L^p estimates for joint eigenfunctions

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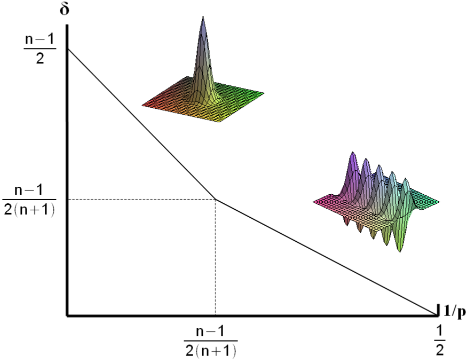
L^p estimates for eigenfunctions

$$(h^2\Delta - 1)u = 0$$

u obeys the estimates due to Sogge

$$\|u\|_{L^p} \lesssim h^{-\delta(n,p)} \|u\|_{L^2}$$

Known to be sharp for spherical harmonics



Joint eigenfunctions

In a letter to Morawetz, Sarnak asks whether improvements can be made to L^∞ estimates for u

$$P_i u = 0 \quad \Delta = P_1, \dots, P_r$$

where the P_i are a set of pseudodifferential operators.

Non example on sphere S^2

$$P_1 = \Delta \quad P_2 = \partial_\varphi$$

must obey Sogge's estimates for $p \geq 6$ as zonal harmonics are invariant under rotations around the north pole.

Positive result, if M is a rank r symmetric space

$$\|u\|_{L^\infty} \lesssim h^{-\frac{n-r}{2}} \|u\|_{L^2}$$

L^p results on symmetric spaces

For eigenfunctions of Δ on symmetric spaces Marshall obtains

$$\|u\|_{L^p} \lesssim h^{-r\delta(n/r,p)} \|u\|_{L^2}$$

(except at $p = \frac{2(n+r)}{n-r}$ where there is logarithmic loss)

Example to keep in mind

$$S^{n/r} \times S^{n/r} \times \cdots \times S^{n/r}$$

$$u = \phi_1(x_1)\phi_2(x_2)\cdots\phi_r(x_r)$$

where each ϕ_i is a spherical harmonic. So eigenfunctions on symmetric spaces have L^p growth no worse than this example.

More general case?

Assume u is a joint quasimode of r semiclassical ΨDOs

$$p_1(x, hD), \ldots, p_r(x, hD)$$

where $p_1(x, hD)$ is suitably Laplace-like.

$$p_i(x, hD)u = \text{Error}$$

- What do we need to assume about the quasimode error?
- What conditions do we need on the $p_i(x, hD)$ to get improvements?
- What conditions do we need to make $p_1(x, hD)$ suitably Laplace-like?

Quasimode error

Typical to assume

$$p(x, hD)u = O_{L^2}(h \|u\|_{L^2})$$

since this is the error introduced by localisation. However could have

$$v = u + hf$$

where u is an exact solution and f is anything with $\|f\|_{L^2} = 1$.

$$f = h^{-n/2}\chi(h^{-1}x_0)$$

where x_0 is a zero of u, then

$$\|v\|_{L^\infty} > h^{-\frac{n-2}{2}} \|v\|_{L^2}$$

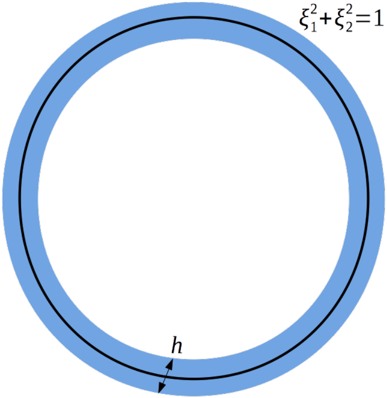
Need a stronger condition. Say u is a strong order h quasimode if

$$p^k(x, hD)u = O_{L^2}(h^k \|u\|_{L^2})$$

Non-degeneracy assumptions on the $p_i(x, hD)$

Look at $2D$ flat model for inspiration

$$p_1(x,\xi) = \xi_1^2 + \xi_2^2 - 1$$



Work on Fourier side

$$\mathcal{F}_h[u] = \frac{1}{(2\pi h)^{n/2}} \int e^{\frac{i}{h}\langle x,\xi\rangle} u(x)dx$$

