BOUNDEDNESS OF MAXIMAL FUNCTIONS ON NON-DOUBLING MANIFOLDS WITH ENDS

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Abstract. Let M be a manifold with ends constructed in [2] and Δ be the Laplace-Beltrami operator on M. In this note, we show the weak type $(1,1)$ and L^p boundedness of the Hardy-Littlewood maximal function and of the maximal function associated with the heat semigroup $\mathcal{M}_\Delta f(x) = \sup_{t>0} |\exp(-t\Delta)f(x)|$ on $L^p(M)$ for $1 < p \leq \infty$. The significance of these results comes from the fact that M does not satisfies the doubling condition.

1. Introduction

The theory of Calderón-Zygmund operators has played a crucial role in harmonic analysis and its wide applications in the last half a century or so. We refer readers to the excellent book [7] and the references therein. In the standard Calderón-Zygmund theory, an essential feature is the so-called doubling condition. Let us recall that a metric space (X, d, μ) equipped with a metric d and a measure μ satisfies the doubling condition if there exists a constant C such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r))$$

for all $x \in X$ and $r > 0$.

Many metric spaces in classical analysis satisfy the doubling condition such as the Euclidean spaces and their smooth domains (with Lebesgue measure), Lie groups and manifolds of polynomial growth. However, there are significant applications for which underlying ambient spaces do not satisfy the doubling condition, for example domains of Euclidean spaces with rough boundaries, Lie groups and manifolds with exponential growth. To these non-doubling spaces, the standard Calderón-Zygmund theory established in the 70's and 80's is not applicable.

Recent works of Nazarov, Treil, Volberg, Tolsa and others, see for example [3, 4, 5, 6, 8, 9] show that a large part of the standard Calderön-Zygmund theory can be adapted to the case of non-doubling spaces which satisfy a mild growth condition. In [1], Duong and A. McIntosh also obtain estimates for certain singular integrals acting on some domains which do not necessarily satisfy the doubling condition. However, the theory of singular integrals

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on non-doubling spaces is far from being complete and there are still many significant open problems in this topic.

In this note, we study the boundedness of certain maximal functions on non-doubling manifolds with ends. More specifically, we will show the weak type $(1,1)$ of the Hardy-Littlewood maximal function and the maximal function associated with the heat semigroup of the Laplace-Beltrami operator as well as L^p boundedness for these maximal operators for $1 < p \le \infty$. Let us recall that the maximal function associated with the heat semigroup is defined by the following formula

$$\mathcal{M}_\Delta f(x) = \sup_{t>0} |\exp(-t\Delta)f(x)| \tag{1}$$

for $f \in L^p(M)$, $1 \le p \le \infty$. The behaviour of the kernels of the semigroup $\exp(-t\Delta)$ on manifolds with ends was studied in [2]. For the convenience of reader, we recall the main result of [2] in the next section as it plays a key role in our estimates of the operator $\mathcal{M}_\Delta$.

2. Manifolds with ends

Let M be a complete non-compact Riemannian manifold. Let $K \subset M$ be a compact set with non-empty interior and smooth boundary such that $M\backslash K$ has k connected components $E_1, \ldots, E_k$ and each E_i is non-compact. We say in such a case that M has k ends with respect to K and refer to K as the central part of M. In many cases, each E_i is isometric to the exterior of a compact set in another manifold M_i. In such case, we write $M = M_1 \sharp M_2 \sharp \cdots \sharp M_k$ and refer to M as a connected sum of the manifolds M_i, $i = 1, 2, \cdots, k$.

Following [2] we consider the following model case. Fix a large integer N (which will be the topological dimension of M) and, for any integer $m \in [2, N]$, define the manifold $\mathcal{R}^m$ by

$$\mathcal{R}^m = \mathbb{R}^m \times \mathbb{S}^{N-m}.$$

The manifold $\mathcal{R}^m$ has topological dimension N but its "dimension at infinity" is m in the sense that $V(x,r) \approx r^m$ for $r \ge 1$, see [2, (1.3)]. Thus, for different values of m, the manifold $\mathcal{R}^m$ have different dimension at infinity but the same topological dimension N, This enables us to consider finite connected sums of the $\mathcal{R}^m$'s.

