Microlocal analysis for Kerr-de Sitter black holes

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This talk is about the key analytic ingredients of the stability of Kerr-de Sitter (KdS) black holes, which are certain Lorentzian manifolds solving Einstein's equation.

Concretely, the stability statement, made more precise in the next talk, is that perturbations of initial data for Schwarzschild-de Sitter, or slowly rotating KdS black holes, will produce global solutions of Einstein's equation which decay at an exponential rate to a KdS metric, with potentially different mass and angular momentum parameters than the perturbed metric.

Variations of the method also show the stability of Minkowski space, but this is somewhat more technical(!), so will be underemphasized.

The linearized version of this equation is hyperbolic, thus an evolution equation, once appropriate gauge terms are introduced. In particular, there is a good finite time solvability theory.

However, stability is a global statement in space-time. We attack this by using *global analysis*, rather than using the finite time solvability and attempting to control it uniformly as time goes to infinity.

Global analysis is standard for elliptic PDE: one cannot solve an elliptic PDE by solving it locally! We thus need to develop tools for non-elliptic global analysis.

One way to do global analysis is to compactify the underlying manifold to a manifold with boundary or corners since this automatically gives uniformity; more on this later!

Typically global analysis proceeds in a Fredholm framework for an operator P on a manifold M, with or without boundary or corners. For us, this will be an operator acting on tensors, but this barely has an effect on the analysis, so we suppress bundles.

Thus, we work with Hilbert spaces X, Y of distributions on M, such that

- $P : X \to Y$ continuously,
- $\mathrm{Ran}\, P$ closed
- $\mathrm{Ker}\, P$, $Y / \mathrm{Ran}\, P$ are finite dimensional.

The desired properties,

- $P : X \to Y$ continuously,
- $\operatorname{Ran} P$ closed
- $\operatorname{Ker} P$, $Y / \operatorname{Ran} P$ are finite dimensional,

are guaranteed by the Fredholm estimates

$$\|u\|_X \leq C(\|Pu\|_Y + \|u\|_{Z_1})$$

and

$$\|v\|_{Y^*} \leq C(\|P^* v\|_{X^*} + \|u\|_{Z_2}),$$

where the inclusion maps $X \to Z_1$ and $Y^* \to Z_2$ are compact.

One often wants actual invertibility; this amounts to being able to drop the relatively compact terms.

The simplest example is elliptic operators on compact manifolds without boundary M, acting between sections of vector bundles (which we suppress), with basic geometric examples being the Laplacian on functions or differential forms, and Dirac operators.

- $P \in \mathrm{Diff}^m(M)$ elliptic (at least principally classical), i.e. $\sigma_m(P)$ invertible on $T^*M \setminus o$,
- $X = H^s = H^s(M)$, $Y = H^{s-m}(M)$, $s \in \mathbb{R}$,
- so $X^* = H^{-s}(M)$, $Y^* = H^{-s+m}(M)$,
- $Z_1 = H^{-N}(M)$, $Z_2 = H^{-N}(M)$, N large.

The Fredholm property follows from the elliptic estimate

$$\|\phi\|_{H^r} \leq C(\|L\phi\|_{H^{r-m}} + \|\phi\|_{H^{-N}}),$$

with $L = P$, $r = s$, resp. $L = P^*$, $r = -s + m$. Note that the choice of s is irrelevant here (elliptic regularity).

The non-elliptic problems we consider are problems in which the elliptic estimate is replaced by estimates of the form

$$\|u\|_{H^s} \leq C(\|Pu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

i.e. with a loss of one derivative relative to the elliptic setting, and

$$\|v\|_{H^{s'}} \leq C(\|P^* v\|_{H^{s'-m+1}} + \|v\|_{H^{-N'}}),$$

with $s' = -s + m - 1$ being the case of interest.

Such estimates imply that $P : X \to Y$ is Fredholm if

$$X = \{u \in H^s : \; Pu \in H^{s-m+1}\}, \; Y = H^{s-m+1}.$$

Here X is a first order coisotropic space associated to the characteristic set of P; it is easy to see that C^∞ is still dense in it.