Preserves L^2 norms. Need

$$(\xi_1^2 + \xi_2^2 - 1)\mathcal{F}_h[u] = O_{L^2}(h)$$

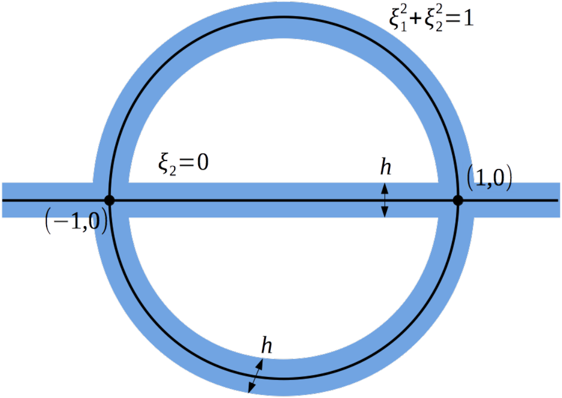
Now we add in a second operator.

$$p_2(x,\xi) = \xi_2 \quad p_2(x,hD) = hD_{x_2}$$

This requires that

$$\xi_2 \mathcal{F}_h[u] = O_{L^2}(h)$$

If u is a joint quasimode $\mathcal{F}_h[u]$ must live in h^2 balls around $(1,0)$ and $(-1,0)$. This mean the L^∞ norm has to be bounded.

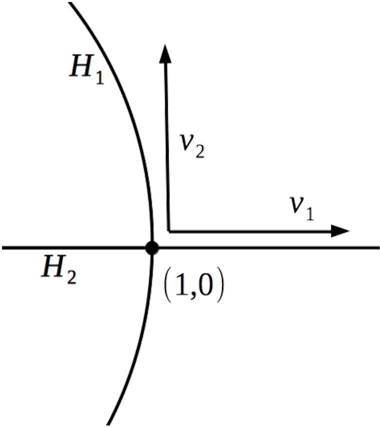


Consider the hypersurfaces

$$H_1 = \{\xi \mid |\xi|^2 - 1 = 0\}$$

$$H_2 = \{\xi \mid \xi_2 = 0\}$$

Their normal vectors at $(1,0)$, ν_1 and ν_2 are perpendicular. So each one constrains $\mathcal{F}_h$ in one direction. Suggests an assumption that all the $p_i(x,\xi)$ have hypersurfaces as characteristic sets and that the normals are linearly independent.



Consider Sarnak's non-example in this framework

$$\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\varphi}\right)u = -h^{-2}u$$

Converting to a semiclassical ΨDO and dropping non-principal terms

$$(h^2\sin^2\theta D_\theta^2 + h^2 D_\varphi^2 - \sin^2\theta)u = 0$$

So

$$p_1(\theta,\varphi,\xi,\eta) = \xi_1^2\sin^2\theta + \eta^2 - \sin^2\theta$$

$$p_2(\theta,\varphi,\xi,\eta) = \eta$$

Now at $\theta = 0$

$$H_1 = \{(\xi,\eta) \mid \eta^2 = 0\} \quad H_2 = \{(\xi,\eta) \mid \eta = 0\}$$

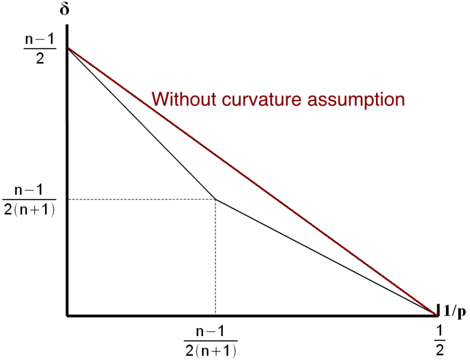
So clearly we need the non-degeneracy to hold for all sets $\{\xi \mid p_i(x_0,\xi) = 0\}$.