Fix N and k integers $N_1, N_2, \ldots, N_k \in [2, N]$ such that

$$N = \max\{N_1, N_2, \ldots, N_k\}.$$

Next consider the manifold

$$M = \mathcal{R}^{N_1} \sharp \mathcal{R}^{N_2} \sharp \cdots \sharp \mathcal{R}^{N_k}.$$

In [2] Grigoryan and Saloff-Coste establish both the global upper bound and lower bound for the heat kernel acting on this model class. Now we recall the first part of their results with the hypothesis that

$$n := \min_{1 \le i \le k} N_i > 2.$$

Let K be the central part of M and $E_1, \ldots, E_k$ be the ends of M so that E_i is isometric to the complement of a compact set in $\mathcal{R}^{N_i}$. Write $E_i = \mathcal{R}^{N_i} \backslash K$.

Thus, $x \in \mathcal{R}^{N_i} \backslash K$ means that the point $x \in M$ belongs to the end associated with $\mathcal{R}^{N_i}$. For any $x \in M$, define

$$|x| := \sup_{z \in K} d(x, z),$$

where $d = d(x, y)$ is the geodesic distance in M. One can see that $|x|$ is separated from zero on M and $|x| \approx 1 + d(x, K)$.

For $x \in M$, let

$$B(x, r) := \{y \in M : d(x, y) < r\}$$

be the geodesic ball with center $x \in M$ and radius $r > 0$ and let $V(x, r) = \mu(B(x, r))$ where μ is a Riemannian measure on M.

Throughout the paper, we take the simple case $k = 2$ for the model of metric spaces with non-doubling measure, i.e., we set $M = \mathcal{R}^n \sharp \mathcal{R}^m$ with $2 < n < m$. Then, from the construction of the manifold M, we can see that

(a) $V(x, r) \approx r^m$ for all $x \in M$, when $r \leq 1$;
(b) $V(x, r) \approx r^n$ for $B(x, r) \subset \mathcal{R}^n$, when $r > 1$; and
(b) $V(x, r) \approx r^m$ for $x \in \mathcal{R}^n \backslash K$, $r > 2|x|$, or $x \in \mathcal{R}^m$, $r > 1$.

It is not difficult to check that M does not satisfy the doubling condition. Indeed, consider a sequence of balls $B(x_k, r_k) \subset \mathcal{R}^n$ such that $r_k = |x_k| > 1$ and $r_k \to \infty$ as $k \to \infty$. Then $V(x_k, r_k) \approx (r_k)^n$. However, $V(x_k, 2r_k) \approx (r_k)^m$ and the doubling condition fails.

Let Δ be the Laplace-Beltrami operator on M and $e^{-t\Delta}$ the heat semigroup generated by Δ. We denote by $p_t(x, y)$ the heat kernel associated to $e^{-t\Delta}$.

We recall here the following theorem which is the main results obtain in [2].

Theorem A. [2] *Let $M = \mathcal{R}^m \sharp \mathcal{R}^n$ with $2 < n < m$. Then the heat kernel $p_t(x, y)$ satisfies the following estimates.*
1. For $t \leq 1$ and all $x, y \in M$,

$$p_t(x, y) \approx \frac{C}{V(x, \sqrt{t})} \exp\Big(-c\frac{d(x, y)^2}{t}\Big).$$

2. For $x, y \in K$ and all $t > 1$,

$$p_t(x, y) \approx \frac{C}{t^{n/2}} \exp\Big(-c\frac{d(x, y)^2}{t}\Big).$$

3. For $x \in \mathcal{R}^m \backslash K$, $y \in K$ and all $t > 1$,

$$p_t(x, y) \approx C\Big(\frac{1}{t^{n/2}|x|^{m-2}} + \frac{1}{t^{m/2}}\Big) \exp\Big(-c\frac{d(x, y)^2}{t}\Big).$$

4. For $x \in \mathcal{R}^n \backslash K$, $y \in K$ and all $t > 1$,

$$p_t(x, y) \approx C\Big(\frac{1}{t^{n/2}|x|^{n-2}} + \frac{1}{t^{n/2}}\Big) \exp\Big(-c\frac{d(x, y)^2}{t}\Big).$$

5. For $x \in \mathcal{R}^m \backslash K$, $y \in \mathcal{R}^n \backslash K$ and all $t > 1$,

$$p_t(x, y) \approx C\Big(\frac{1}{t^{n/2}|x|^{m-2}} + \frac{1}{t^{m/2}|y|^{n-2}}\Big) \exp\Big(-c\frac{d(x, y)^2}{t}\Big)$$