A complication is that H^s is often a *variable order* (or anisotropic) Sobolev space (see e.g. Unterberger, Duistermaat), i.e. s is a real-valued function on S^*M. This space is defined by:

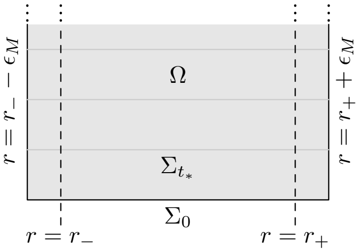
- Let $s_0 = \inf s$, and let $A \in \Psi^s_\delta(M) \subset \Psi^{\sup s}_\delta(M)$, $\delta \in (0, 1/2)$, be elliptic,
- $$H^s = \{u \in H^{s_0} : \ Au \in L^2\},$$
- for instance if g is a Riemannian metric, one can take the principal symbol of A to be $|\xi|^s_g$.

Then $P \in \Psi^m$ maps $P : H^s \to H^{s-m}$ continuously still.

To give a concerete idea for KdS, P could be the Fourier transformed, in t_*, wave operator, so M would be the 'spatial manifold', $(r_- - \epsilon_M, r_+ + \epsilon_M) \times \mathbb{S}^2$, $\epsilon_M > 0$, where r_-, r_+ are the event and cosmological horizons.

Of course, this is not compact, one way to deal with this is to add *Cauchy hypersurfaces* at $r = r_+ + \epsilon_M$ and $r = r_- - \epsilon_M$; another is to double the space and place *complex absorption* near the aforementioned hypersurfaces.

While these are important, they distract from the main point, so I will work in a simpler setting with fewer features.


$r = r_- - \epsilon_M$
$r = r_+ + \epsilon_M$
Ω
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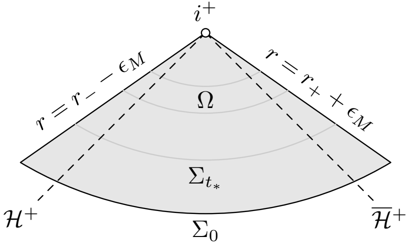

i^+
$r = r_- - \epsilon_M$
$r = r_+ + \epsilon_M$
Ω
Σ_{t_*}
$\mathcal{H}^+$
$\overline{\mathcal{H}}^+$
Σ_0

Another example would be the dynamical systems problems, see Faure-Sjöstrand, Dyatlov-Zworski, Dyatlov-Faure-Guillarmou,...

Of course, we need to *prove* these estimates. The tool we use is *microlocal analysis*. A key point is that this is both *perturbation stable*, and *works in a limited regularity setting*, with work on this going back to Beals and Reed in the 1980s.

The most basic non-elliptic phenomenon, when $P \in \Psi^m$ has real scalar principal symbol $p\,\mathrm{Id}$, is propagation of singularities along the bicharacteristics, i.e. integral curves of H_p (due to Hörmander):

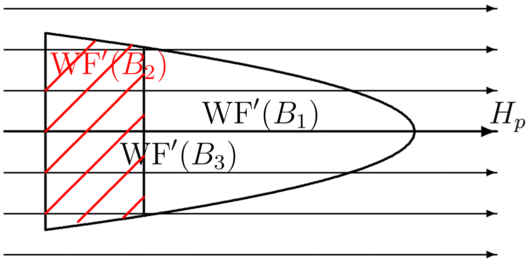
- $\mathrm{WF}^s(u) \subset S^*M$ measures whether u is microlocally in H^s, i.e. a point $\alpha \in S^*M$ is *not* in $\mathrm{WF}^s(u)$ if there is $A \in \Psi^0(M)$ such that A is elliptic at α and $Au \in H^s$.
- Away from $\mathrm{Char}(P) = \{p = 0\}$, i.e. in the elliptic set $\mathrm{Ell}(P)$, one has microlocal elliptic regularity, i.e. the Sobolev wave front set, $\mathrm{WF}^s(u)$, satisfies $\mathrm{WF}^s(u) \setminus \mathrm{Char}(P) \subset \mathrm{WF}^{s-m}(Pu)$.
- In $\mathrm{Char}(P) \setminus \mathrm{WF}^{s-m+1}(Pu)$, $\mathrm{WF}^s(u)$, is a union of maximally extended bicharacteristics.

