundary conditions on wave spectrum and stability
- Develop codes to treat dynamics in full 3D:
  time-domain evolution code SPDC
  frequency-domain code SPECN to calculate linear normal modes
- Comparison of results with experimental and observational data

Hidden Symmetries and Fusion Energy

Simons Foundation Collaboration on Hidden Symmetries and Fusion Energy, directed by Amitava Bhattacharjee of Princeton University.
July 24, 2018 https://hiddensymmetries.princeton.edu/

$2m USD per year over 9 institutions: Princeton University, Cornell University, Max Planck Institute for Plasma Physics Greifswald, New York University, Columbia University, University of Maryland, University of Texas (Austin), Warwick University and the University of Colorado, Boulder

Collaborating institutions: Princeton Plasma Physics Laboratory, Oak Ridge National Laboratory, EPFL Lausanne, University of Wisconsin



Hidden Symmetries and Fusion Energy

Multi-disciplinary team: Mathematics + Physics

- *Princeton University*: A. Bhattacharjee (*Collaboration Director*), P. Constantin (*Founding PI*)
- *Australian National University*: R. Dewar (*Founding PI*), M. Hole
- *Columbia University*: A. Boozer (*Founding PI*)
- *Cornell University*: D. Bindel (*Founding PI*)
- *EPFL, Lausanne*: J. Loizu

Courant Institute of Mathematical Sciences

- *New York University*: G. Stadler (*Founding PI*), A. Cerfon, M. O'Neil, H. Weitzner
- *Princeton Plasma Physics Laboratory:* S. Hudson, H. Mynick and A. Reiman
- *University of Colorado-Boulder*: J. Meiss (*Founding PI*)
- *University of Maryland*: L.-M. Imbert-Gerard (*Founding PI*), M. Landreman
- *University of Texas at Austin*: O. Ghattas (*Founding PI*)
- *Max Planck Institute for Plasma Physics*: P. Helander (*Founding PI*), T. Pedersen, Frank Jenko
- *Oak Ridge National Laboratory:* D. Spong
- *University of Warwick*: R. MacKay (*Founding PI*), J. Robinson, J. Rodrigo
- *University of Wisconsin*: C. Hegna

Advisory Board : *C. Fefferman* (Chair), J. Cary, D. Keyes, B. Khesin, B. Wohlmuth

Hidden Symmetries and Fusion Energy

Objective: To create and exploit an effective mathematical and computational framework for the design of stellarators with hidden symmetries.

Research challenges: Finding optimum magnetic fields with hidden symmetries is an interdisciplinary challenge straddling optimization theory, plasma physics, dynamical systems and the analysis of partial differential equations (PDEs).

Deliverables: Optimum design principles of a stellarator, a modern optimization code (SIMSOPT, SIMonS OPTimization code) that can exploit the full power of petascale and exascale computers, and designs of next-generation stellarator experiments

Features: Support of the Exascale Computing Project of the US Dep. Of Energy. Aims to harness Aurora, first exascale computer in the US

Hidden Symmetries and Fusion Energy

Some mathematical challenges

- Magnetic field line flow produce a sea of good surfaces and chaos. *How do we quantify non-integrability of field lines and control it? How well do particles track field lines? How are particles transported across "cantori"*?

- MHD describes the magnetized plasma as a fluid. The 3D MHD problem is not known to be globally well-posed even with dissipation added. *What are the special solutions enabled by hidden symmetries?*

- *Formulate the design problem as a constrained, risk-averse, multi-objective stochastic optimization problem.* Seek Pareto-optimal solutions. *This cannot be done in a physics-agnostic way.* Existing stochastic optimization methods are not able to address high-dimensional problems.

Collaboration Opportunities within MSI

Research – drawing on MSI skill set

- Computational mathematics: sparse grids, optimisation
- Astrophysics: MHD, shocks,
- PDEs: properties and existence
- Inverse problems

Teaching

- M. J. Hole: Physics Honours Advanced Electromagnetic Theory (Jackson) and Statistical Mechanics; Wave Theory (transmission line theory)
- R. L. Dewar: Dynamical Systems, Fluid Dynamics, Classical Mechanics

ANU Grand Challenge(s)

1. Clean Energy from Fusion

- For over a decade, M. J. Hole has promoted Australian participation in ITER (Australian ITER Forum)
- Dewar and Hole have significant international profile
- ANU also has world-recognised profile in advanced diagnostics and materials
- Wider potential to further engage CECS, CAPS

2. Model / Data Fusion (not nuclear fusion)

- Develop algorithms / framework to constrain the information-explosion with high-fidelity simulation
- *Referred to as "integrated modelling" in fusion science*
- Multiple fields of application: fusion science, climate science, astronomy, computer-network security?

Summary

- Plasma Theory and Modelling: a vibrant ANU pursuit.
- Very strong international collaboration (people & experiments)
- Important student / post-doc research training dimension
- Current Research areas
  - Burning plasma physics: anisotropy and flow, energetic particle driven modes
  - Fully 3D toroidal physics and MRxMHD.
    Impacts of 3D structure on plasma.
    Dynamics of MRxMHD plasmas,
  - Bayesian inference of configurations
    Bayesian inference (inversion) for particle velocity distribution
  - *wakefield accelerator physics, ELM sandpile models*