Control from an Interior Hypersurface

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Outline

General questions

Eigenfunctions
- A very weak lower bound
- A GCC-type result

Wave equation
- Two results
- Solution to the wave equation?
- Controllability/observability for the wave equation

Questions

Setting:

- (M, g) be a compact n dimensional Riemannian manifold, possibly with boundary ∂M,
- Δ_g the (non-positive) Laplace-Beltrami operator on M,
- Σ is an *interior hypersurface* ($\dim \Sigma = n - 1$, $\Sigma \cap \partial M = \emptyset$, global normal vector field ∂_ν).

General questions:

- What is the "mass" / "energy" left by solutions to $-\Delta_g - \lambda^2$ or $\partial_t^2 - \Delta_g$ or $\partial_t - \Delta_g$ on the hypersurface Σ?
- Can we observe solutions to $-\Delta_g - \lambda^2$ or $\partial_t^2 - \Delta_g$ or $\partial_t - \Delta_g$ on the hypersurface Σ?
- Can we "control" solutions to $\partial_t^2 - \Delta_g$ or $\partial_t - \Delta_g$ from the hypersurface Σ?

$\rightsquigarrow$ Usual situations: open set $\omega \subset M$, or open subset of the boundary $\Gamma \subset \partial M$.

First Break

Eigenfunctions: some classical results

$$(-\Delta_g - \lambda^2)\phi = 0, \qquad \phi|_{\partial M} = 0. \tag{Eig}$$

- $\omega \subset M$, open
- Study: $\|\phi\|_{L^2(\omega)}$,
- Weak lower bound ($\omega \neq \emptyset$): $\|\phi\|_{L^2(\omega)} \geq ce^{-\lambda/c}\|\phi\|_{L^2(M)}$,
- [Donnelly-Fefferman '88]

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- $\Gamma \subset \partial M$, open
- Study: $\|\partial_n \phi|_\Gamma\|_{L^2(\Gamma)}$
- Weak lower bound ($\Gamma \neq \emptyset$): $\|\partial_n \phi|_\Gamma\|_{L^2(\Gamma)} \geq ce^{-\lambda/c}\|\phi\|_{L^2(M)}$,
- [Lebeau-Robbiano '95]

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Improve under a geometric condition on ω?

Eigenfunctions: some classical results

$$(-\Delta_g - \lambda^2)\phi = 0, \qquad \phi|_{\partial M} = 0. \tag{Eig}$$

Assumption (GCC): there exists $L > 0$ such that all (generalized) geodesics of length $\geq L$ intersect ω.

Then $\|\phi\|_{L^2(\omega)} \geq c\|\phi\|_{L^2(M)}$, [Bardos-Lebeau-Rauch '92]

Sometimes but not always necessary for eigenfunctions (always necessary for $o(\lambda^{-1})$-quasimodes)

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Sometimes but not always necessary for eigenfunctions (always necessary for $o(\lambda^{-1})$-quasimodes)

Assumption (GCC): there exists $L > 0$ such that all (generalized) geodesics of length $\geq L$ intersect Γ (at a non-diffractive point).

Then $\|\langle\lambda\rangle^{-1}\partial_n\phi|_\Gamma\|_{L^2(\Gamma)} \geq c\|\phi\|_{L^2(M)}$, [Bardos-Lebeau-Rauch '92]

Eigenfunctions: a very weak lower bound

$$(-\Delta_g - \lambda^2)\phi = 0, \qquad \phi|_{\partial M} = 0. \tag{Eig}$$

Theorem

Assume M is connected and $\mathrm{Int}(\Sigma)$ is nonempty. Then there exists $c > 0$ so that for all $\lambda \geq 0$ and $\phi \in L^2(M)$ solutions to (Eig), we have

$$\|\phi|_\Sigma\|_{L^2(\Sigma)} + \|\partial_\nu \phi|_\Sigma\|_{L^2(\Sigma)} \geq ce^{-\lambda/c}\|\phi\|_{L^2(M)}.$$

$\rightsquigarrow$ Recall [Donnelly-Fefferman '88], [Lebeau-Robbiano '95]:

- $\|\phi\|_{L^2(\omega)} \geq ce^{-\lambda/c}\|\phi\|_{L^2(M)}$,
- $\|\partial_n \phi|_\Gamma\|_{L^2(\Gamma)} \geq ce^{-\lambda/c}\|\phi\|_{L^2(M)}$.

