Inverse boundary problems for elliptic PDE in low regularity setting

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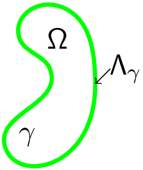
The Calderón problem, 1980

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with smooth boundary. Consider the boundary value problem,

$$\begin{cases} L_\gamma u = \nabla \cdot (\gamma \nabla u) = 0 \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = f. \end{cases}$$

Here $\gamma = \gamma(x)$ is the electrical conductivity, $\gamma \in W^{1,\infty}(\Omega)$, $\gamma > 0$ on $\overline{\Omega}$, f represents the imposed voltage potential at the boundary.



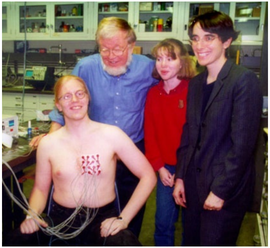
The Dirichlet–to–Neumann map: $\Lambda_\gamma(f) = (\gamma \partial_\nu u)|_{\partial\Omega}$, where ν is the unit outer normal to $\partial\Omega$.

We apply the voltage potential on $\partial\Omega$, measure the resulting current flux at $\partial\Omega$, and encode this information into the Dirichlet–to–Neumann map.

The Calderón problem: Does Λ_γ determine γ in Ω?

Applications:

- Medical imaging (electrical impedance tomography)



- Non-destructive testing (corrosion, cracks)

  ⋮

- Geophysical exploration (oil prospecting)

First global uniqueness result in dimension $n \geq 3$:

Theorem (Sylvester–Uhlmann, 1987)

Let $0 < \gamma_j \in C^{1,1}(\overline{\Omega})$, $j = 1, 2$. *If* $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ *then* $\gamma_1 = \gamma_2$ *in* Ω.

Idea of proof

Step 1. Reduction of the conductivity equation to the Schrödinger equation,

$$\gamma_j^{-1/2} \circ L_{\gamma_j} \circ \gamma_j^{-1/2} = \Delta - q_j, \quad q_j = \frac{\Delta \gamma_j^{1/2}}{\gamma_j^{1/2}} \in L^\infty(\Omega), \quad j = 1, 2.$$

Step 2. $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Longrightarrow$ the integral identity:

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0.$$

Step 3. Construction of special solutions to $(\Delta - q_j)u_j = 0$, called complex geometric optics (CGO) solutions:

$$u_j(x; h) = e^{\frac{x \cdot \zeta_j}{h}} (1 + r_j(x; h)), \quad j = 1, 2,$$

for all $h > 0$ small enough.

Here $\zeta_j \in \mathbb{C}^n$, $\zeta_j \cdot \zeta_j = 0$, $|\zeta_j| \sim 1 \Longrightarrow -\Delta(e^{\frac{x \cdot \zeta_j}{h}}) = 0$, r_j: remainder which tends to zero in a suitable sense, as $h \to 0$.

Step 4. Testing the integral identity against complex geometric optics solutions.

For any $\xi \in \mathbb{R}^n$, find $\zeta_1, \zeta_2 \in \mathbb{C}^n$ such that $\frac{\zeta_1+\zeta_2}{h} = i\xi$.

Substituting the CGO solutions

$$u_1(x) = e^{\frac{x\cdot\zeta_1}{h}}(1 + r_1(x; h)), \quad u_2(x) = e^{\frac{x\cdot\zeta_2}{h}}(1 + r_2(x; h)),$$

into the integral identity, we get

$$\int_\Omega (q_1 - q_2)e^{i\xi\cdot x}(1 + r_1 + r_2 + r_1 r_2)dx = 0.$$

Letting $h \to 0$, we conclude that

$$\mathcal{F}(q_1 - q_2)(-\xi) = 0, \quad \forall \xi \in \mathbb{R}^n.$$

Thus, $q_1 = q_2$ in Ω, and therefore, $\gamma_1 = \gamma_2$ in Ω. This completes the proof.

The issue of regularity of conductivity

- Sylvester–Uhlmann, 1987: $\gamma \in C^{1,1}(\overline{\Omega})$;
- Brown, 1996: $\gamma \in C^{1,\frac{1}{2}+\delta}(\overline{\Omega})$, $\delta > 0$;
- Päivärinta–Panchenko–Uhlmann, 2003: $\gamma \in W^{\frac{3}{2},\infty}(\Omega)$;
- Brown–Torres, 2003: $\gamma \in W^{\frac{3}{2},p}(\Omega)$, $p > 2n$;
- Haberman-Tataru, 2013: $\gamma \in C^1(\overline{\Omega})$ and $\gamma \in W^{1,\infty}(\Omega)$ with $\|\nabla \log \gamma\|_{L^\infty}$ small;
- Caro-Rogers 2016: $\gamma \in W^{1,\infty}(\Omega)$;
- Haberman, 2016, n=3,4: $\gamma \in W^{1,n}(\Omega)$ (γ need not be continuous).

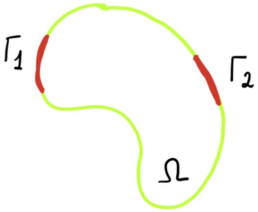
The Calderón problem with partial data

In practice impedance tomography measurements cannot be taken on the entire boundary due to limitations in resources or obstructions from natural obstacles.

This leads us to consider the Calderón problem with partial data.

Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary open non-empty. The partial Dirichlet–to–Neumann map,

$$\Lambda_{\gamma,\Gamma_1,\Gamma_2}(f) = (\gamma\partial_\nu u)|_{\Gamma_2}, \quad \text{supp}\ (f) \subset \Gamma_1.$$



The Calderón problem with partial data: Does $\Lambda_{\gamma,\Gamma_1,\Gamma_2}$ determine γ in Ω? Open in general.