Laplace-like condition for $p_1(x, hD)$

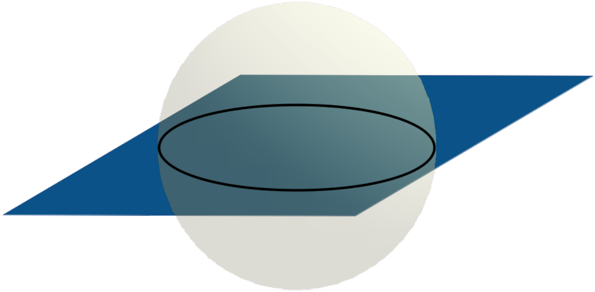
Distinctive piecewise behaviour comes from the curvature of

$$\{\xi \mid |\xi|_{g(x)} = 1\}$$

Connected to the classical theory of restriction conjectures. Need to maintain this curvature assumption.



Cross sections



Joint eigenfunction condition leads to taking cross-sections. So need to maintain curvature for these.

Need that

$$\{\xi \mid p_1(x_0, \xi)\}$$

has positive definite second fundamental form.

Theorem (T 2018)

Suppose u is a strong, joint $O_{L^2}(h)$ quasimode of a set of $p_1(x, hD), \dots, p_r(x, hD)$ where

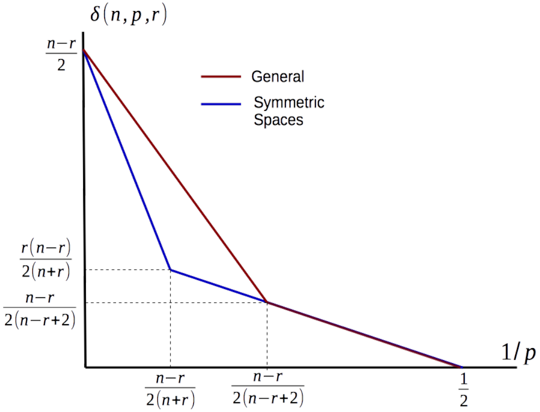
- *For any x_0, $\{\xi \mid p_i(x_0, \xi) = 0\}$ is a hypersurface $i = 1, \dots, r$.*
- *For any x_0, $\{\xi \mid p_1(x, \xi) = 0\}$ has positive definite second fundamental form.*
- *If $\nu_i(x, \xi)$ is the normal to $\{\xi \mid p_i(x, \xi) = 0\}$ then the set of $\nu_i(x, \xi)$ are linearly independent.*

Then

$$\|u\|_{L^p} \lesssim h^{-\delta(n,p,r)} \|u\|_{L^2}$$

$$\delta(n,p,r) = \begin{cases} \frac{n-r}{2} - \frac{n-r+1}{p} & \frac{2(n-r+2)}{n-r} \leq p \leq \infty \\ \frac{n-r}{4} - \frac{n-r}{2p} & 2 \leq p \leq \frac{2(n-r+2)}{n-r}. \end{cases}$$

Comparison to symmetric spaces



- Agrees for $p = \infty$ and for $p \leq \frac{2(n-r+2)}{n-r}$.
- Symmetric spaces enjoy slightly better estimates between $\frac{2(n-r+2)}{n-r}$ and ∞.
- Might ask whether better estimates are possible in the general case.

Sharpness

Consider the case

$$p_1(x,\xi) = |\xi|^2 - 1 \quad p_i(x,\xi) = \xi_i, \text{ for } i = 2,\dots,r$$

Then we are essentially looking at solutions

$$u(x) = v(x_1, x_{r+1}, \dots, x_n)$$

where v satisfies the $n - r + 1$ dimensional eigenfunction equation. So we can apply Sogge's bounds in dimension $n - r + 1$. These agree with $\delta(n, p, r)$.

Proof sketch

Will define a process for successively factoring out r of the ξ_i effectively making the problem a $n - r + 1$ dimensional one.

Some Notation

$$\tilde{\xi}^i = (\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n)$$

$$\tilde{\xi}^{(i,j)} = (\xi_1, \ldots, \xi_{i-1}, \xi_{j+1}, \ldots, \xi_n)$$

same notation for $\tilde{x}^i$, $\tilde{x}^{(i,j)}$.

Using the fact that u is a strong $O_{L^2}(h)$ quasimode we can restrict our attention to regions of $\mathbb{R}^{2n}$ near a point (x_0, ξ_0) where all the $p_i(x, \xi)$ are zero.