This is proved by positive commutator estimates, using that $\imath\sigma_{m+m'-1}([P, A]) = H_p a$, arranging a definite sign for the latter where one wants to have the conclusion.

This gives an estimate

$$\|B_1 u\|_{H^s} \leq C(\|B_2 u\|_{H^s} + \|B_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

$B_j \in \Psi^0$, provided $\mathrm{WF}'(B_1) \subset \mathrm{Ell}(B_3)$, and all bicharacteristics from points in $\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)$ reach the elliptic set $\mathrm{Ell}(B_2)$ of B_2 while remaining in $\mathrm{Ell}(B_3)$.

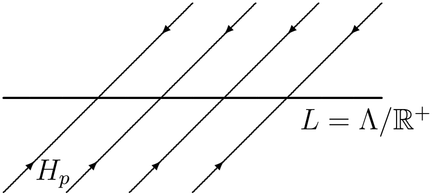


The same estimate is valid if s is variable, provided one either restricts to *forward* bicharacteristics and requires $H_p s \leq 0$, or to *backward* bicharacteristics and requires $H_p s \geq 0$.

The basic problem with this estimate is the term $\|B_2 u\|_{H^s}$ on the right hand side – how does one control this?

One option is *complex absorption*, this allows the use of constant order Sobolev spaces. The point is then that bicharacteristics reach the elliptic set of an operator Q with real principal symbol, and one works with $P - iQ$.

A more natural option is to have some structure of the bicharacteristic flow: we need that there are submanifolds L of S^*M which act as sources/sinks in the normal direction.



- The most frequent place these arise is *radial sets*, i.e. points in T^*M where H_p is tangent to the dilation orbits. Note that Hörmander's theorem provides no information here. E.g. conormal bundle of horizons of SdS.
- Another example: horizons of KdS: now non-trivial flow within Λ.
- In non-degenerate settings, i.e. when H_p is non-zero, the biggest possible dimension of a radial set is that of M, in which case it is a conic Lagrangian submanifold of T^*M.
- In this case, they act as source or sink within $\mathrm{Char}(P)$; in the source case H_p flows to the zero section within Λ, in the sink case from the zero section: red shift/blue shift.

Let $\tilde{p}$ be the principal symbol of $\frac{1}{2i}(P - P^*) \in \Psi^{m-1}$, and define $\tilde{\beta}$ by

$$\tilde{p}|_\Lambda = \tilde{\beta}\frac{H_p\rho}{\rho},$$

where ρ is an elliptic homogeneous degree 1 function, which is independent of choices (even that of the metric defining the adjoint!).

In this case there is an analogue of the propagation of singularities theorem, but there is a threshold, $(m-1)/2 - \tilde{\beta}$:

- if the Sobolev order is higher than this, one can propagate estimates from $L = \Lambda/\mathbb{R}^+$, without needing a priori control like $B_2 u$,
- if the Sobolev order is below this, one can propagate estimates to L, needing control in a punctured neighborhood of L.

- If $s \geq s_0 > (m-1)/2 - \tilde{\beta}$, then

$$\|B_1 u\|_{H^s} \leq C(\|B_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{s_0}}),$$

$B_j \in \Psi^0$ elliptic on L, provided $\mathrm{WF}'(B_1) \subset \mathrm{Ell}(B_3)$, and all bicharacteristics from points in $\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)$ tend to L while remaining in $\mathrm{Ell}(B_3)$.