Results for $C^{1,1}$ conductivities

Let $\gamma_1, \gamma_2 \in C^{1,1}(\overline{\Omega})$. If $\Lambda_{\gamma_1,\Gamma_1,\Gamma_2} = \Lambda_{\gamma_2,\Gamma_1,\Gamma_2}$ then $\gamma_1 = \gamma_2$ in Ω.

- Ammari–Uhlmann, 2004: $\gamma_1 = \gamma_2$ near $\partial\Omega$, $\Gamma_1 = \Gamma_2 \subset \partial\Omega$ arbitrary
- Isakov, 2007: $\Gamma_1 = \Gamma_2 = \Gamma$ and $\partial\Omega \setminus \Gamma$ is either a part of a hyperplane or a sphere

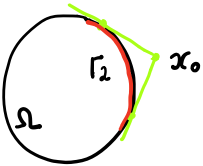
A characteristic feature of these results: reduction of the partial data problem to a full data one using unique continuation and symmetry type arguments.

- Bukhgeim–Uhlmann, 2002:

  $$\Gamma_1 = \partial\Omega, \quad \Gamma_2 = \{x \in \partial\Omega : \xi \cdot \nu(x) < \varepsilon\}, \quad \xi \in \mathbb{S}^{n-1}, \quad \varepsilon > 0.$$

  Note: Γ_2 is slightly more than a half of the boundary
- Kenig–Sjöstrand–Uhlmann, 2007:

  $$\Gamma_1 = \partial\Omega, \quad \Gamma_2 = \{x \in \partial\Omega : \frac{(x - x_0)}{|x - x_0|} \cdot \nu(x) < \varepsilon\},$$
  $$x_0 \notin \overline{ch(\Omega)}, \quad \varepsilon > 0.$$



  Note: when Ω is strictly convex, Γ_2 could be arbitrarily small
- Kenig–Salo, 2014: unifies approaches of Kenig–Sjöstrand–Uhlmann and Isakov and extends both of them

Linearized case: $\Gamma_1 = \Gamma_2$ arbitrary

- Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann, 2009: linearization at the zero potential
- Sjöstrand–Uhlmann, 2016: linearization at a real analytic potential

The issue of regularity of conductivity in the partial data problem

- In the Bukhgeim–Uhlmann result:

$$\Gamma_1 = \partial\Omega, \quad \Gamma_2 = \{x \in \partial\Omega : \xi \cdot \nu(x) < \varepsilon\}, \quad \xi \in \mathbb{S}^{n-1}, \quad \varepsilon > 0.$$

  - Knudsen, 2006: $\gamma \in W^{\frac{3}{2}+\delta,2n}(\Omega)$, $\delta > 0$
  - Zhang, 2012: $\gamma \in C^{1,\delta}(\overline{\Omega}) \cap H^{\frac{3}{2}}(\Omega)$, $\delta > 0$

- In the Kenig–Sjöstrand–Uhlmann result:

$$\Gamma_1 = \partial\Omega, \quad \Gamma_2 = \{x \in \partial\Omega : \frac{(x - x_0)}{|x - x_0|} \cdot \nu(x) < \varepsilon\},$$
$$x_0 \notin \overline{ch(\Omega)}, \quad \varepsilon > 0.$$

Note: when Ω is strictly convex, Γ_2 could be arbitrarily small

Theorem (K.–Uhlmann, 2016)

Let $\gamma_1, \gamma_2 \in C^{1,\delta}(\overline{\Omega}) \cap H^{\frac{3}{2}}(\Omega)$, $\delta > 0$ arbitrarily small. Assume that $\gamma_1, \gamma_2 > 0$ in $\overline{\Omega}$, $\gamma_1 = \gamma_2$ and $\partial_\nu \gamma_1 = \partial_\nu \gamma_2$ on $\partial\Omega \setminus \Gamma_2$. If $\Lambda_{\gamma_1,\Gamma_1,\Gamma_2} = \Lambda_{\gamma_2,\Gamma_1,\Gamma_2}$ then $\gamma_1 = \gamma_2$ in Ω.

Remark. K.–Uhlmann, 2016: the result holds also for $\gamma_1, \gamma_2 \in W^{1,\infty}(\Omega) \cap H^{\frac{3}{2}+\delta}(\Omega)$, $\delta > 0$.

Remark. Rodriguez, 2016 : $\gamma_1, \gamma_2 \in W^{\frac{3}{2}+\delta,2n}(\Omega)$, $\delta > 0$ (independently)

By Sobolev embedding,

$$W^{\frac{3}{2}+\delta,2n}(\Omega) \subset C^{1,\delta}(\overline{\Omega}) \cap H^{\frac{3}{2}}(\Omega).$$

Outline of the proof

Step 1. Complex geometric optics solutions for Lipschitz continuous conductivities

Let $0 < \gamma \in W^{1,\infty}(\Omega)$ and extend it to a function on $\mathbb{R}^n$ so that the extension $0 < \gamma \in W^{1,\infty}(\mathbb{R}^n)$ and $\gamma = 1$ near infinity.

We reduce the conductivity equation to the Schrödinger equation:

$$\gamma^{-1/2} \circ L_\gamma \circ \gamma^{-1/2} = \Delta - q,$$

$$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} = -\nabla\gamma^{1/2} \cdot \nabla\gamma^{-1/2} + \frac{1}{2}\Delta \log \gamma \in (H^{-1} \cap \mathcal{E}')(\mathbb{R}^n).$$

Define the "multiplication by q" map

$$m_q : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$$

by

$$\langle m_q(u), v\rangle_{\mathbb{R}^n} = -\int_{\mathbb{R}^n} (\nabla\gamma^{1/2} \cdot \nabla\gamma^{-1/2}) uv dx - \frac{1}{2}\int_{\mathbb{R}^n} \nabla \log \gamma \cdot \nabla(uv) dx,$$

for $u, v \in H^1(\mathbb{R}^n)$. Here $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ is the distribution duality on $\mathbb{R}^n$.