Step 1: Choose an appropriate coordinate system

Since $\{\xi \mid p_1(x_0, \xi) = 0\}$ is a hypersurface, we know

$$\nabla_\xi p_i(x_0, \xi_0) \neq 0.$$

Pick ξ_1 so

$$\partial_{\xi_1} p(x_0, \xi_0) \neq 0 \quad \nabla_{\tilde{\xi}^1} p_1(x_0, \xi_0) = 0$$

In these coordinates $\nu_1(x_0, \xi_0) = \xi_1$

Since $\nu_1(x_0, \xi_0), \nu_2(x_0, \xi_0)$ are LI we cannot have

$$\nabla_{\tilde{\xi}^1} p_2(x_0, \xi_0) = 0.$$

Pick ξ_2 so that

$$\partial_{\xi_2} p_2(x_0, \xi_0) \neq 0 \quad \nabla_{\tilde{\xi}^{(1,2)}} p_2(x_0, \xi_0) = 0$$

Keep doing this up to ξ_r.

Step 2: Factorising out ξ_r

We start with ξ_r. Since $\partial_{\xi_r} p_r(x_0, \xi_0) \neq 0$ we write

$$p_r(x,\xi) = e_r(x,\xi)(\xi_r - a_r(x,\tilde{\xi}^r))$$

where $|e_r(x,\xi)| > c > 0$ so $e(x,hD)$ is invertible. That is

$$(hD_{x_r} - a_r(x, hD_{\tilde{x}^r}))u = O_{L^2}(h)$$

Now for x near x_0 the hypersurface $\{\xi \mid p_r(x,\xi) = 0\}$ is defined by $\xi_r = a_r(x,\tilde{\xi}^r)$. For $i = 1, \ldots, r-1$ set

$$p_{r,i}(x,\tilde{\xi}^r) = p_i(x,\xi_1,\ldots,\xi_{r-1},a_r(x,\tilde{\xi}^r),\xi_{r+1},\ldots,\xi_n)$$

and claim that u is still a quasimode

$$p_{r,i}(x,hD)u = O_{L^2}(h) \quad i = 1,\ldots,r-1$$

We can say that if

$$q_i(x,\xi) = p_i(x,\xi) - p_{r,i}(x,\tilde{\xi}^r)$$

then

$$q(x,\xi) = (\xi_r - a_r(x,\tilde{\xi}^r))r(x,\xi)$$

so by composition of semiclassical ΨDOs

$$q_i(x,hD)u = O_{L^2}(h\|u\|_{L^2})$$

So indeed

$$p_{r,i}(x,hD)u = O_{L^2}(h\|u\|_{L^2})$$

Step 3: Inductive process to factor out $\xi_2, \dots, \xi_{r-1}$

Note that

$$\partial_{\xi_j} p_{r,i}(x, \tilde{\xi}^r) = \left(\partial_{\xi_j} p_i + \partial_{\xi_r} p_i \partial_{\xi_j} a_r\right)(x_0, \tilde{\xi}_0^r)$$

So since $\partial_{\xi_r} p_i(x_0, \xi_0) = 0$ we retain

$$\partial_{\xi_i} p_{r,i}(x_0, \tilde{\xi}_0^r) \neq 0 \quad \partial_{\xi_j} p_{r,i}(x_0, \tilde{\xi}_0^r) = 0 \;\; j > i.$$

So we can repeat the process factorising $p_{r,r-1}(x, \tilde{\xi}^r)$ to write

$$\xi_{r-1} = a_{r-1}(x, \tilde{\xi}^{(r-1,r)})$$

$$p_{r,r-1,i}(x, \tilde{\xi}^{(r-1,r)}) = p_{r,i}(x, \xi_1, \dots, \xi_{r-2}, a_{r-1}(x, \tilde{\xi}^{(r-1,r)}), \xi_{r+1}, \dots, \xi_n)$$

Keep doing this in an inductive fashion until ξ_2 is factored out.

