THE LILEY FORM FOR THE PRESSURE TENSOR IN MAGNETISED PLASMA

Roger J. Hosking

School of Mathematical Sciences

University of Adelaide

Email: marjhg@gmail.com

Contents

1. Quirks of History

"*Transport Processes in a Plasma*" (Braginskii, 1965) is one of the most cited articles in plasma physics. Clarity throughout its English translation provided the Western research community with an early insight into the basis of macroscopic plasma models beyond the ideal or even resistive formulations previously widely used.

The microscopic foundation is a Boltzmann-type kinetic equation that alternatively may be written

$$\frac{\partial f_s}{\partial t} + \mathbf{c}\cdot\nabla f_s + \mathbf{a}_s(\mathbf{r}, \mathbf{c}, t)\cdot\nabla_{\mathbf{c}} f_s = \frac{\delta f_s}{\delta t} , \qquad (1)$$

defining the time evolution of a velocity distribution function $f_s(\mathbf{r}, \mathbf{c}, t)$ in phase space $(\mathbf{r}, \mathbf{c})$ for each particle species s, where $\mathbf{a}_s$ is the particle acceleration.

For a particle mass m_s and charge e_s the acceleration is

$$\mathbf{a}_s(\mathbf{r}, \mathbf{c}, t) = \mathbf{g} + \frac{e_s}{m_s}[\mathbf{E}(\mathbf{r}, t) + \mathbf{c} \times \mathbf{B}(\mathbf{r}, t)] \ ,$$

if both gravitational and electromagnetic components are included (where $\mathbf{g}$, $\mathbf{E}$ and $\mathbf{B}$ denote gravitational, electric and magnetic fields) – cf. Chapman & Cowling (1970).

The symbol $\boldsymbol{\nabla}_{\mathbf{c}}$ denotes the gradient operator relative to the independent velocity vector $\mathbf{c}$, analogous to $\boldsymbol{\nabla}$ relative to the position vector $\mathbf{r}$ in phase space, and the term on the right-hand side $\delta f_s/\delta t$ represents the time rate of change of the velocity distribution function due to the microscopic particle collisions.

Any macroscopic field equation of interest corresponds to taking some moment of equation (1). If $\mathbf{w} = \mathbf{c} - \mathbf{c_0}$ denotes the peculiar velocity for the species relative to some reference velocity $\mathbf{c_0}$, the moment corresponding to any related function $\boldsymbol{\Psi}(\mathbf{w})$ is defined by

$$n_s \langle \boldsymbol{\Psi} \rangle = \int f_s(\mathbf{r}, \mathbf{c}, t)\, \boldsymbol{\Psi}(\mathbf{w})\, d\mathbf{c}\ , \tag{2}$$

including the particle number density (number of particles in a unit volume)

$$n_s(\mathbf{r}, t) = \int f_s(\mathbf{r}, \mathbf{c}, t)\, d\mathbf{c}\ .$$

One may identify $\mathbf{c_0}$ with mass-weighted mean velocity $\mathbf{v}$ defined via the total density and total mass flux

$$\rho = \Sigma_s \rho_s \ , \quad \rho \mathbf{v} = \Sigma_s \rho_s \mathbf{v}_s \ ,$$

where the summation is over all species with density ρ_s and mean flow velocity $\mathbf{v}_s = \langle \mathbf{c} \rangle$. Then we may invoke

$$\begin{aligned}
&\frac{\partial (n_s \langle \boldsymbol{\Psi} \rangle)}{\partial t} + \boldsymbol{\nabla} \cdot (n_s \langle (\mathbf{v} + \mathbf{w})\, \boldsymbol{\Psi} \rangle) \\
&- n_s \langle (\mathbf{f}_s + \frac{e_s}{m_s} \mathbf{w} \times \mathbf{B} - \mathbf{w} \cdot \boldsymbol{\nabla} \mathbf{v}) \cdot \boldsymbol{\nabla}_{\mathbf{w}} \boldsymbol{\Psi} \rangle \;=\; \mathcal{C}_s(\boldsymbol{\Psi}) \ , \\
&\text{with } \mathbf{f}_s = \mathbf{g} + \frac{e_s}{m_s} [\mathbf{E} + \mathbf{v} \times \mathbf{B}] - \frac{d\mathbf{v}}{dt} \, , \ \mathcal{C}_s(\boldsymbol{\Psi}) = \int \frac{\delta f_s}{\delta t} \boldsymbol{\Psi} \, d\mathbf{c} \ ,
\end{aligned} \tag{3}$$

which is a generalised equation of change from (1).