Construct complex geometric optics solutions for the Schrödinger equation with a singular potential,

$$-\Delta u + m_q(u) = 0 \quad \text{in} \quad \Omega.$$

Following Kenig–Sjöstrand–Uhlmann, 2007, we rely on Carleman estimates for the semiclassical Laplace operator $-h^2\Delta$, $0 < h \to 0$.

Carleman estimates with limiting Carleman weights

Let $\varphi \in C^\infty(\overline{\Omega}, \mathbb{R})$. Consider the conjugated operator

$$P_\varphi = e^{\frac{\varphi}{h}}(-h^2\Delta)e^{-\frac{\varphi}{h}},$$

with the semiclassical principal symbol

$$p_\varphi(x,\xi) = \xi^2 + 2i\nabla\varphi\cdot\xi - |\nabla\varphi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

Definition (Kenig–Sjöstrand–Uhlmann, 2007)

$\varphi \in C^\infty(\overline{\Omega}, \mathbb{R})$ is a limiting Carleman weight for $-h^2\Delta$ if $\nabla\varphi \neq 0$ and the Poisson bracket of $\operatorname{Re} p_\varphi$ and $\operatorname{Im} p_\varphi$ satisfies,

$$\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\}(x,\xi) = 0 \quad \text{when} \quad p_\varphi(x,\xi) = 0, \quad (x,\xi) \in \Omega \times \mathbb{R}^n.$$

Note: if φ is a LCW then so is $-\varphi$.

Example:

- linear weights $\varphi(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}^n$, $|\alpha| = 1$,
- logarithmic weights $\varphi(x) = \log |x - x_0|$, with $x_0 \notin \overline{\Omega}$.

Dos Santos Ferreira–Kenig–Salo–Uhlmann, 2009: Complete local classification of liming Carleman weights on $\mathbb{R}^n$.

Proposition (Salo–Tzou, 2009, Kenig–Sjöstrand–Uhlmann, 2007)

Let φ be a limiting Carleman weight for $-h^2\Delta$, and let $\tilde{\varphi} = \varphi + \frac{h}{2\varepsilon}\varphi^2$. Then for $0 < h \ll \varepsilon \ll 1$ and $s \in \mathbb{R}$, we have the following Carleman estimate with a gain of 2 derivatives,

$$\frac{h}{\sqrt{\varepsilon}}\|u\|_{H^{s+2}_{scl}(\mathbb{R}^n)} \le C\|e^{\tilde{\varphi}/h}(-h^2\Delta)e^{-\tilde{\varphi}/h}u\|_{H^s_{scl}(\mathbb{R}^n)}, \quad C > 0,$$

for all $u \in C_0^\infty(\Omega)$.

Here

$$\|u\|_{H^s_{\mathrm{scl}}(\mathbb{R}^n)} = \|\langle hD\rangle^s u\|_{L^2(\mathbb{R}^n)}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2}.$$

Recalling that

$$m_q : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n),$$

we should use the Carleman estimate with $s = -1$ and $\varepsilon > 0$ sufficiently small but fixed.

Proposition

For all $h > 0$ sufficiently small, we have

$$h\|u\|_{H^1_{scl}(\mathbb{R}^n)} \leq C\|e^{\varphi/h}(-h^2\Delta + h^2 m_q)e^{-\varphi/h}u\|_{H^{-1}_{scl}(\mathbb{R}^n)},$$

for all $u \in C_0^\infty(\Omega)$.

The formal $L^2(\Omega)$ adjoint of the operator

$$e^{\varphi/h}(-h^2\Delta + h^2 m_q)e^{-\varphi/h}$$

is of the form

$$e^{-\varphi/h}(-h^2\Delta + h^2 m_q)e^{\varphi/h}$$

and therefore, the same Carleman estimate holds for the adjoint.

The Carleman estimate for the adjoint implies the following solvability result.

Proposition

If $h > 0$ is small enough, then for any $v \in H^{-1}(\Omega)$, there is a solution $u \in H^1(\Omega)$ of the equation

$$e^{\varphi/h}(-h^2\Delta + h^2 m_q)e^{-\varphi/h}u = v \quad in \quad \Omega,$$

which satisfies

$$\|u\|_{H^1_{scl}(\Omega)} \leq \frac{C}{h}\|v\|_{H^{-1}_{scl}(\Omega)}.$$

Complex WKB method

Fix a point $x_0 \notin \overline{\mathrm{ch}(\Omega)}$ and let $\varphi(x) = \log|x - x_0|$.
We wish to construct complex geometric optics solutions to $-\Delta u + m_q(u) = 0$, which are of the form

$$u(x;h) = e^{\frac{\varphi + i\psi}{h}}(a(x) + r(x;h)).$$

Here $\psi \in C^\infty(\overline{\Omega}, \mathbb{R})$ should solve the eikonal equation:

$$|\nabla\psi|^2 = |\nabla\varphi|^2, \quad \nabla\varphi \cdot \nabla\psi = 0,$$

and the amplitude $a \in C^\infty(\overline{\Omega})$ should satisfy the transport equation:

$$2(\nabla\varphi + i\nabla\psi) \cdot \nabla a + (\Delta\varphi + i\Delta\psi)a = 0.$$

Kenig–Sjöstrand–Uhlmann, 2007: the eikonal and transport equations have global smooth solutions.

The remainder r should satisfy

$$e^{-\frac{(\varphi+i\psi)}{h}}(-h^2\Delta + h^2 m_q)(e^{\frac{\varphi+i\psi}{h}} r) = h^2\Delta a - h^2 m_q(a).$$

The solvability result together with standard L^2 smoothing estimates for $\nabla \log\gamma \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n) \Longrightarrow \exists r \in H^1(\Omega)$ such that

$$\|r\|_{H^1_{\mathrm{scl}}(\Omega)} = o(1), \quad h \to 0.$$

Such remainder estimates are not strong enough to solve the inverse problem, even in the full data case.