Step 4: Express $p_{r,\dots,2,1}(x, hD)$ as an evolution equation

From our inductive process we have a $p_{r,\dots,2,1}(x, \tilde{\xi}^{(2,\dots,r)})$ so that

$$p_{r,\dots,2,1}(x, hD)u = O_{L^2}(h)$$

and

$$\partial_{\xi_1} p_{r,\dots,2,1}(x, \tilde{\xi}^{(2,\dots,r)}) \neq 0$$

Write

$$p_{r,\dots,2,1}(x, \tilde{\xi}^{(2,\dots,r)}) = e(x, \tilde{\xi}^{(2,\dots,r)})(\xi_1 - b(x, \tilde{\xi}^{(1,\dots,r)}))$$

Adopt notation $x = (t, y, z)$, $\tilde{\xi}^{(2,\dots,r)} = (\xi_1, \eta)$. Then we have

$$(hD_t - b(t, y, z, hD_z))u = O_{L^2}(h \|u\|_{L^2})$$

and the curvature assumption on $p_1(x, \xi)$ gives us that $\frac{\partial^2 b}{\partial \eta_i \partial \eta_j}$ is a positive definite matrix.

Step 5: Reduce to an $n - r + 1$ problem

Since $b(t, y, z, hD_z)$ has no derivatives in y we can treat y as a parameter.

The factorisation of each $p_i(x, \xi)$ allows us to reduce the problem to estimating

$$L_y^p L_t^2 L_z^2 \to L_y^p L_t^p L_z^p$$

and

$$L_y^2 L_t^2 L_z^2 \to L_y^p L_t^2 L_z^2$$

separately. The first the same as the Koch-Tataru-Zworski semiclassical estimates in dimension $n - r + 1$. The second follows from factorising each $p_{r,r-1,\ldots,i}(x, \xi)$.

$L^p_y L^2_t L^2_z \to L^p_y L^p_t L^p_z$ estimates

Use Duhamel to write

$$u = U(t,0)u + \frac{1}{h}\int_0^t U(t-s,s)E[u]$$

$$E[u] = (hD_t + b(t,y,z,hD_z))u$$

$$\begin{cases} (hD_t - b(t+s,y,z,hD_z))U(t,s) = 0 \\ U(0,s) = \mathrm{Id} \end{cases}$$

Then write $U(t,s)$ has a semiclassical FIO

$$U(t,s) = \frac{1}{(2\pi h)^{n-r+1}} \int e^{\frac{i}{h}(\phi(t,y,z,\xi) - \langle v,\xi\rangle)} b(t,y,z,\xi) d\xi$$

$$\phi_t - b(t,y,z,\nabla_z\phi) = 0 \quad \phi(0,y,z) = \langle z,\xi\rangle$$

Estimates follow from a $TT^\star$ argument.

$$\|U(t,s)U^\star(\tau,s)\|_{L^1_z\to L^\infty_z} \lesssim h^{-\frac{n-r+1}{2}}(h+|t-\tau|)^{-\frac{n-r+1}{2}}$$

$$\|U(t,s)U^\star(\tau,s)\|_{L^2_z\to L^2_z} \lesssim 1$$

- The $(h+|t-\tau|)^{-\frac{n-r+1}{2}}$ arises because of the curvature.
- Estimates follow from interpolation then application of Hardy-Littlewood-Sobolev.

$L^2_y L^2_t L^2_z \to L^p_y L^2_t L^2_z$ estimates

We have $p_{r,\dots,3,2}(x, \tilde{\xi}^{(3,r)}$ so that

$$\partial_{\xi_2} p_{r,\dots,3,2}(x_0, \tilde{\xi}_0^{(3,r)}) \neq 0$$

so factorisation gives that

$$(hD_{x_2} - a_2(x, hD_{\tilde{x}^{3,r}})u = O_{L^2}(h\|u\|)$$

Use Duhamel again

$$u = W(x_2, 0)u + \frac{1}{h}\int_0^{x_2} W(x_2 - s, s)E[u]ds$$

So $L^p_{x_2}$ is bounded by $L^2_{x_2}$. Then work inductively through $x_3, \dots, x_r$.

Further work

- Is there any assumption on a general set $p_1(x, hD), \ldots, p_r(x, hD)$ that is strictly weaker than assuming M is a symmetric space but also implies

$$\|u\|_{L^p} \lesssim h^{r\delta(n/r,p)} \|u\|_{L^2}?$$

- Can we use weaker assumptions on the intersections of the $\{\xi \mid p_i(x,\xi) = 0\}$ to get improvements for some p?

In particular consider the case $p_1 = |\xi|^2 - 1$ and $p_2 = \xi_1 - 1$. Expect improvement for high p but not low p (where tube concentration saturates estimates).