- If $s < (m-1)/2 - \tilde{\beta}$ then

$$\|B_1 u\|_{H^s} \leq C(\|B_2 u\|_{H^s} + \|B_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

$B_j \in \Psi^0$ elliptic on L, provided $\mathrm{WF}'(B_1) \subset \mathrm{Ell}(B_3)$, and all bicharacteristics from points in $(\mathrm{WF}'(B_1) \cap \mathrm{Char}(P)) \setminus L$ reach the elliptic set $\mathrm{Ell}(B_2)$ of B_2 while remaining in $\mathrm{Ell}(B_3)$.

Replacing P by P^* changes the sign of $\tilde{\beta}$, and it naturally leads to estimates on the required dual spaces.

As a consequence, if there are radial sets L_1, L_2 such that all bicharacteristics in $\mathrm{Char}(P) \setminus (L_1 \cup L_2)$ escape to L_1 in one of the directions along the bicharacteristics and to L_2 in the other, one has the required Fredholm estimate provided one can arrange the Sobolev spaces so that

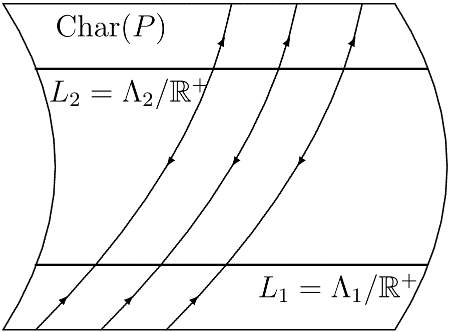
- at L_1 the Sobolev order is above the threshold for P,
- at L_2 the Sobolev order is above the threshold for P^*,
- the Sobolev order is monotone decreasing from L_1 to L_2.

Namely,

$$\|u\|_{H^s} \leq C(\|Pu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

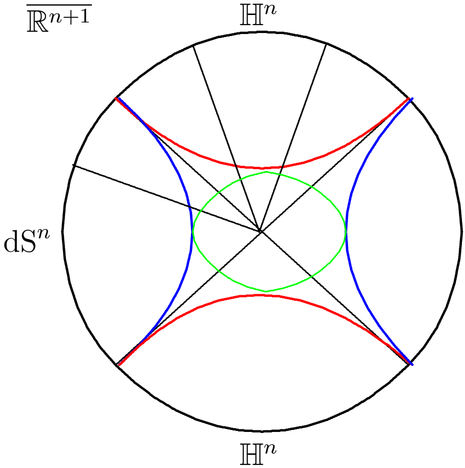
$$\|v\|_{H^{s'}} \leq C(\|P^* v\|_{H^{s'-m+1}} + \|v\|_{H^{-N'}}),$$

with $s' = -s + m - 1$.


$\mathrm{Char}(P)$
$L_2 = \Lambda_2/\mathbb{R}^+$
$L_1 = \Lambda_1/\mathbb{R}^+$

A non-KdS example is as follows:

- Let $\tilde{M} = \mathbb{R}^{n+1}$ with the Minkowski metric and $\Box$ be the wave operator.
- Let ρ be a homogeneous degree 1 positive function, e.g. a Euclidean distance from the origin.
- The conjugate of $\rho^2\Box$ by the Mellin transform along the dilation orbits gives a family of operators P_σ, σ the Mellin dual parameter, on $\mathbb{S}^n$ (smooth transversal to the dilation orbits).
- P_σ is elliptic inside the light cone, but are Lorentzian outside the light cone.
- The conormal bundle of the light cone consists of radial points.
- The characteristic set has two components, and there are four components of the radial set: a future and a past component within each component of the characteristic set.