Improvement for $\gamma \in W^{1,\infty}(\Omega) \cap H^{\frac{3}{2}}(\Omega)$

In this case,

$$\|r\|_{H^1_{\mathrm{scl}}(\Omega)} = o(h^{1/2}), \quad h \to 0.$$

Where does this improvement come from?

- $\gamma \in W^{1,\infty} \implies A = \nabla \log \gamma \in L^\infty \cap \mathcal{E}'$ and therefore,

  $$\|A - A * \Psi_\tau\|_{L^2} = o(1), \quad \tau \to 0.$$

  Here Ψ_τ is the standard mollifier.
- $\gamma \in W^{1,\infty} \cap H^{\frac{3}{2}} \implies A = \nabla \log \gamma \in H^{\frac{1}{2}}$ and therefore,

  $$\|A - A * \Psi_\tau\|_{L^2} = o(\tau^{1/2}), \quad \tau \to 0.$$

Step 2. Boundary Carleman estimates with limiting Carleman weights

Long tradition in PDE: Lebeau–Robbiano, 1994, Burq, 2002, Fursikov–Imanuvilov, 1996, Koch–Tataru, 2001, ..., Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann, 2007.

Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann, 2007 gives boundary Carleman estimates for $-h^2\Delta$ with a gain of one derivative for functions $u \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$.

m_q is too singular to be absorbed $\Longrightarrow$ we work directly with the conductivity equation

$$-\Delta u - A \cdot \nabla u = 0, \tag{1}$$

where $A = \nabla \log \gamma \in L^\infty \cap H^{\frac{1}{2}}$.

We would like to use the Carleman estimates of Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann for (1) to control boundary terms over the inaccessible portion of the boundary $\partial\Omega \setminus \Gamma_2$.

$\Longrightarrow$ encounter uncontrollable terms of magnitude

$$\frac{1}{h^{1/2}}\|\nabla \log \gamma_1 - \nabla \log \gamma_2\|_{L^2}, \quad h \to 0.$$

To overcome this difficulty, we shall follow an idea of Päivärinta–Panchenko–Uhlmann, 2003, Knudsen, 2006:

replace the conductivity equation (1) by its conjugated version:

$$e^{\frac{w_h}{2}}(-\Delta - A\cdot\nabla)e^{-\frac{w_h}{2}}u = -\Delta u + (A_h - A)\cdot\nabla u + V_h u = 0,$$

where

$$w_h = \log\gamma * \Psi_h, \quad A_h = \nabla w_h = A * \Psi_h \in C_0^\infty,$$
$$V_h = \frac{\nabla\cdot A_h}{2} - \frac{A_h^2}{4} + \frac{A\cdot A_h}{2} \in L^\infty \cap H^{\frac{1}{2}}.$$

Advantage:

- $A - A_h$ is small: $\|A - A_h\|_{L^2} = o(h^{1/2})$,
- V_h is not too large: $\|V_h\|_{L^\infty} = \mathcal{O}(h^{-1})$.

The price to pay for working with the conjugated equation:

need to extend the boundary Carleman estimates of Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann to functions which need not vanish along $\partial\Omega$.

Consider

$$-\Delta + X \cdot \nabla + W,$$

where $X \in L^\infty$, $W \in L^\infty$,

$$\|X\|_{L^\infty} = \mathcal{O}(1), \quad \|W\|_{L^\infty} = \mathcal{O}\left(\frac{1}{h}\right), \quad h \to 0.$$

Boundary Carleman estimates

For all $u \in H^2(\Omega)$ and all $h > 0$ small enough,

$$
\begin{aligned}
&\mathcal{O}(h)\|e^{-\frac{\varphi}{h}}u\|^2_{L^2(\partial\Omega)} + \mathcal{O}(h^2)\int_{\partial\Omega} e^{-\frac{2\varphi}{h}}|\partial_\nu u||u|dS \\
&\quad + \mathcal{O}(h^3)\int_{\partial\Omega_-}(-\partial_\nu\varphi)e^{-\frac{2\varphi}{h}}|\partial_\nu u|^2 dS \\
&\quad + \mathcal{O}(h^3)\|e^{-\frac{\varphi}{h}}\nabla_t u\|^2_{L^2(\partial\Omega)} + \mathcal{O}(h^3)\int_{\partial\Omega} e^{-\frac{2\varphi}{h}}|\nabla_t u||\partial_\nu u|dS \\
&\quad + \mathcal{O}(1)\|e^{-\varphi/h}(-h^2\Delta + hX\cdot h\nabla + h^2 W)u\|^2_{L^2(\Omega)} \\
&\quad \geq h^2(\|e^{-\frac{\varphi}{h}}u\|^2_{L^2(\Omega)} + \|e^{-\frac{\varphi}{h}}h\nabla u\|^2_{L^2(\Omega)}) \\
&\quad + h^3\int_{\partial\Omega_+}(\partial_\nu\varphi)e^{-\frac{2\varphi}{h}}|\partial_\nu u|^2 dS
\end{aligned}
$$

Here

$$\partial\Omega_\pm = \{x \in \partial\Omega : \pm\partial_\nu\varphi(x) \geq 0\},$$

Note: can control the integral over the inaccessible boundary portion $\partial\Omega \setminus \Gamma_2 \subset \partial\Omega_+$.

Idea of the proof: work with the convexified weight $\varphi + \frac{h}{2\varepsilon}\varphi^2$ and integrate by parts.