6. For $x, y \in \mathcal{R}^m \backslash K$ and all $t > 1$,

$$p_t(x,y) \approx \frac{Ct^{-n/2}}{|x|^{m-2}|y|^{m-2}} \exp\Big(-c\frac{|x|^2+|y|^2}{t}\Big) + \frac{C}{t^{m/2}} \exp\Big(-c\frac{d(x,y)^2}{t}\Big)$$

7. For $x, y \in \mathcal{R}^n \backslash K$ and all $t > 1$,

$$p_t(x,y) \approx \frac{Ct^{-n/2}}{|x|^{n-2}|y|^{n-2}} \exp\Big(-c\frac{|x|^2+|y|^2}{t}\Big) + \frac{C}{t^{n/2}} \exp\Big(-c\frac{d(x,y)^2}{t}\Big).$$

3. The boundedness of Hardy-Littlewood maximal function

In this section we consider $M = \mathcal{R}^m \sharp \mathcal{R}^n$ for $m > n > 2$. A main difficulty which we encounter in our study is that the doubling condition fails in this setting. However, local doubling still holds, i.e. the doubling condition holds for a ball $B(x,r)$ under the additional assumption $r \le 1$.

Let us recall next the standard definition of uncentered Hardy–Littlewood Maximal function. For any $p \in [1,\infty]$ and any function $f \in L^p$ let

$$\mathcal{M}f(x) = \sup_{y\in M,\ r>0} \left\{ \frac{1}{V(y,r)} \int_{B(y,r)} |f(z)|dz \colon x \in B(y,r) \right\}.$$

Also we have the centered Hardy–Littlewood Maximal function. For any $p \in [1,\infty]$ and any function $f \in L^p$ we set

$$\mathcal{M}_c f(x) = \sup_{r>0} \frac{1}{V(x,r)} \int_{B(x,r)} |f(z)|dz.$$

It is straightforward to see that $\mathcal{M}_c f(x) \le \mathcal{M}f(x)$ for all x. Moreover in the doubling setting

$$\mathcal{M}(f) \le C\mathcal{M}_c(f), \tag{2}$$

where C is the same constant as in the doubling condition. However, we point out that estimate (2) does not hold in the setting $M = \mathcal{R}^m \sharp \mathcal{R}^n$ with $m > n > 2$. More specifically, one has the following proposition.

Proposition 1. *In the setting $M = \mathcal{R}^m \sharp \mathcal{R}^n$ with $m > n > 2$, the estimate $\mathcal{M}(f) \le C\mathcal{M}_c(f)$ fails for any constant C.*

Proof. Denote the characteristic functions of the sets $\mathcal{R}^m \backslash K$, $\mathcal{R}^n \backslash K$ and K by χ_1, χ_2 and χ_3, respectively. Let $f = \chi_2$. Then for any fixed $x \in \mathcal{R}^m$, we first note that

$$\frac{1}{V(B)} \int_B \chi_2(y)dy \le 1$$

for any $B \ni x$. Furthermore, we can construct balls $B \ni x$ such that the ball B with centre z, radius r, lying mostly in $\mathcal{R}^n$ by choosing $z \in \mathcal{R}^n$, r large enough and $d(z,x) = r - \epsilon$ for ϵ sufficiently small. This implies that

$$\mathcal{M}(f)(x) = \sup_{B\ni x} \frac{1}{V(B)} \int_B \chi_2(y)dy = 1.$$

Now consider the centered Hardy–Littlewood Maximal function $\mathcal{M}_c(f)$. By the definition for any $r > 0$,

$$\frac{1}{V(x,r)} \int_{B(x,r)} f(z)dz = \frac{C}{r^m} \int_{B(x,r)\cap(\mathcal{R}^n\backslash K)} dz.$$

This implies that $r > |x|$ and the term $\frac{C}{r^m}\int_{B(x,r)\cap(\mathcal{R}^n\backslash K)} dz$ is comparable to $\frac{(r-|x|)^n}{r^m}$.