There are 2^2 Fredholm problems: in each component of the characteristic set one can choose at which of the two radial sets within it the Sobolev order is high. (F, F*, A, R)

Concretely, in one component Σ_+ of the characteristic set, the bicharacteristics go from the past component of the radial set L_{+-} to the future one L_{++}; in the other component Σ_- they go from the future component of the radial set L_{-+} to the past one L_{--}. So reasonable choices of Fredholm problems:

- Make the Sobolev spaces high regularity at the past radial sets and low at the future radial sets: this is the *forward propagator*.
- Make the Sobolev spaces low regularity at the past radial sets and high at the future radial sets: this is the *backward propagator*.
- Make the Sobolev spaces high regularity at the sources L_{+-} and L_{-+} and low regularity at the sinks, or vice versa. These are the Feynman propagators, and they propagate estimates for P_σ in the direction of the Hamilton flow in the first case, and against the Hamilton flow in the second.
- The adjoint always propagates estimates in the *opposite* direction as the operator itself.

Another similar case is totally characteristic, or b-, pseudodifferential operators $P \in \Psi_b^m(M)$ on manifolds with boundary (Melrose). Here, writing x, y_j are local coordinates, x a boundary defining function, and $x = e^{-t_*}$,

- $\mathcal{V}_b(M)$ is the Lie algebra of C^∞ vector fields tangent to ∂M, locally spanned by $x\partial_x = -\partial_{t_*}$ and ∂_{y_j} over $C^\infty(M)$,
- $\mathrm{Diff}_b(M)$ is the associated algebra of differential operators, $\Psi_b(M)$ the corresponding pseudodifferential algebra,
- With $x = e^{-t_*}$, *locally* the ps.d.o. algebra $\Psi_b(M)$ is (almost) Hörmander's uniform algebra in a 'cylindrical set', $(t_*, y) \in (1, \infty) \times O$, $\overline{O} \subset \mathbb{R}^{n-1}$ compact.
- $\mathcal{V}_b(M)$ is the set of all sections of a vector bundle ${}^bTM$, with dual bundle ${}^bT^*M$ – a local basis for ${}^bT^*M$ is $\frac{dx}{x} = -dt_*, dy_j$,
- the principal symbol p of $P \in \Psi_b^m(M)$ is now a function on ${}^bT^*M \setminus o$,
- ellipticity is the invertibility of the principal symbol.

In the KdS setting, x would be e^{-t_*}, y would be the spatial variables, in a region that extends beyond the horizons.

The elliptic estimate in this setting is

$$\|\phi\|_{H_b^{s,r}} \leq C(\|P\phi\|_{H_b^{s-m,r}} + \|\phi\|_{H_b^{-N,r}}).$$

Even for $s > -N$, the inclusion $H_b^{s,r} \to H_b^{-N,r}$ is *not* compact, so this is not sufficient for Fredholm estimates.

Example: manifolds with cylindrical ends, i.e. with Riemannian metrics on ${}^bTM$, which at ∂M have the form $\frac{dx^2}{x^2} + h = dt_*^2 + h$, h a metric on $T\partial M$, and P is for instance the Laplacian.

Another example is asymptotically Euclidean spaces, but here one needs to work with $x^{-2}\Delta_g$!

- To fix the problem of non-relatively compact errors, one needs the normal operator $N(P) \in \Psi_b^m(\partial M \times \mathbb{R}^+)$, which is dilation invariant in the second factor, which arises by 'freezing the coefficients' of P at ∂M.
- It is this required asymptotic behavior, modulo $O(x^\delta) = O(e^{-\delta t_*})$, $\delta > 0$, that strengthens the uniform algebra.
- The Mellin transform of $N(P)$, $\hat{N}(P)(\sigma)$, $\sigma \in \mathbb{C}$, is the indicial family.
- If P is elliptic, so is $\hat{N}(P)$, which is now on a manifold without boundary, so forms an analytic Fredholm family.
- If invertible at one point, $\hat{N}(P)$ is meromorphic, with finite rank poles σ_j, called indicial roots or resonances.
- If ϕ_j is in the nullspace of $\hat{N}(P)(\sigma)$, then $x^{\imath\sigma_j}\phi_j$ is a 'quasinormal mode' of $N(P)$ and P; $\operatorname{Im}\sigma_j > 0$ means it is *growing* as $x \to 0$.