Eigenfunctions: a very weak lower bound

$$(-\Delta_g - \lambda^2)\phi = 0, \qquad \phi|_{\partial M} = 0. \tag{Eig}$$

Proposition (Optimality in general)

Consider the manifold

$$M = [-\pi, \pi]_z \times \mathbb{T}^1_\theta, , \qquad g(z, \theta) = dz^2 + R(z)^2 d\theta^2.$$

Assume that R is even and has two bumps. Let $\Sigma = \{z = 0\} \times \mathbb{T}^1 \subset M$. *Then, there exist* $C, c > 0$ *and sequences* $\lambda_j^{e/o} \to +\infty$ *and* $\phi_j^{e/o} \in L^2(M)$ *such that*

$$(-\Delta - (\lambda_j^{e/o})^2)\phi_j^{e/o} = 0, \quad \|\phi_j^{e/o}\|_{L^2(M)} = 1, \quad \phi_j^{e/o}|_{\partial M} = 0,$$

with

$$\partial_\nu \phi_j^e|_\Sigma = 0, \quad \|\phi_j^e|_\Sigma\|_{L^2(\Sigma)} \le Ce^{-c\lambda_j^e},$$

$$\phi_j^o|_\Sigma = 0, \quad \|\partial_\nu \phi_j^o|_\Sigma\|_{L^2(\Sigma)} \le Ce^{-c\lambda_j^o}.$$

Eigenfunctions: a very weak lower bound

Open question: can we remove one of the two terms?

Conjecture (?)

Assume Σ has positive definite second fundamental form. Then there exists $C, c, \lambda_0 > 0$ so that for all $(\lambda, \phi) \in [\lambda_0, \infty) \times L^2(M)$ satisfying (Eig), we have

$$\|\phi|_\Sigma\|_{L^2(\Sigma)} \geq Ce^{-c\lambda}\|\phi\|_{L^2(M)}, \quad \textit{and} \quad \|\lambda^{-1}\partial_\nu\phi|_\Sigma\|_{L^2(\Sigma)} \geq Ce^{-c\lambda}\|\phi\|_{L^2(M)}.$$

This would have applications (cf Toth-Zelditch '17 for nodal domains)

Eigenfunctions: a GCC-type result

An improved lower bound under a geometric condition.

$$(-\Delta_g - \lambda^2)\phi = 0, \qquad \phi|_{\partial M} = 0. \tag{Eig}$$

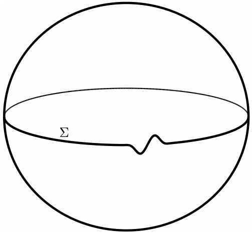
Assumption ($\mathcal{T}$GCC)

There is $L > 0$ s.t. all (generalized) geodesics of length $\leq L$ cross Σ transversally.

Theorem

Assume ($\mathcal{T}$GCC) then there is $c > 0$ so that for all $\lambda \geq 0$ and $\phi \in L^2(M)$ solutions to (Eig)*, we have*

$$\|\phi|_\Sigma\|_{L^2(\Sigma)} + \|\langle\lambda\rangle^{-1}\partial_\nu\phi|_\Sigma\|_{L^2(\Sigma)} \geq c\|\phi\|_{L^2(M)}. \tag{1}$$



$\rightsquigarrow$ [BLR '92]: Under "GCC"

- $\|\phi\|_{L^2(\omega)} \geq c\|\phi\|_{L^2(M)}$,
- $\|\langle\lambda\rangle^{-1}\partial_n\phi|_\Gamma\|_{L^2(\Gamma)} \geq c\|\phi\|_{L^2(M)}$.

Eigenfunctions: a GCC-type result

- Removing "$\mathcal{T}$" in ($\mathcal{T}$GCC)?

Proposition

Assume $M = S^2$ and Σ is a great circle. Then there exists a sequence (λ_j, ϕ_j) satisfying $(-\Delta_g - \lambda_j^2)\phi_j = 0$ together with $\lambda_j \to +\infty$ and

$$\phi_j|_\Sigma = 0, \quad \|\lambda_j^{-1}\partial_\nu \phi_j|_\Sigma\|_{L^2(\Sigma)} \leq \lambda_j^{-1/4}\|\phi_j\|_{L^2(M)}.$$

- Beware that $\|\phi|_\Sigma\|_{L^2(\Sigma)}$ might be unbounded as $\lambda \to +\infty$: take $M = S^2$ and Σ is a great circle. There is another sequence s.t.

$$\|\phi_j|_\Sigma\|_{L^2(\Sigma)} \sim c\lambda_j^{1/4}$$

- different regularity/growth on the glancing set! [Galkowski '16]

Eigenfunctions: a GCC-type result

Open question: can we remove one of the two terms?