It is easy to check that the maximal value of the above term is comparable to

$$(\frac{n|x|}{m-n})^n/(\frac{m|x|}{m-n})^m,$$

which shows that $\mathcal{M}(f)$ is not pointwise bounded by any multiple of $\mathcal{M}_c(f)$ since the maximal value depends on x and tends to zero when $|x|$ goes to ∞. This proves Proposition 1. □

Theorem 2. *The maximal function $\mathcal{M}(f)$ is of weak type $(1,1)$ and bounded on all L^p spaces for $1 < p \le \infty$.*

Proof. Here and throughout the paper, for the sake of simplicity we use $|\cdot|$ to denote the measure of the sets in M. It is straightforward that the maximal function $\mathcal{M}(f)$ is bounded on L^∞. We will show that the weak type $(1,1)$ estimate

$$|\{x\colon \mathcal{M}f(x) > \alpha\}| \le C\frac{\|f\|_1}{\alpha}$$

holds, then the L^p boundedness of $\mathcal{M}(f)$ follows from the Marcinkiewicz interpolation theorem.

We consider two cases:

Case 1: $\frac{\|f\|_1}{\alpha} < 1$.

Following the standard proof of weak type for Maximal operator we note that for any $x \in \{x\colon \mathcal{M}f(x) > \alpha\}$ there exist a ball such that $x \in B(y,r)$ and

$$\frac{1}{V(y,r)}\int_{B(y,r)} |f(z)|dz > \alpha. \tag{3}$$

This implies

$$\|f\|_1 = \int_M |f(z)|dz \ge \int_{B(y,r)} |f(z)|dz > \alpha V(y,r).$$

Therefore $1 > \frac{\|f\|_1}{\alpha} > V(y,r)$, hence $r \le 1$ and the ball $B(y,r)$ satisfies doubling condition so one can use standard Vitali covering argument to prove weak type $(1,1)$ estimate in this case.

Case 2 : $\frac{\|f\|_1}{\alpha} \ge 1$.

First we split M into three components $\mathcal{R}^m\backslash K$, $\mathcal{R}^n\backslash K$ and K, and denote their characteristic functions by χ_1, χ_2 and χ_3, respectively. Since the maximal function $\mathcal{M}(f)$ is sublinear, it is enough to show that each of the three terms $\mathcal{M}(\chi_1 f)$, $\mathcal{M}(\chi_2 f)$ and $\mathcal{M}(\chi_3 f)$ is of weak type $(1,1)$.

We first consider $\mathcal{M}(\chi_1 f)$. Then

$$\begin{aligned}&|\{x : \mathcal{M}(\chi_1 f)(x) > \alpha\}| \le |\{x \in \mathcal{R}^m\backslash K : \mathcal{M}(\chi_1 f)(x) > \alpha\}|\\ &\quad + |\{x \in \mathcal{R}^n\backslash K : \mathcal{M}(\chi_1 f)(x) > \alpha\}| + |\{x \in K : \mathcal{M}(\chi_1 f)(x) > \alpha\}|\\ &\quad =: I_1 + I_2 + I_3.\end{aligned}$$

The estimate for I_1 follows from the classical weak type $(1,1)$ estimate since $\chi_1 f$ is a function on $\mathcal{R}^m\backslash K$ and the measure on $\mathcal{R}^m\backslash K$ satisfies the doubling condition.

To estimate I_2, we note that for all $x \in \mathcal{R}^n\backslash K$,

$$\sup\left\{\frac{1}{|B(y,r)|} : r > d(x,y) \quad \text{and} \quad B(y,r)\cap(\mathcal{R}^m\backslash K) \neq \emptyset\right\} \leq C\frac{1}{|x|^n}.$$

The above inequality implies that

$$\mathcal{M}\chi_1 f(x) \leq C\frac{\|\chi_1 f\|_1}{|x|^n} \quad \forall x \in \mathcal{R}^n\backslash K. \tag{4}$$

Hence

$$I_2 \leq |\{x \in \mathcal{R}^n\backslash K : C\frac{\|\chi_1 f\|_1}{|x|^n} > \alpha\}| \leq C\frac{\|\chi_1 f\|_1}{\alpha} \leq C\frac{\|f\|_1}{\alpha}.$$