- As $\hat{N}(P)(\sigma)$ is a 'large parameter' family (Shubin), or after rescaling a 'semiclassical family' with $h = |\sigma|^{-1}$, then one has large parameter/semiclassical elliptic estimates in strips $|\operatorname{Im}\sigma| < c$,

$$\|\psi\|_{H_h^s} \leq C|\sigma|^{-m}(\|\hat{N}(P)(\sigma)\psi\|_{H_h^{s-m}} + h^N\|\psi\|_{H_h^{-N}}),$$

  which gives invertibility for small h.

- If r is such that $r \neq -\operatorname{Im}\sigma_j$ for any j, this gives

$$\|\phi\|_{H_b^{s,r}} \leq C\|N(P)\phi\|_{H_b^{s-m,r}}$$

  and if adjoints are with respect to a product-type metric,

$$\|\phi\|_{H_b^{s',r}} \leq C\|N(P^*)\phi\|_{H_b^{s'-m,r}}.$$

Combining all pieces:

$$\|\phi\|_{H_b^{s,r}} \leq C(\|P\phi\|_{H_b^{s-m,r}} + \|\phi\|_{H_b^{-N,r-1}}),$$

with a similar estimate for the adjoint, so

$$P : H_b^{s,r} \to H_b^{s-m,r}$$

is Fredholm.

Furthermore, solutions have an expansion in terms of the resonant states: roughly (ignoring multiplicities) $Pu = f$, $u \in H_b^{s,r}$, $f \in H_b^{s-m,r'}$, $r' > r$, implies that $u = \sum_j x^{\imath\sigma_j}\phi_j + u'$, $u' \in H_b^{s,r'}$.

For the Laplacian on Euclidean space, this is the spherical harmonics expansion at infinity.

There is a completely analogous framework in the Lorentzian setting, such as KdS, assuming that H_p flow has a structure analogous to the compact manifold setting.

A case that does *not* require additional structures is Minkowski space, radially compactified, or its generalizations to *'Lorentzian scattering metrics'*. (More precisely, one is working with $P = x^{-(n-2)/2} x^{-2} \Box_g x^{(n-2)/2}$, $x = \rho^{-1}$.)

Another case which applies with either complex absorption or Cauchy hypersurfaces added is the static patch of de Sitter space, or its perturbations.

In KdS there is in addition trapping, which affects the large parameter estimates on *decaying spaces*, but not on *growing spaces*.

- In these cases both the (symbolic!) propagation of singularities and the radial estimates have b-analogues.
- As for elliptic problems, this is not sufficient: one needs to work with the Mellin transformed normal operator family, which was described before.
- Again, (for appropriate Sobolev orders, with restrictions arising from the radial points, as well as potential trapping) $\hat{N}(P)(\sigma)$ is a Fredholm family, with finitely many poles in strips $|\operatorname{Im}\sigma| < c$.
- If $r \neq -\operatorname{Im}\sigma_j$ for any of the poles σ_j of $\hat{N}(P)(\sigma)^{-1}$ then $P : X \to Y$,

$$X = \{u \in H_b^{s,r} : \; Pu \in H_b^{s-1,r}\}, \; Y = H_b^{s-1,r}, \text{ is Fredholm.}$$

- If the space are chosen to correspond to the *forward in time* problem, then for $\operatorname{Im}\sigma \gg 1$, $\hat{N}(P)(\sigma)^{-1}$ is invertible; this gives the *forward in time* solution operator for P.

- One also gets an expansion of the solution in terms of resonances, as in the elliptic case.
- If there is trapping, the large parameter and b-microlocal estimates become lossy in terms of derivatives in $\operatorname{Im}\sigma \leq 0$, resp. $r \geq 0$.
- In KdS, these can be handled in a bigger region, $\operatorname{Im}\sigma \geq -c$, $c > 0$, resp. $r < r_0$, $r_0 > 0$, for appropriate c, r_0, due to the work of Wunsch-Zworski and Dyatlov who analyzed *normally hyperbolic trapping*.
- This suffices to show that solutions of the KdS wave equations have an expansion in terms of resonances *up to exponential decay*.