Important point: cancellation of boundary terms containing $\partial_\nu^2 u$ (as such terms cannot be handled)

Step 3. Converting $\Lambda_{\gamma_1,\Gamma_1,\Gamma_2} = \Lambda_{\gamma_2,\Gamma_1,\Gamma_2}$ into an integral identity

Let $u_j \in H^1(\Omega)$ satisfy $L_{\gamma_j} u_j = 0$ in Ω, $j = 1, 2$, and let $\widetilde{u}_1 \in H^1(\Omega)$ be an auxiliary solution such that $L_{\gamma_1} \widetilde{u}_1 = 0$ with $\widetilde{u}_1 = u_2$ on $\partial\Omega$. Then

$$\int_\Omega \Big(-\nabla \gamma_1^{1/2} \cdot \nabla(\gamma_2^{1/2} u_1 u_2) + \nabla \gamma_2^{1/2} \cdot \nabla(\gamma_1^{1/2} u_1 u_2) \Big)\, dx \\ = \int_{\partial\Omega \setminus \Gamma_2} (\Lambda_{\gamma_1} \widetilde{u}_1 - \Lambda_{\gamma_2} u_2) u_1 \, dS.$$

Long tradition in inverse boundary problems ... Brown, 1996

Step 4. Testing the integral identity against complex geometric optics solutions

We can extend γ_j to all of $\mathbb{R}^n$ so that $\gamma_j - 1 \in C^{1,\delta}(\mathbb{R}^n) \cap H^{3/2}(\mathbb{R}^n)$, with $\gamma_1 = \gamma_2$ on $\mathbb{R}^n \setminus \Omega$ (thanks to the boundary determination).

Let us substitute the complex geometric optics solutions

$$u_1(x;h) = \gamma_1^{-1/2} e^{-\frac{(\varphi + i\psi)}{h}} \left(a_1(x) + r_1(x;h)\right),$$

$$u_2(x;h) = \gamma_2^{-1/2} e^{\frac{\varphi + i\psi}{h}} \left(a_2(x) + r_2(x;h)\right),$$

into the integral identity and pass to the limit $h \to 0$.

As $h \to 0$, the LHS of the integral identity $\to$

$$\int_\Omega \left(-\nabla \gamma_1^{1/2} \cdot \nabla(\gamma_1^{-1/2} a_1 a_2) + \nabla \gamma_2^{1/2} \cdot \nabla(\gamma_2^{-1/2} a_1 a_2)\right) dx.$$

Here the improved remainder estimates, available for $\gamma_j \in W^{1,\infty} \cap H^{3/2}$,

$$\|r_j\|_{H^1_{\mathrm{scl}}(\Omega)} = o(h^{1/2})$$

are vital.

What about the RHS in the limit $h \to 0$?

Main Lemma: We have

$$\int_{\partial\Omega \setminus \Gamma_2} (\Lambda_{\gamma_1} \widetilde{u}_1 - \Lambda_{\gamma_2} u_2) u_1 \, dS = o(1), \quad h \to 0.$$

When proving this result, we work with the functions $e^{w_{1,h}/2}\widetilde{u}_1$, $e^{w_{2,h}/2}u_2$, solving the conjugated conductivity equations. We apply our boundary Carleman estimates to the function

$$e^{w_{1,h}/2}\widetilde{u}_1 - e^{w_{2,h}/2}u_2 \quad (\neq 0 \quad \text{along} \quad \partial\Omega)$$

and the conjugated conductivity operator

$$-\Delta + (A_{1,h} - A_1) \cdot \nabla + V_{1,h},$$

making systematic use of the full strength of the estimates available.

Step 5. Recovering the conductivity

We have

$$\int_\Omega \left(-\nabla\gamma_1^{1/2}\cdot\nabla(\gamma_1^{-1/2}a_1a_2)+\nabla\gamma_2^{1/2}\cdot\nabla(\gamma_2^{-1/2}a_1a_2)\right)\,dx=0,$$

for all $a_j\in C^\infty(\mathrm{neigh}(\overline{\Omega}),\mathbb{R}^n)$ solving the transport equation

$$(\nabla\varphi+i\nabla\psi)\cdot\nabla a_j+\frac{1}{2}\left(\Delta\varphi+i\Delta\psi\right)a_j=0.$$

Recalling that

$$q_j=\frac{\Delta\gamma_j^{1/2}}{\gamma_j^{1/2}}\in\left(H^{-1}\cap\mathcal{E}'\right)(\mathbb{R}^n),$$

we get with $q=q_1-q_2\in\mathcal{E}'(\overline{\Omega})$,

$$\langle q,a_1a_2\rangle_{\mathcal{E}',C^\infty}=0.$$

Recalling that $\varphi(x) = \log|x - x_0|$, $x_0 \notin \overline{\text{ch}(\Omega)}$ and varying x_0 slightly, we can replace q by its translates $\tau_y q$, $y \in \mathbb{R}^n$ small, and hence we get for all τ small,

$$\langle q * \Psi_\tau, a_1 a_2 \rangle_{\mathcal{E}', C^\infty} = 0.$$

Here $q * \Psi_\tau$ is smooth and hence the analysis of Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann, 2007, based on the microlocal Helgason and microlocal Holmgren theorems, applies $\Longrightarrow q * \Psi_\tau = 0$.

Thus, $q = 0 \Longrightarrow \gamma_1 = \gamma_2$.

Remark. Can we go below $3/2$ derivatives in the partial data Calderón problem?

- Haberman–Tataru, 2013, Caro–Rogers, 2016: to go below $3/2$ derivatives in the full data case, exploit averaging arguments depending crucially on the linear nature of the limiting Carleman weights.
- Kenig–Sjöstrand–Uhlmann, 2007: a key point in the partial data problem is to use non-linear weights

To reach lower regularity in the partial data problem, it seems therefore that a new approach would be needed.

The Magnetic Schrödinger operator

It is of great interest in inverse problems to consider other elliptic equations and more general geometries.

Let (M, g) be a compact smooth Riemannian manifold with boundary of dimension $n \geq 3$.

Let $A \in C^\infty(M, T^*M)$ be a complex valued 1-form (magnetic potential) and let $q \in C^\infty(M, \mathbb{C})$ (electric potential). Consider the magnetic Schrödinger operator

$$L_{A,q} = d_A^* d_A + q,$$

where d_A is the magnetic de Rham differential

$$d_A = d + iA : C^\infty(M) \to C^\infty(M, T^*M),$$

and

$$d_A^* = d^* - i\langle \overline{A}, \cdot \rangle_g : C^\infty(M, T^*M) \to C^\infty(M)$$

is its formal L^2 adjoint. Here $\langle \cdot, \cdot \rangle_g$ is the pointwise scalar product in the space of 1–forms.