Familiar fundamental field quantities are related to the lower moments defined by (2) for each species s of particle mass m_s — viz.

$$\begin{aligned}
\rho_s &\equiv n_s m_s && \text{(density)} \\
\mathbf{u}_s &\equiv \langle \mathbf{w} \rangle && \text{(mean relative velocity)} \\
p_s &\equiv n_s \langle \frac{1}{3} m_s w^2 \rangle && \text{(pressure)} \\
\mathbf{p}_s &\equiv n_s \langle m_s \mathbf{w}\mathbf{w} \rangle && \text{(pressure tensor)} \\
\mathbf{q}_s &\equiv n_s \langle \frac{1}{2} m_s w^2 \mathbf{w} \rangle && \text{(thermal flux)} \, .
\end{aligned}$$

Other higher moments are much less familiar!

Thus if $\mathbf{\Psi}$ is identified with the quantities $m_s, m_s\mathbf{w}, \frac{1}{2}m_s w^2, m_s\{\mathbf{ww}\}, \frac{1}{2}m_s w^2\mathbf{w}$ etc. successively, then for each species the general equation of change (3) produces the basic equations of continuity (mass), momentum, thermal energy, the traceless part of the pressure tensor $\mathbf{t}_s$, heat conduction vector $\mathbf{q}_s$, etc. etc.

Now let us proceed to the far less cited results due to Liley, Hosking & Marinoff and Callen *et al.*, as has now been presented in considerable detail in Chapter 2 of Hosking & Dewar ("*Fundamental Fluid Mechanics and Magnetohydrodynamics*", Springer, 2016).

2. Plasma Pressure Tensor

Various representations for the collisional terms on the right-hand sides of the moment equations have been considered, but on tensor rank alone two simple linear expressions for each species s are

$$\mathcal{C}_s(m_s\{\mathbf{ww}\}) = -\sum_j \frac{\vartheta_{sj}}{\tau_{sj}}\mathbf{t}_j \ ,$$

$$\mathcal{C}_s\left(\frac{1}{2}m_s(w^2 - \frac{5kT_s}{m_s})\mathbf{w}\right) = -\sum_j \left(\frac{1}{\tau_{sj}}\mathbf{R}_j + \zeta_{sj}\rho_j\mathbf{u}_j\right) \ ,$$

with the coefficients $\{\tau_{sj}, \vartheta_{sj}, \zeta_{sj}\}$ functions of number density and temperature.

An asymptotic closure like Chapman-Enskog for neutral gas transport (Chapman & Cowling, 1970) or one of various parameter ordering procedures for plasmas may be considered, but there is another "careful and elegant" approach (Hazeltine & Meiss, 2003) – notably the Grad "thirteen moment" approximation, which corresponds to a truncation of the hierarchy of moment equations up to the equation for the heat flux $\mathbf{q}_s$ where terms involving two higher level quantities are omitted.

[Similar higher "n moment" ($n > 13$) closures have also been considered, including even more of the hierarchy of moment equations and then neglecting terms involving even higher moments beyond the corresponding set.]

It is convenient to introduce a characteristic frequency ω to represent the time derivative term in the resulting moment equation for the traceless component of the plasma pressure tensor $\mathbf{t}$, which may then be written in the notationally convenient form

$$\mathbf{t} - 2\{\mathbf{t}\times\mathbf{a}\} = -2\mu\mathbf{s} \tag{4}$$

where the symmetric and traceless generalised rate of deformation (or strain) tensor $\mathbf{s}$ is given by

$$\begin{aligned} 2p\,\mathbf{s} = {} & -2\rho\{\mathbf{f}\,\mathbf{u}\} + 2p\{\nabla\mathbf{v}\} + \mathbf{t}\nabla\cdot\mathbf{v} \\ & + 2\{\mathbf{t}\cdot\nabla\mathbf{v}\} + \nabla\cdot\mathbf{h} + \sum_{j\neq s}\frac{\vartheta_{sj}}{\tau_{sj}}\mathbf{t}_j\ , \end{aligned} \tag{5}$$

with coefficient $\mu = \dfrac{p\,\tau}{\omega\tau + \vartheta}$ and vector $\mathbf{a} = \dfrac{e\mathbf{B}}{m}\dfrac{\tau}{\omega\tau + \vartheta}$.