To estimate I_3, we note that the measure of K is finite. Therefore

$$|I_3| \leq |K| \leq C\frac{\|f\|_1}{\alpha}.$$

To prove the weak $(1,1)$ estimate of $\mathcal{M}(\chi_2 f)$ we note that

$$\begin{aligned}|\{x : \mathcal{M}(\chi_2 f)(x) > \alpha\}| &\leq |\{x \in \mathcal{R}^m\backslash K : \mathcal{M}(\chi_2 f)(x) > \alpha\}| \\ &+ |\{x \in \mathcal{R}^n\backslash K : \mathcal{M}(\chi_2 f)(x) > \alpha\}| + |\{x \in K : \mathcal{M}(\chi_2 f)(x) > \alpha\}| \\ &=: II_1 + II_2 + II_3.\end{aligned}$$

II_2 and II_3 can be verified following the same steps as for I_1 and I_3, respectively. To estimates II_1 we observe that

$$\mathcal{M}\chi_2 f(x) \leq C\frac{\|\chi_2 f\|_1}{|x|^m} \quad \forall x \in \mathcal{R}^m\backslash K. \tag{5}$$

Hence $II_2 \leq C\frac{\|f\|_1}{\alpha}$.

Similarly, to deal with $\mathcal{M}(\chi_3 f)$ we note that

$$\begin{aligned}|\{x : \mathcal{M}(\chi_3 f)(x) > \alpha\}| &\leq |\{x \in \mathcal{R}^m\backslash K : \mathcal{M}(\chi_3 f)(x) > \alpha\}| \\ &+ |\{x \in \mathcal{R}^n\backslash K : \mathcal{M}(\chi_3 f)(x) > \alpha\}| + |\{x \in K : \mathcal{M}(\chi_3 f)(x) > \alpha\}| \\ &=: III_1 + III_2 + III_3.\end{aligned}$$

The estimate of III_1 follows immediately since the measure on $(\mathcal{R}^m\backslash K)\cup K$ satisfies the doubling condition. The estimate of III_3 is the same as that of I_3 or II_3. Next to estimates III_2 we further decompose $\{x \in \mathcal{R}^n\backslash K\}$ into two parts $\{x \in \mathcal{R}^n\backslash K : |x| \leq 2\}$ and $\{x \in \mathcal{R}^n\backslash K : x > 2\}$. For the first part we directly have

$$|\{x \in \mathcal{R}^n\backslash K : |x| \leq 2,\ \mathcal{M}(\chi_3 f)(x) > \alpha\}| \leq C \leq C\frac{\|f\|_1}{\alpha}.$$

For the second part, similar to the estimate of I_2, we note that for all $x \in \mathcal{R}^n\backslash K$ and $|x| > 2$,

$$\sup\left\{\frac{1}{|B(y,r)|} : r > d(x,y) \quad \text{and} \quad B(y,r)\cap K \neq \emptyset\right\} \leq C\frac{1}{|x|^n}.$$

Hence,

$$\mathcal{M}\chi_3 f(x) \leq C\frac{\|\chi_3 f\|_1}{|x|^n} \quad \forall x \in \mathcal{R}^n\backslash K \text{ and } |x| > 2,$$

which implies that

$$|\{x \in \mathcal{R}^n \backslash K : |x| > 2,\ \mathcal{M}(\chi_3 f)(x) > \alpha\}| \leq C\frac{\|f\|_1}{\alpha}.$$

Combining the estimates of $\mathcal{M}(\chi_1 f)$, $\mathcal{M}(\chi_2 f)$ and $\mathcal{M}(\chi_3 f)$ we verify (3). The proof of Theorem 2 is now complete.

□

4. The boundedness of the maximal function $\mathcal{M}_\Delta$

In this section we prove that the heat maximal operator satisfies weak type $(1,1)$ and is bounded on L^p for $1 < p \leq \infty$.

We note that when the heat semigroup has a Gaussian upper bound, then the maximal function corresponding to heat semigroup is pointwise dominated by the Hardy-Littlewood maximal operator. In this case, the weak type $(1,1)$ estimate of $\mathcal{M}_\Delta$ follows from the weak type $(1,1)$ estimate of the Hardy-Littlewood maximal function. However, in considered setting this is no longer the case and the operator $\mathcal{M}_\Delta$ can not be controlled by the Hardy-Littlewood maximal function. We can see this via the estimates of the heat semigroup in the proof of Theorem 3 below where we give a direct proof of the weak type estimates of the heat maximal operator.

The following theorem is the main result of this section.