$L_{0,0} = -\Delta$ and $L_{A,q}$ is a 1st order perturbation of $-\Delta$.

The set of the Cauchy data for solutions of the magnetic Schrödinger equation:

$$C_{A,q} := \{(u|_{\partial M}, \langle d_A u, \nu\rangle_g|_{\partial M}) : u \in H^1(M^0), L_{A,q}u = 0\},$$

ν is the unit outer normal to ∂M.

Inverse problem: $C_{A,q} \Longrightarrow A$ and q in M?

An obstruction to uniqueness: If $F \in C^\infty(M)$ is a non-vanishing function, we have

$$F^{-1} \circ L_{A,q} \circ F = L_{A-iF^{-1}dF,q}.$$

Furthermore, if $F|_{\partial M} = 1$ then

$$C_{A,q} = C_{A-iF^{-1}dF,q}.$$

Hope: $C_{A,q} \Longrightarrow$ the magnetic field dA and q in M.

Main tool in proving global uniqueness results

Complex CGO solutions to $L_{A,q}u = 0$:

$$u = e^{\frac{1}{h}(\varphi + i\psi)}(a(x) + r(x; h)), \quad 0 < h \ll 1,$$

where $\varphi \in C^\infty(M, \mathbb{R})$ is a limiting Carleman weight.

- Dos Santos Ferreira–Kenig–Salo-Uhlmann, 2009: Existence of LCWs requires a certain conformal symmetry, such as: (M, g) is conformally transversally anisotropic (CTA) if $(M, g) \subset (\mathbb{R} \times M_0, g)$ where $g = c(e \oplus g_0)$.

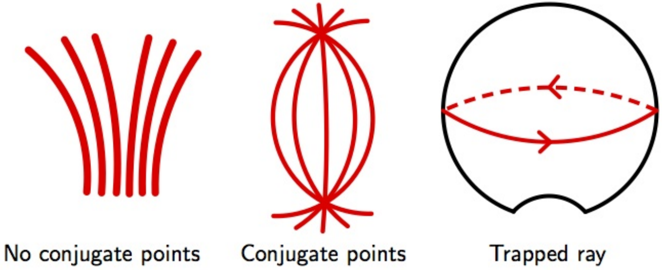
(M, g) is admissible if

(i) (M, g) is CTA,

(ii) transversal manifold (M_0, g_0) is simple.

(M_0, g_0) is simple if for any $p \in M_0$, the exponential map $\exp_p$ is a diffeomorphism onto M_0 and ∂M_0 is strictly convex.

Hence, M_0 has no conjugate points and no trapped geodesics.



Theorem (Dos Santos Ferreira–Kenig–Salo–Uhlmann, 2009)

Let (M, g) be admissible. Let $A^{(1)}, A^{(2)} \in C^\infty(M, T^*M)$ be complex valued 1-forms, and $q^{(1)}, q^{(2)} \in C^\infty(M, \mathbb{C})$. If $C_{A^{(1)},q^{(1)}} = C_{A^{(2)},q^{(2)}}$, then $dA^{(1)} = dA^{(2)}$ and $q^{(1)} = q^{(2)}$.

How about weakening the regularity assumptions of the potentials?

- Dos Santos Ferreira–Kenig–Salo, 2013: $A = 0$, $q \in L^{\frac{n}{2}}$.

Theorem (K–Uhlmann, 2017)

Let (M, g) be admissible. Let $A^{(1)}, A^{(2)} \in L^\infty(M, T^*M)$ be complex valued 1-forms, and $q^{(1)}, q^{(2)} \in L^\infty(M, \mathbb{C})$. If $C_{A^{(1)},q^{(1)}} = C_{A^{(2)},q^{(2)}}$, then $dA^{(1)} = dA^{(2)}$ and $q^{(1)} = q^{(2)}$.

Difficulty when A is only L^∞

$$
\begin{aligned}
L_{A,q}u &= (d_A^* d_A + q)u \\
&= -\Delta_g u + id^*(Au) - i\langle A, du\rangle_g + (\langle A, A\rangle_g + q)u.
\end{aligned}
$$

When $A \in L^\infty(M, T^*M)$, $d^*(Au) \in H^{-1}(M^0)$ and thus,

$$L_{A,q} : C_0^\infty(M^0) \to H^{-1}(M^0).$$

Need to construct CGO solutions to

$$L_{A,q}u = 0$$

when $L_{A,q}$ has singular coefficients and show that these solutions suffice to solve the inverse problem.

Constructing CGO solutions

Use Carleman estimates with limiting Carleman weights!

Example: If (M, g) is CTA then the globally defined function

$$\varphi(x) = x_1$$

is a LCW.

Let $M \subset U \subset N$, U open, N compact manifold without boundary, $\dim(N) = \dim(M)$.

Proposition (K–Uhlmann, 2017)

Let φ be a limiting Carleman weight for $-h^2\Delta$ and let $\tilde{\varphi} = \varphi + \frac{h}{2\varepsilon}\varphi^2$. Then for all $0 < h \ll \varepsilon \ll 1$ and $s \in \mathbb{R}$, we have a Carleman estimate with a gain of two derivatives,

$$\frac{h}{\sqrt{\varepsilon}}\|u\|_{H^{s+2}_{scl}(N)} \leq C\|e^{\frac{\tilde{\varphi}}{h}}(-h^2\Delta)e^{-\frac{\tilde{\varphi}}{h}}u\|_{H^s_{scl}(N)}, \quad C > 0,$$

for all $u \in C_0^\infty(M^0)$.

$$\|u\|_{H^s_{\mathrm{scl}}(N)} = \|(1 - h^2\Delta)^{\frac{s}{2}}u\|_{L^2(N)}.$$

Hence, we get a solvability result: if $h > 0$ is small enough, then for any $v \in H^{-1}(M^0)$, there is a solution $u \in H^1(M^0)$ of the equation

$$e^{\frac{\varphi}{h}}(h^2 L_{A,q})e^{-\frac{\varphi}{h}}u = v \quad \text{in} \quad M^0,$$

which satisfies

$$\|u\|_{H^1_{\mathrm{scl}}(M^0)} \leq \frac{C}{h}\|v\|_{H^{-1}_{\mathrm{scl}}(M^0)}.$$

WKB analysis on an admissible manifold

Let $\varphi(x) = x_1$. We want

$$u(x) = e^{-\frac{\rho(x)}{h}}(a(x) + r_0(x,h)), \quad \rho = \varphi + i\psi,$$

to solve $L_{A,q}u = 0$.