Conjecture (?)

Assume further that Σ has positive definite second fundamental form. Then there exists $C, c, \lambda_0 > 0$ so that for all $(\lambda, \phi) \in [\lambda_0, \infty) \times L^2(M)$ satisfying (Eig), *we have*

$$\|\phi\|_{L^2(M)} \leq C\|\phi|_\Sigma\|_{L^2(\Sigma)}, \quad \textit{and} \quad \|\phi\|_{L^2(M)} \leq C\|\lambda^{-1}\partial_\nu\phi|_\Sigma\|_{L^2(\Sigma)}.$$

Resolvent estimates/quasimodes

For all $\lambda \geq 0$ and all $u \in H^2(M) \cap H_0^1(M)$ we have

$$\|u\|_{L^2(M)} \leq Ce^{c\lambda}\big(\|u|_\Sigma\|_{L^2(\Sigma)} + \|\langle\lambda\rangle^{-1}\partial_\nu u|_\Sigma\|_{L^2(\Sigma)} + \|(-\Delta_g - \lambda^2)u\|_{L^2(M)}\big).$$

Under $(\mathcal{T}\mathrm{GCC})$

$$\|u\|_{L^2(M)} \leq C\big(\|u|_\Sigma\|_{L^2(\Sigma)} + \|\langle\lambda\rangle^{-1}\partial_\nu u|_\Sigma\|_{L^2(\Sigma)} + \langle\lambda\rangle^{-1}\|(-\Delta_g - \lambda^2)u\|_{L^2(M)}\big).$$

Related questions/works

- General upper bounds:
  [Tataru '98], L^p estimates [Burq-Gérard-Tzvetkov '07] [Tacy '10-14]
  [Christianson-Hassell-Toth '15]
  Near the glancing set [Galkowski '16]
- Lower bounds on $M = \mathbb{T}^2, \mathbb{T}^3$:
  Bourgain and Rudnick [Bourgain-Rudnick '09-12] If Σ is a real analytic hypersurface with nonvanishing curvatures, then

$$C^{-1}\|\phi\|_{L^2(M)} \leq \|\phi|_\Sigma\|_{L^2(\Sigma)} \leq C\|\phi\|_{L^2(M)}$$

- Density one subsequences of eigenfunctions equidistribute:
  Quantum Ergodic Restriction [Toth-Zelditch '12-'13-'17]
  [Dyatlov-Zworski '12],
  Torus [Hezari-Rivière '18]

Second break

The wave equation controlled from Σ

- Goal: "control" the wave equation from Σ:

$$\begin{cases} \Box v = f_0 \delta_\Sigma + f_1 \delta'_\Sigma & \text{on } (0,T) \times \mathrm{Int}(M), \\ v = 0 & \text{on } (0,T) \times \partial M, \\ (v, \partial_t v)|_{t=0} = (v_0, v_1) & \text{in } \mathrm{Int}(M). \end{cases} \tag{Wave}$$

where

$$\langle f_0 \delta_\Sigma, \varphi \rangle = \int_\Sigma f_0 \varphi d\sigma, \qquad \langle f_1 \delta'_\Sigma, \varphi \rangle = -\int_\Sigma f_1 \partial_\nu \varphi d\sigma.$$

- More precisely: Given (v_0, v_1) and $T > 0$, find (f_0, f_1) such that

$$(v, \partial_t v)|_{t=T} = (0, 0).$$

(equivalent to control to any target)

- Question 0: what is a solution? well-posedness?