Note that **a** proportional to the gyrofrequency firstly reflects the time derivative and secondly the collisional terms, with one or the other predominant when the parameter $\omega\tau$ is respectively sufficiently large or small. Note also we refer here to collisions between particles of the particular species under consideration (with its appropriate factor ϑ typically of order one), and any collisional coupling between the different species is represented by the summation for $j \neq s$ incorporated in **s**. The form (4) isolates the term $\{\mathbf{t}\times\mathbf{a}\}$ proportional to the magnetic field **B**, which enables us to obtain **t** as an explicit function of **s** below — even though there are two terms involving **t** in the generalised deformation tensor (5), a point examined further in the case of a simple ion-electron plasma in Hosking & Dewar (2016).

Using tensor identities, we may re-express (4) as

$$\mathbf{t} = -\,\frac{2\mu}{(1+|\mathbf{a}|^2)(1+4|\mathbf{a}|^2)}\,\big[\,(\mathbf{s}+2\{\mathbf{s}\times\mathbf{a}\})(1+|\mathbf{a}|^2) + 6\,(\{\mathbf{s}\cdot\mathbf{aa}\}+2\{\{\mathbf{s}\cdot\mathbf{aa}\}\times\mathbf{a}\})+6\,\mathbf{s}{:}\mathbf{aa}\{\mathbf{aa}\}\,\big]\,. \quad (6)$$

The Cartesian representation of this **Liley form** (6) appears in Chapman & Cowling (1970), who assumed a uniform magnetic field **B** — but we now recognise that (6) is an **invariant result (valid in any system of coordinates) without any restriction on the magnetic field (even null points)**. Eq. (4) and the result (6) obviously always reduce to $\mathbf{t} = -2\mu\mathbf{s}$ for a neutral species and for charged species if there is no magnetic field (i.e. for $\mathbf{a} = 0$), when we recover the familiar shear viscosity form on identifying $\mathbf{s} = \{\nabla\mathbf{v}\}$.

However, the terms involving the vector $\mathbf{a}$ usually dominate for charged particles in the presence of a magnetic field, producing characteristically anisotropic plasma viscosity contributions. Thus on noting

$$-\frac{2\mu}{(1+|\mathbf{a}|^2)(1+4|\mathbf{a}|^2)} = \frac{\mu}{2}\left[-|\mathbf{a}|^{-4} + \frac{5}{4}|\mathbf{a}|^{-6} + \cdots\right] ,$$

in magnetised plasma wherever $|\mathbf{a}| \gg 1$ the general explicit form (6) may be expanded to obtain

$$\mathbf{t} = \mathbf{t}_{\parallel} + \mathbf{t}_{\mathrm{g}} + \mathbf{t}_{\perp} + \cdots \ , \qquad (7)$$

successively specifying the leading *parallel*, *cross* or *transverse* (*gyroviscous* or "FLR") and *perpendicular* viscosity components.

Consequently, we obtain the quite succinct forms

$$\mathbf{t}_{\|} = -3\mu\,\mathbf{s}{:}\hat{\mathbf{b}}\hat{\mathbf{b}}\{\hat{\mathbf{b}}\hat{\mathbf{b}}\}\ ,$$

$$\mathbf{t}_{\mathrm{g}} = -\frac{\mu}{|\mathbf{a}|}\left\{\mathbf{s}\times\hat{\mathbf{b}} + 6\{\mathbf{s}\cdot\hat{\mathbf{b}}\hat{\mathbf{b}}\}\times\hat{\mathbf{b}}\right\}\ ,$$

$$\mathbf{t}_{\perp} = -\frac{\mu}{2|\mathbf{a}|^2}\left[\mathbf{s} + 6\{\mathbf{s}\cdot\hat{\mathbf{b}}\hat{\mathbf{b}}\} - \frac{15}{2}\mathbf{s}{:}\hat{\mathbf{b}}\hat{\mathbf{b}}\{\hat{\mathbf{b}}\hat{\mathbf{b}}\}\right]\ .$$

Note: the operator $\{\ \}$ is defined by

$$\{\mathbf{F}\} \equiv \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \frac{1}{3}\mathbf{F}{:}\mathbf{I}\,\mathbf{I}$$

for any dyadic $\mathbf{F}$, where $\mathbf{F}^T$ denotes its transpose. As $\mathrm{Tr}\,\mathbf{F} \equiv \mathbf{F}{:}\mathbf{I}$ and $\mathrm{Tr}\,\mathbf{I} = 3$, the result is a traceless symmetric dyadic.