Theorem 3. *Let $\mathcal{M}_\Delta$ be the operator defined by* (1). *Then $\mathcal{M}_\Delta$ is weak type* $(1,1)$ *and for any function $f \in L^p$, $1 < p \leq \infty$, the following estimates hold*

$$\|\mathcal{M}_\Delta f\|_{L^p(M)} \leq C\|f\|_{L^p(M)}.$$

Proof. We first show that $\mathcal{M}_\Delta$ is weak type $(1,1)$, i.e., we need to prove that there exists a positive constant C such that for any $f \in L^1(M)$ and for any $\lambda > 0$,

$$\left|\left\{x \in M : \sup_{t>0} |\exp(-t\Delta)f(x)| > \lambda\right\}\right| \leq \frac{C}{\lambda}\|f\|_{L^1(M)}. \tag{6}$$

Fix $f \in L^1(M)$. Similarly as in Section 3 we set $f_1(x) = f(x)\chi_{\mathcal{R}^m\backslash K}(x)$, $f_2(x) = f(x)\chi_{\mathcal{R}^n\backslash K}(x)$ and $f_3(x) = f(x)\chi_K(x)$, where K is the center of M. To prove (6), it suffices to verify that the following three estimates hold:

$$\left|\left\{x \in \mathcal{R}^m\backslash K : \sup_{t>0} |\exp(-t\Delta)f(x)| > \lambda\right\}\right| \leq \frac{C}{\lambda}\|f\|_{L^1(M)}; \tag{7}$$

$$\left|\left\{x \in \mathcal{R}^n\backslash K : \sup_{t>0} |\exp(-t\Delta)f(x)| > \lambda\right\}\right| \leq \frac{C}{\lambda}\|f\|_{L^1(M)}; \tag{8}$$

$$\left|\left\{x \in K : \sup_{t>0} |\exp(-t\Delta)f(x)| > \lambda\right\}\right| \leq \frac{C}{\lambda}\|f\|_{L^1(M)}. \tag{9}$$

We first consider (7). Since $\mathcal{M}_\Delta$ is a sublinear operator, we have

$$\begin{aligned}
&\big|\big\{x \in \mathcal{R}^m\backslash K : \sup_{t>0} |\exp(-t\Delta)f(x)| > \lambda\big\}\big| \\
&\le \big|\big\{x \in \mathcal{R}^m\backslash K : \sup_{t>0} |\exp(-t\Delta)f_1(x)| > \lambda\big\}\big| \\
&\quad + \big|\big\{x \in \mathcal{R}^m\backslash K : \sup_{t>0} |\exp(-t\Delta)f_2(x)| > \lambda\big\}\big| \\
&\quad + \big|\big\{x \in \mathcal{R}^m\backslash K : \sup_{t>0} |\exp(-t\Delta)f_3(x)| > \lambda\big\}\big| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

To estimate I_1 we consider two cases.

Case 1: $t > 1$. By Theorem A Point 6

$$\begin{aligned}
|\exp(-t\Delta)f_1(x)| \le C \int_{\mathcal{R}^m\backslash K} \Big(&\frac{1}{t^{\frac{n}{2}}|x|^{m-2}|y|^{m-2}} \exp(-\frac{c(|x|^2+|y|^2)}{t}) \\
&+ \frac{1}{t^{\frac{m}{2}}} \exp(-\frac{cd(x,y)^2}{t})\Big)|f(y)|dy \\
=: I_{11} + I_{12}.&
\end{aligned}$$

To estimate I_{11} we note that

$$\begin{aligned}
\frac{t^{-n/2}}{|x|^{m-2}|y|^{m-2}} \exp(-\frac{c(|x|^2+|y|^2)}{t}) &\le C \frac{t^{-n/2}}{|x|^{m-2}|y|^{m-2}} \frac{t^{\frac{n}{2}}}{(t+|x|^2+|y|^2)^{\frac{n}{2}}} \\
&\le \frac{1}{|x|^{m-2+n}} \le \frac{1}{|x|^m}
\end{aligned}$$

since $|y| \ge 1$ and $n > 2$. Hence,

$$I_{11} \le C \int_{\mathcal{R}^m\backslash K} \frac{1}{|x|^{m-2+n}} f(y)dy \le C \frac{\|f\|_{L^1(M)}}{|x|^m}.$$

To estimate I_{12} we note that if $x \in \mathcal{R}^m\backslash K$ then

$$\int_{\mathcal{R}^m\backslash K} \frac{1}{t^{\frac{m}{2}}} \exp(-\frac{cd(x,y)^2}{t})|f(y)|dy \le C\mathcal{M}_{\mathcal{R}^m\backslash K}(f)(x)$$

where $\mathcal{M}_{\mathcal{R}^m\backslash K}(f)(x)$ is the Hardy-Littlewood maximal function acting on $\mathcal{R}^m\backslash K$.