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$\rightsquigarrow$ [Bardos Lebeau Rauch '92]:

- $\Box v = \mathbb{1}_\omega f$,
- $\Box v = 0, \quad v|_{\partial M} = \mathbb{1}_\Gamma f$

The wave equation controlled from Σ

General spirit [J.L. Lions '70-'90]: "duality"

- Well-posedness $\Longleftrightarrow$ regularity estimate for the free equation (direct inequality)
- Controllability $\Longleftrightarrow$ observability estimate for the free equation (reverse inequality)

The wave equation controlled from Σ

Normal coordinates: $x = (x_1, x')$, $\Sigma = \{x_1 = 0\}$, $|x_1| = \mathrm{dist}_g(x, \Sigma)$, $\partial_\nu = \partial_{x_1}$,

$$\sigma(-\Delta_g) = \xi_1^2 + r(x_1, x', \xi'),$$

(think $r(x_1, x', \xi') = |\xi'|^2$)

$$\sigma(\Box) = -\tau^2 + \sigma(-\Delta_g) = -\tau^2 + \xi_1^2 + r(x_1, x', \xi').$$

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Conical subsets of $T^*_{\mathbb{R}\times \text{Int}(\Sigma)}(\mathbb{R} \times M) \setminus 0$

$$\mathcal{E} = \{(t, 0, x', \tau, \xi_1, \xi') \mid \xi_1^2 + r(0, x', \xi') \neq \tau^2\},$$
$$\mathcal{T} = \{(t, 0, x', \tau, \xi_1, \xi') \mid \xi_1^2 + r(0, x', \xi') = \tau^2,\ \xi_1^2 > 0\},$$
$$\mathcal{G} = \{(t, 0, x', \tau, \xi_1, \xi') \mid \xi_1^2 + r(0, x', \xi') = \tau^2,\ \xi_1^2 = 0\}.$$

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Conical subsets of $\in T^*(\mathbb{R} \times \mathrm{Int}(\Sigma)) \setminus 0$

$$\mathcal{E}^\Sigma := \{(t, x', \tau, \xi') \mid r(0, x', \xi') > \tau^2\},$$
$$\mathcal{T}^\Sigma := \{(t, x', \tau, \xi') \mid r(0, x', \xi') < \tau^2\},$$
$$\mathcal{G}^\Sigma := \{(t, x', \tau, \xi') \mid r(0, x', \xi') = \tau^2\}.$$

The wave equation controlled from Σ: well-posedness

$$\begin{cases} \Box v = f_0 \delta_\Sigma + f_1 \delta'_\Sigma & \text{on } \mathbb{R}_+^* \times \mathrm{Int}(M), \\ v = 0 & \text{on } \mathbb{R}_+^* \times \partial M, \\ (v, \partial_t v)|_{t=0} = (v_0, v_1) & \text{in } \mathrm{Int}(M). \end{cases} \quad \text{(Wave)}$$

Theorem

For all $(v_0, v_1) \in L^2(M) \times H^{-1}(M)$ *and for all* $f_0 \in H^{-1}_{\mathrm{comp}}(\mathbb{R}_+^* \times \mathrm{Int}(\Sigma))$ *and* $f_1 \in L^2_{\mathrm{comp}}(\mathbb{R}_+^* \times \mathrm{Int}(\Sigma))$ *such that*

$$\mathrm{WF}^{-\frac{1}{2}}(f_0), \mathrm{WF}^{\frac{1}{2}}(f_1) \cap \mathcal{G}^\Sigma = \emptyset,$$

there exists a unique $v \in L^2_{\mathrm{loc}}(\mathbb{R}_+^*; L^2(M))$ *solution of* (Wave).

⇝ Well-posedness OK if (f_0, f_1) are

- $H^{-1}(\mathbb{R} \times \Sigma) \times L^2(\mathbb{R} \times \Sigma)$ overall
- $H^{-\frac{1}{2}} \times H^{\frac{1}{2}}$ microlocally near the glancing set $\mathcal{G}^\Sigma$

⇝ Poor regularity in time of the solution!

The wave equation controlled from Σ: control!

Definition

(Σ, T) satisfies ($\mathcal{T}$GCC) if every (generalized) bicharacteristic of $\Box$ intersects $T^*_{(0,T)\times \mathrm{Int}(\Sigma)}(\mathbb{R}\times M)\setminus \mathcal{G}$.