Choose $\psi \in C^\infty(M, \mathbb{R})$ satisfying the eikonal equation:

$$|\nabla\psi|^2 = |\nabla\varphi|^2, \quad \langle\nabla\varphi, \nabla\psi\rangle = 0,$$

and the amplitude $a \in C^\infty(M)$ satisfying the transport equation:

$$(2i\langle A_\tau, d\rho\rangle_g + 2\nabla\rho)a + (\Delta\rho)a = 0.$$

Here $A_\tau \in C^\infty(M)$ is a regularization of A so that $\|A - A_\tau\|_{L^2(M)} \to 0$ as $\tau \to 0$.

The transversal manifold M_0 is simple $\Longrightarrow$ the eikonal and transport equations have global smooth solutions!

Let (r, θ) be (global) polar normal coordinates on M_0 centered at some $\omega \in \partial M_0$. Then take $\psi = r$ and the transport equation becomes a $\overline{\partial}$–equation with $\overline{\partial} = \frac{1}{2}(\partial_{x_1} + i\partial_r)$.

Determination of the remainder r:

$$e^{\frac{\rho}{h}} h^2 L_{g,A,q}(e^{-\frac{\rho}{h}} r) = -h^2 L_{A,q} a + 2ih\langle A - A_\tau, d\rho\rangle_g a \in H^{-1}(M^0).$$

The solvability result gives a family of CGO solutions:

$$u = e^{-\frac{1}{h}(x_1+ir)}(|g|^{-1/4} c^{1/2} e^{i\Phi_h} e^{i\lambda(x_1+ir)} b(\theta) + r_0),$$

with $\|r_0\|_{H^1_{\mathrm{scl}}(M)} \to 0$ as $h \to 0$.

Here $\lambda \in \mathbb{R}$, $b(\theta)$ is smooth and $\Phi_h \to \overline{\partial}^{-1}(A_1 + iA_r)$.

The integral identity and its consequences

- Convert $C_{A^{(1)},q^{(1)}} = C_{A^{(2)},q^{(2)}}$ into the integral identity:

$$\int_M i\langle A^{(1)} - A^{(2)}, u_1 du_2 - u_2 du_1\rangle_g dV_g + \int_M (\langle A^{(1)}, A^{(1)}\rangle_g - \langle A^{(2)}, A^{(2)}\rangle_g + q^{(1)} - q^{(2)}) u_1 u_2 dV_g = 0,$$

  which holds for any $u_1, u_2 \in H^1(M^0)$ satisfying $L_{A^{(1)},q^{(1)}} u_1 = 0$ and $L_{-A^{(2)},q^{(2)}} u_2 = 0$.
- Test the integral identity against CGO solutions of the form:

$$u_1 = e^{-\frac{1}{h}(x_1 + ir)}(a_1 + r_1),$$

$$u_2 = e^{\frac{1}{h}(x_1 + ir)}(a_2 + r_2),$$

  and let $h \to 0$.

Inverting the attenuated geodesic ray transform

- We get that the following family of integrals vanishes, for $\omega \in \partial M_0$:

$$\int_{\mathbb{S}^{n-2}} \int_0^{\tau(\omega,\theta)} [f(\gamma_\theta(r)) + \alpha(\dot{\gamma}_\theta(r))] e^{-\lambda r} b(\theta) dr d\theta = 0.$$

Here $\gamma_\theta(r)$ is the geodesic on M_0 given by $r \mapsto (r, \theta)$ and $\tau(\omega, \theta)$ is the time when γ_θ exits M_0.

The function $f \in L^\infty(M_0)$ and the 1-form $\alpha \in L^\infty(M_0, T^*M_0)$ are obtained from $A^{(1)}$ and $A^{(2)}$ by the partial Fourier transform in x_1.

If $A^{(1)}, A^{(2)}$ are smooth $\Longrightarrow$ the attenuated geodesic ray transform vanishes,

$$I_\lambda(f, \alpha)(\omega, \theta) := \int_0^{\tau(\omega,\theta)} [f(\gamma_\theta(r)) + \alpha(\dot{\gamma}_\theta(r))] e^{-\lambda r} dr = 0,$$

for all $\omega \in \partial M_0$, $\theta \in \mathbb{S}^{n-2}$ and the attenuation $\lambda \in \mathbb{R}$.

This transform always has a non-trivial kernel,

$$I_\lambda(-\lambda p, dp) = 0$$

for all $p \in C^\infty(M_0)$ with $p|_{\partial M_0} = 0$.

- Mukhometov, 1977, Sharafutdinov, 1994,... Dos Santos Ferreira–Kenig–Salo–Uhlmann, 2009: When M_0 is simple, the attenuated geodesic ray transform is injective, up to the natural obstruction:

$$I_\lambda(f, \alpha) = 0 \Longrightarrow f = -\lambda p, \quad \alpha = dp.$$

Thus, if $A^{(1)}, A^{(2)}$ are smooth, we conclude that $dA^{(1)} = dA^{(2)}$.