Case 2: $t \le 1$. By Theorem A Point 1

$$|\exp(-t\Delta)f_1(x)| \le \int_{\mathcal{R}^m\backslash K} \frac{1}{t^{\frac{m}{2}}} \exp(-\frac{cd(x,y)^2}{t})|f(y)|dy.$$

Again the right-hand side of above estimate is bounded by $\mathcal{M}_{\mathcal{R}^m\backslash K}(f)(x)$. These estimates prove weak type $(1,1)$ for I_1 since $\mathcal{R}^m\backslash K$ satisfies doubling condition.

Next we show weak type estimates for I_2. We also consider two cases.

Case 1: $t > 1$. By Theorem A Point 5

$$|\exp(-t\Delta)f_2(x)| \le C\int_{\mathcal{R}^n\backslash K}\Big(\frac{1}{t^{\frac{n}{2}}|x|^{m-2}} + \frac{1}{t^{\frac{m}{2}}|y|^{n-2}}\Big) \exp(-\frac{cd(x,y)^2}{t})|f(y)|dy$$
$$=: I_{21} + I_{22}.$$

Similarly as in the estimate for I_{11} we get

$$\begin{aligned} I_{21} &\le C\int_{\mathcal{R}^n\backslash K}\frac{1}{t^{\frac{n}{2}}|x|^{m-2}}\frac{t^{\frac{n}{2}}}{(t+d(x,y)^2)^{\frac{n}{2}}}|f(y)|dy \\ &\le C\int_{\mathcal{R}^n\backslash K}\frac{1}{|x|^{m-2+n}}|f(y)|dy \le C\frac{\|f\|_1}{|x|^m}, \end{aligned}$$

since $n > 2$, $|x| \ge 1$ and in this case, $d(x,y) \ge |x|$.

To estimate I_{22} we note that

$$\begin{aligned} I_{22} &\le C\int_{\mathcal{R}^n\backslash K}\frac{1}{t^{\frac{m}{2}}|y|^{n-2}}\frac{t^m}{(t+d(x,y)^2)^m}|f(y)|dy \\ &\le C\int_{\mathcal{R}^n\backslash K}\frac{t^{\frac{m}{2}}}{(t+d(x,y)^2)^m}|f(y)|dy \\ &\le C\int_{\mathcal{R}^n\backslash K}\frac{\sqrt{t}^m}{(\sqrt{t}+d(x,y))^{2m}}|f(y)|dy \end{aligned}$$

since $|y| \ge 1$. By decomposing the Poisson kernel $\frac{\sqrt{t}^m}{(\sqrt{t}+d(x,y))^{2m}}$ into annuli, it is easy to see that the last term of the above inequality is bounded by $C\mathcal{M}_{\mathcal{R}^n\backslash K}(f)(x)$.

Case 2: $t \le 1$. Again by Theorem A Point 1

$$|\exp(-t\Delta)f_2(x)| \le C\int_{\mathcal{R}^n\backslash K}\frac{1}{t^{\frac{m}{2}}}\exp(-\frac{cd(x,y)^2}{t})|f(y)|dy.$$

Hence it is bounded by $C\mathcal{M}(f)(x)$.

Similar to I_1, we have

$$I_2 \le C\frac{\|f\|_1}{\lambda}.$$

Now we consider I_3.