Theorem

Assume (Σ, T) satisfies ($\mathcal{T}$GCC). Then for any $(v_0, v_1) \in L^2(M)\times H^{-1}(M)$ there exist $(f_0, f_1) \in H^{-1}_{\mathrm{comp}}((0,T)\times \mathrm{Int}(\Sigma)) \times L^2_{\mathrm{comp}}((0,T)\times \mathrm{Int}(\Sigma))$ with

$$\mathrm{WF}(f_0), \mathrm{WF}(f_1) \cap (\mathcal{G}^\Sigma \cup \mathcal{E}^\Sigma) = \emptyset,$$

so that the solution to (Wave) *has $v \equiv 0$ for $t \geq T$.*

Solution of the wave equation controlled from Σ

Assume v, u, F solve (in a reasonable sense):

$$\Box v = f_0 \delta_\Sigma + f_1 \delta'_\Sigma \text{ in } \mathscr{D}'((0,T) \times \mathrm{Int}(M)), \quad \text{and} \quad \Box u = F.$$

Multiply by u and integrate by parts

$$\left[(\partial_t v, u)_{L^2(M)} - (v, \partial_t u)_{L^2(M)}\right]_0^T + (v, F)_{L^2((0,T)\times M)} = \int_{(0,T)\times\Sigma} \left(f_0 u|_\Sigma - f_1 \partial_\nu u|_\Sigma\right) dt d\sigma.$$

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For this to make sense as a definition (in a weak sense):

- take F's as test functions (Riesz Representation theorem)
- the regularity f_0 should fit that of $u|_\Sigma$ (u sol free wave)
- $F \in L^2((0,T) \times M)$, $(u, \partial_t u) \in H_0^1 \times L^2$ will yield a solution $v \in L^2((0,T) \times M)$ for data $(v, \partial_t v)|_{t=0} \in L^2 \times H^{-1}$

Main question: what is the regularity of $u|_\Sigma, \partial_\nu u|_\Sigma$ (for u sol of $\Box u = F, F \in L^2((0,T) \times M), (u, \partial_t u) \in H_0^1 \times L^2$?
$\rightsquigarrow$ Then, take this as a definition of a solution!

Solution of the wave equation controlled from Σ

Question: regularity of $u|_\Sigma, \partial_\nu u|_\Sigma$ for u sol of

$$\Box u = F, F \in L^2((0,T) \times M), \quad (u, \partial_t u) \in H_0^1 \times L^2?$$

Solution of the wave equation controlled from Σ

Question: regularity of $u|_\Sigma, \partial_\nu u|_\Sigma$ for u sol of

$$\Box u = F, F \in L^2((0,T) \times M), \quad (u, \partial_t u) \in H_0^1 \times L^2?$$

- $u \in H^1$ overall (Cauchy problem)

Solution of the wave equation controlled from Σ

Question: regularity of $u|_\Sigma, \partial_\nu u|_\Sigma$ for u sol of

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- $u \in H^1$ overall (Cauchy problem)
- $u|_\Sigma \in H^{1/2-\varepsilon}$, $\partial_\nu u|_\Sigma \in H^{-1/2-\varepsilon}$ overall (usual trace theorem)

Solution of the wave equation controlled from Σ

Question: regularity of $u|_\Sigma, \partial_\nu u|_\Sigma$ for u sol of

$$\Box u = F, F \in L^2((0,T)\times M), \quad (u, \partial_t u) \in H^1_0 \times L^2?$$

- $u \in H^1$ overall (Cauchy problem)
- $u|_\Sigma \in H^{1/2-\varepsilon}$, $\partial_\nu u|_\Sigma \in H^{-1/2-\varepsilon}$ overall (usual trace theorem)
- $u|_\Sigma \in H^{1/2}$, $\partial_\nu u|_\Sigma \in H^{-1/2}$ overall ($\Box u = F$, $\Box$ elliptic on the conormal)

Solution of the wave equation controlled from Σ

Question: regularity of $u|_\Sigma, \partial_\nu u|_\Sigma$ for u sol of

$$\Box u = F, F \in L^2((0,T) \times M), \quad (u, \partial_t u) \in H_0^1 \times L^2?$$

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- $u|_\Sigma \in H^{1/2-\varepsilon}$, $\partial_\nu u|_\Sigma \in H^{-1/2-\varepsilon}$ overall (usual trace theorem)
- $u|_\Sigma \in H^{1/2}$, $\partial_\nu u|_\Sigma \in H^{-1/2}$ overall ($\Box u = F$, $\Box$ elliptic on the conormal)
- Near $\mathcal{E}^\Sigma$ (elliptic region) $u|_\Sigma \in H^{3/2}$, $\partial_\nu u|_\Sigma \in H^{1/2}$