To handle the non-smooth case, following Dos Santos Ferreira–Kenig–Salo, 2013, and Assylbekov–Yang, 2017, we use duality and the ellipticity of the normal operator $I_\lambda^* I_\lambda$.

Removing the simplicity assumption on M_0

Need limiting Carleman weights $\Longrightarrow$ still assume that

(M,g) is CTA if $(M,g) \subset (\mathbb{R} \times M_0, g)$ where $g = c(e \oplus g_0)$.

Now M_0 need not be simple but demand instead that

Assumption (A): The geodesic ray transform on M_0

$$I(f,\alpha)(x,\xi) = \int_0^{\tau(x,\xi)} \left[f(\gamma_{x,\xi}(t)) + \alpha(\dot{\gamma}_{x,\xi}(t)) \right] dt$$

is injective up to the natural obstruction.

Here $x \in \partial M_0$, $\langle \xi, \nu(x) \rangle < 0$ and the exit time $\tau(x,\xi) < +\infty$.

Theorem (Dos Santos Ferreira–Kurylev–Lassas–Salo, 2016)

Let (M,g) be CTA and assume that (A) holds. Let $q^{(1)}, q^{(2)} \in C(M,\mathbb{C})$. If $C_{0,q^{(1)}} = C_{0,q^{(2)}}$ then $q^{(1)} = q^{(2)}$.

Theorem (Cekić, 2016)

Let (M,g) be CTA and assume that (A) holds. Let $A^{(1)}, A^{(2)} \in C^\infty(M, T^*M)$. If $C_{A^{(1)},0} = C_{A^{(2)},0}$ then $dA^{(1)} = dA^{(2)}$.

Theorem (K–Uhlmann, 2017)

Let (M,g) be CTA and assume that (A) holds. Let $A^{(1)}, A^{(2)} \in C(M, T^*M)$ be complex valued 1-forms, and $q^{(1)}, q^{(2)} \in L^\infty(M,\mathbb{C})$. If $C_{A^{(1)},q^{(1)}} = C_{A^{(2)},q^{(2)}}$, then $dA^{(1)} = dA^{(2)}$. Assuming furthermore that $q^{(1)} = q^{(2)}$, we have

$$A^{(2)} = A^{(1)} - iF^{-1}dF,$$

for some $F \in C^1(M,\mathbb{C})$ non-vanishing with $F|_{\partial M} = 1$.

Examples of non-simple manifolds satisfying (A)

- Uhlmann–Vasy, 2016, Stefanov–Uhlmann–Vasy, 2016: M_0 has strictly convex boundary and is foliated by strictly convex hypersurfaces.
- Guillarmou, 2017: M_0 is negatively curved with strictly convex boundary.
- Guillarmou–Mazzucchelli–Tzou, 2017: M_0 has no conjugate points, has a hyperbolic trapped set.

Some words about the proof

(M, g) is CTA $\Longrightarrow$ still have a notion of LCW $\varphi(x) = x_1$ and a solvability result with a gain of two derivatives.

Following Dos Santos Ferreira–Kurylev–Lassas–Salo, 2016, to construct CGO solutions, replace a global WKB construction by a more general quasimode construction:

$$u = e^{-\frac{x_1}{h}}(v + r_0),$$

where $v \in C^\infty(M)$ is a quasimode such that

$$\|e^{\frac{x_1}{h}} h^2 L_{A^{(1)},q^{(1)}} e^{-\frac{x_1}{h}} v\|_{H^{-1}_{\mathrm{scl}}(M^0)} = o(h), \quad \|v\|_{L^2(M)} = \mathcal{O}(1), \quad h \to 0,$$

and r_0 is the remainder with $\|r_0\|_{H^1_{\mathrm{scl}}(M)} = o(1)$.

How do we construct the quasimode v?

We construct v as a Gaussian beam quasimode concentrated near a non-tangential geodesic

$$\gamma : [0, L] \to M_0.$$

Long tradition: Babich, 1968, Ralston, 1977,...

Locally near a point on γ, we have

$$v(x) = h^{-\frac{n-2}{4}} e^{\frac{i\psi(x')}{h}} a(x; h), \quad x = (x_1, x') \in \mathbb{R} \times M_0,$$

where $\psi \in C^\infty$ complex-valued with $\operatorname{Im} \psi \geq 0$, $\operatorname{Im} \psi|_\gamma = 0$ and $\operatorname{Im} \psi(x') \sim \mathrm{dist}(x', \gamma)^2$. The phase ψ satisfies the eikonal equation:

$$\langle d\psi(x'), d\psi(x') \rangle_{g_0} = 1 + \mathcal{O}(\mathrm{dist}(x', \gamma)^N), \quad N \text{ large.}$$

The amplitude $a \in C^\infty$ solves the transport equation:

$$2\partial_{x_1} a - 2i\langle d\psi, da \rangle_{g_0} - i(\Delta_{g_0}\psi)a + 2a\langle d\psi, A_\tau \rangle_{g_0} + 2i(A_\tau)_1 a$$
$$= \mathcal{O}(\tau^{-1}\mathrm{dist}(x', \gamma)),$$

Once CGO solutions have been constructed, as in the simple manifold case, we test the integral identity against these solutions and pass to the limit $h \to 0$.

Thanks to the concentration property of the quasimodes, we get the vanishing of the attenuated geodesic ray transform,

$$\int_0^L [f(\gamma(t)) + \alpha(\dot{\gamma}(t))] e^{-\lambda t} dt = 0,$$

for any non-tangential geodesic $\gamma : [0, L] \to M_0$ and $\lambda \in \mathbb{R}$.

Here the function $f \in C(M_0)$ and the 1-form $\alpha \in C(M_0, T^* M_0)$ are obtained from $A^{(1)}$ and $A^{(2)}$ by the partial Fourier transform in x_1.

Differentiating with respect to λ and using the injectivity of the unattenuated geodesic ray transform we conclude that $dA^{(1)} = dA^{(2)}$.

THANK YOU VERY MUCH FOR YOUR ATTENTION!