Case 1: $t > 1$. By Theorem A Point 3

$$\begin{aligned} |\exp(-t\Delta)f_3(x)| &\le C\int_K\Big(\frac{1}{t^{\frac{n}{2}}|x|^{m-2}} + \frac{1}{t^{\frac{m}{2}}}\Big)\exp(-\frac{cd(x,y)^2}{t})|f(y)|dy \\ &=: I_{31} + I_{32}. \end{aligned}$$

To estimate I_{31} we note that

$$\begin{aligned} I_{31} &\le C\int_K\frac{1}{t^{\frac{n}{2}}|x|^{m-2}}\frac{t^{\frac{n}{2}}}{(t+d(x,y)^2)^{\frac{n}{2}}}|f(y)|dy \le C\frac{\|f\|_1}{|x|^{m+n-2}} \\ &\le C\frac{\|f\|_1}{|x|^m}, \end{aligned}$$

where we use the facts that $n > 2$, $|x| > 1$ and that in this case, $d(x, y) \approx |x|$. Similarly,

$$I_{32} \leq C \int_K \frac{1}{t^{\frac{m}{2}}} \frac{t^{\frac{m}{2}}}{(t + d(x,y)^2)^{\frac{m}{2}}} |f(y)| dy \leq C \frac{\|f\|_1}{|x|^m}.$$

Case 2: $t \leq 1$. By Theorem A Point 1

$$|\exp(-t\Delta) f_3(x)| \leq \int_K \frac{1}{t^{\frac{m}{2}}} \exp(-\frac{cd(x,y)^2}{t}) |f(y)| dy.$$

Hence it is bounded by $C\mathcal{M}(f)(x)$.

Combining the estimates of the two cases, we obtain

$$I_3 \leq C \frac{\|f\|_{L^1(M)}}{\lambda}.$$

The estimates of I_1, I_2 and I_3 together imply (7).

We now turn to the estimate of (8). Similarly to the proof of (7), we have

$$\begin{aligned}
&\big|\big\{x \in \mathcal{R}^n \backslash K : \sup_{t>0} |\exp(-t\Delta) f(x)| > \lambda\big\}\big| \\
&\leq \big|\big\{x \in \mathcal{R}^n \backslash K : \sup_{t>0} |\exp(-t\Delta) f_1(x)| > \lambda\big\}\big| \\
&\quad + \big|\big\{x \in \mathcal{R}^n \backslash K : \sup_{t>0} |\exp(-t\Delta) f_2(x)| > \lambda\big\}\big| \\
&\quad + \big|\big\{x \in \mathcal{R}^n \backslash K : \sup_{t>0} |\exp(-t\Delta) f_3(x)| > \lambda\big\}\big| \\
&=: II_1 + II_2 + II_3.
\end{aligned}$$

We note that the estimate of II_1 is similar to that of I_2, while the estimate of II_2 is similar to that of I_1. Moreover, the estimate of II_3 is similar to that of I_3. Therefore we can verify that (8) holds.

Finally, we turn to the estimate of (9). We have

$$\begin{aligned}
&\big|\big\{x \in K : \sup_{t>0} |\exp(-t\Delta) f(x)| > \lambda\big\}\big| \\
&\leq \big|\big\{x \in K : \sup_{t>0} |\exp(-t\Delta) f_1(x)| > \lambda\big\}\big| \\
&\quad + \big|\big\{x \in K : \sup_{t>0} |\exp(-t\Delta) f_2(x)| > \lambda\big\}\big| \\
&\quad + \big|\big\{x \in K : \sup_{t>0} |\exp(-t\Delta) f_3(x)| > \lambda\big\}\big| \\
&=: III_1 + III_2 + III_3.
\end{aligned}$$

Also, we point out that the estimate of III_1 is similar to that of I_3 and that the estimate of III_2 is similar to that of II_3.

Concerning the term III_3, we first note that in this case $x \in K$. We have

$$|\exp(-t\Delta) f_3(x)| \leq C \int_K \frac{1}{t^{\frac{m}{2}}} \exp(-\frac{cd(x,y)^2}{t}) |f(y)| dy.$$

It is easy to see that the right-hand side of the above inequality is bounded by $C\mathcal{M}(f)(x)$. Thus, we have

$$III_3 \leq C \frac{\|f\|_1}{\lambda}.$$

Hence, we can see that (9) holds. Now (7), (8) and (9) together imply that (6) holds, i.e., $\mathcal{M}_\Delta$ is of weak type (1,1).

Next, note that the semigroup $\exp(-t\Delta)$ is submarkovian so $\mathcal{M}_\Delta$ is bounded on $L^\infty(M)$. This together with (6), implies that $\mathcal{M}_\Delta$ is bounded on $L^p(M)$ for all $1 < p < \infty$.

The proof of Theorem 3 is complete.

□

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