Solution of the wave equation controlled from Σ

Question: regularity of $u|_\Sigma, \partial_\nu u|_\Sigma$ for u sol of

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- $u \in H^1$ overall (Cauchy problem)
- $u|_\Sigma \in H^{1/2-\varepsilon}$, $\partial_\nu u|_\Sigma \in H^{-1/2-\varepsilon}$ overall (usual trace theorem)
- $u|_\Sigma \in H^{1/2}$, $\partial_\nu u|_\Sigma \in H^{-1/2}$ overall ($\Box u = F$, $\Box$ elliptic on the conormal)
- Near $\mathcal{E}^\Sigma$ (elliptic region) $u|_\Sigma \in H^{3/2}$, $\partial_\nu u|_\Sigma \in H^{1/2}$
- Near $\mathcal{T}^\Sigma$ (hyperbolic/transversal region) $u|_\Sigma \in H^1$, $\partial_\nu u|_\Sigma \in L^2$ (Cauchy problem w.r.t. Σ)

Solution of the wave equation controlled from Σ

Question: regularity of $u|_\Sigma, \partial_\nu u|_\Sigma$ for u sol of

$$\Box u = F, F \in L^2((0,T)\times M), \quad (u, \partial_t u) \in H_0^1 \times L^2?$$

- $u \in H^1$ overall (Cauchy problem)
- $u|_\Sigma \in H^{1/2-\varepsilon}$, $\partial_\nu u|_\Sigma \in H^{-1/2-\varepsilon}$ overall (usual trace theorem)
- $u|_\Sigma \in H^{1/2}$, $\partial_\nu u|_\Sigma \in H^{-1/2}$ overall ($\Box u = F$, $\Box$ elliptic on the conormal)
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- Near $\mathcal{T}^\Sigma$ (hyperbolic/transversal region) $u|_\Sigma \in H^1$, $\partial_\nu u|_\Sigma \in L^2$ (Cauchy problem w.r.t. Σ)

Conclusion:

- $u|_\Sigma \in H^{1/2}$, $\partial_\nu u|_\Sigma \in H^{-1/2}$ overall
- $u|_\Sigma \in H^1$, $\partial_\nu u|_\Sigma \in L^2$ away from $\mathcal{G}^\Sigma$

Solution of the wave equation controlled from Σ

Question: regularity of $u|_\Sigma, \partial_\nu u|_\Sigma$ for u sol of

$$\Box u = F, F \in L^2((0,T)\times M), \quad (u, \partial_t u) \in H_0^1 \times L^2?$$

Conclusion:

- $u|_\Sigma \in H^1$, $\partial_\nu u|_\Sigma \in L^2$ away from $\mathcal{G}^\Sigma$
- $u|_\Sigma \in H^{1/2}$, $\partial_\nu u|_\Sigma \in H^{-1/2}$ near $\mathcal{G}^\Sigma$ (overall)

In view of

$$\left[(\partial_t v, u)_{L^2(M)} - (v, \partial_t u)_{L^2(M)}\right]_0^T + (v, F)_{L^2((0,T)\times M)} = \int_{(0,T)\times\Sigma} \left(f_0 u|_\Sigma - f_1 \partial_\nu u|_\Sigma\right) dt d\sigma,$$

we should take:

- $f_0 \in H^{-1}$, $f_1 \in L^2$ away from $\mathcal{G}^\Sigma$
- $f_0 \in H^{-1/2}$, $f_1 \in H^{1/2}$ near $\mathcal{G}^\Sigma$

Remark: any improvement of restriction estimates near $\mathcal{G}^\Sigma$ lowers the regularity requirements for f_0, f_1 for well-posedness

The wave equation controlled from Σ: controllability

- Controllability $\iff$ an observability estimate
- (spirit: "A is onto $\iff \|v\| \leq C\|A^* v\|$")
- (Usual case: "Controllability from $\omega \iff$ $\|(u_0, u_1)\|_{H^1 \times L^2} \leq C\|u\|_{L^2((0,T);H^1(\omega))}$ if $\Box u = 0$")

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Theorem (Observability estimate)

Under Assumption ($\mathcal{T}GCC$), there exists $\delta > 0$ s.t. with

- $A_\delta \in \Psi^0_{\mathrm{phg}}((0,T)\times \mathrm{Int}(\Sigma))$ *with principal symbol as on the picture;*
- $\varphi_\delta \in C_c^\infty((0,T)\times \mathrm{Int}(\Sigma))$ *with* $\varphi_\delta \equiv 1$ *on* $[\delta, T-\delta]\times \Sigma_\delta$

we have, with $\Box u = F$, $(u, \partial_t u)|_{t=0} = (u_0, u_1)$,

$$c_N\|(u_0,u_1)\|^2_{H^1\times L^2} \leq \|A_\delta(\partial_\nu u|_\Sigma)\|^2_{L^2(\mathbb{R}\times\Sigma)} + \|A_\delta(u|_\Sigma)\|^2_{H^1(\mathbb{R}\times\Sigma)} + \|F\|^2_{L^2((0,T)\times M)}$$

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- Controllability $\Longleftrightarrow$ an observability estimate
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The wave equation controlled from Σ: observability

Goal: Prove

$$c_N \|(u_0,u_1)\|^2_{H^1\times L^2} \leq \|A_\delta(\partial_\nu u|_\Sigma)\|^2_{L^2(\mathbb{R}\times\Sigma)} + \|A_\delta(u|_\Sigma)\|^2_{H^1(\mathbb{R}\times\Sigma)} + \|F\|^2_{L^2((0,T)\times M)}$$
$$+ \|\varphi_\delta \partial_\nu u|_\Sigma\|^2_{H^{-N}(\mathbb{R}\times\Sigma)} + \|\varphi_\delta u|_\Sigma\|^2_{H^{-N}(\mathbb{R}\times\Sigma)}$$

Three steps:

- ($\mathcal{T}$GCC) $\implies$ ($\mathcal{T}$GCC) δ uniformly away from $\mathcal{G}$
- High frequency estimate using ($\mathcal{T}$GCC) δ
- Low frequencies: unique continuation from Σ

The wave equation controlled from Σ: observability at high frequency

- Near $\mathcal{T}$, we have $\tau^2 > r(x, \xi')$:
- $\sigma(\Box) = \xi_1^2 + r(x, \xi') - \tau^2 = \left(\xi_1 - \sqrt{\tau^2 - r(x, \xi')}\right)\left(\xi_1 + \sqrt{\tau^2 - r(x, \xi')}\right)$
- $\Box \approx (D_{x_1} - \mathrm{Op}(\lambda))(D_{x_1} + \mathrm{Op}(\lambda))$ with $\lambda = \sqrt{\tau^2 - r(x, \xi')}$

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Then solve the Cauchy problem in the x_1-variable:

$$(D_{x_1} \pm \mathrm{Op}(\lambda))w = f,$$

implies

$$\|w(x_1, \cdot)\|_{L^2((-\varepsilon,\varepsilon)\times\mathbb{R}^d)} \leq C(\|w(0, \cdot)\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^2((-\varepsilon,\varepsilon)\times\mathbb{R}^d)})$$

$\implies$ boundary data $u|_\Sigma, \partial_\nu u|_\Sigma$ dominate u near $\mathcal{T}$.

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$\Longrightarrow$ boundary data $u|_\Sigma, \partial_\nu u|_\Sigma$ dominate u near $\mathcal{T}$.

Then: propagation argument: u near $\mathcal{T}$ dominates u everywhere (using ($\mathcal{T}$GCC))

Conclusion and open problems

Further result: controllability of the heat equation/exponential lower bound for traces of eigenfunctions
$\rightsquigarrow$ Carleman estimates à la Lebeau-Robbiano

Some open questions:

- Fine analysis near the glancing set!
- Continuity of the solution $v \in L^2_{\text{loc}}(\mathbb{R}^+; L^2(M))$?
- (non-)Genericity of the ($\mathcal{T}$GCC) among surfaces satisfying (GCC)?
- Geometric condition for having both:
  $\|\phi|_\Sigma\|_{L^2(\Sigma)} \geq Ce^{-c\lambda}\|\phi\|_{L^2(M)}, \quad \text{and} \quad \|\partial_\nu\phi|_\Sigma\|_{L^2(\Sigma)} \geq Ce^{-c\lambda}\|\phi\|_{L^2(M)}$?
- Geometric condition for having both:
  $\|\phi|_\Sigma\|_{L^2(\Sigma)} \geq C\|\phi\|_{L^2(M)}, \quad \text{and} \quad \|\lambda^{-1}\partial_\nu\phi|_\Sigma\|_{L^2(\Sigma)} \geq C\|\phi\|_{L^2(M)}$?
  for $\lambda \geq \lambda_0$.

Thank you!