The result (6) was originally obtained by Bruce Liley when I first worked with him at ANU in 1968, and the above successive components in a sufficiently large magnetic field ($|\mathbf{a}| \gg 1$) were afterwards identified at Culham (Hosking & Marinoff, 1973). Alternative forms later obtained by Callen *et al.* (1987) immediately follow by expanding the above forms, and exhibit all of the terms in the traceless component of the tensor in Braginskii (1965) – i.e. where (apart from his Delphic coefficients) the W_0 corresponds to $\mathbf{t}_{\|}$, the sum of W_1 and W_2 to $\mathbf{t}_{\perp}$, and the sum of W_3 and W_4 to $\mathbf{t}_g$. Incidentally, in the absence of collisions, an anisotropic plasma pressure $\mathbf{p} = p_{\|}\hat{\mathbf{b}}\hat{\mathbf{b}} + p_{\perp}\mathbf{I}_{\perp}$ was earlier discussed by Chew, Goldberger & Low (1956).

As previously indicated, the Grad "thirteen moment" approximation renders another moment equation that readily produces an explicit result for the heat flux vector $\mathbf{q}$ (Herdan & Liley, 1960). Extensions of the hierarchy to higher moments include contributions by Ramos (2005, 2007), who followed Braginskii and some others in defining the moments with reference to the species flow velocity $\mathbf{v}_s$. However, in a simple plasma consisting of electrons and only one ion species, the mean velocity we used is effectively the ion velocity that corresponds to the predominant plasma viscosity contribution invoked (i.e. $\mathbf{v} \simeq \mathbf{v}_i$ since $m_e \ll m_i$).

3. Applications

A simple variational form demonstrates that the growth rates of ideal and resistive instabilities in magnetised plasma may be reduced by viscosity (Hosking & Dewar, 2016). An early calculation showed "resistive-g modes" with azimuthal mode numbers $m \geq 1$ may be stabilised in the reverse field pinch by parallel viscosity (Hosking & Robinson, 1979), and further nonlinear calculations addressed the $m = 0$ mode (Hender & Robinson, 1981).

Parallel viscosity has been referred to as "Braginskii viscosity" by various authors working in the field of solar physics, following Hollweg (1985, 1986) who explored the Braginskii form W_0 with the collisional coefficient $\eta_0 = 0.96 n_i T_i \tau_i$ for application to the solar corona.

Craig and Litvinenko at the University of Waikato have *inter alia* shown that energy dissipation due to parallel viscosity during magnetic merging can substantially exceed that due to resistivity, formerly considered to explain the observed energy release in solar flares but was “found severely wanting”. For example, this was shown in transient reconnecting plasmas (Craig, 2010), with a formal exact solution for steady state merging (Craig & Litvinenko, 2009), and dynamic coalescence (Craig & Litvinenko, 2010).

However, adoption of the Liley form to deal with the inherent magnetic nulls could prove to be a significant new feature in such calculations.

4. Epilogue

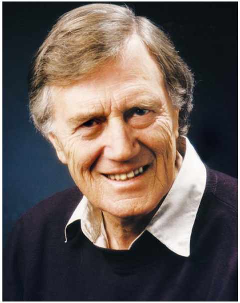
Bruce Liley's PhD thesis was the source of the article Herdan & Liley (1960), which represents the beginning of the theoretical work that eventually led him to the Liley form (6). When I joined him in 1968 under special leave for a few months from the Flinders University of South Australia, Bruce simply gave me his result on a scrap of paper and mentioned several tensor identities used in its derivation – and asked me to check it out independently. The first two Exercises in Hosking & Dewar (2016) illustrate my subsequent derivation; and when he later transferred to the University of Waikato, I had become so interested I chose to follow him there!

Our families had quickly become good friends – but when Bruce and his wife Margaret stayed at our house (he had came to Flinders to give a seminar), he said he was surprised at that possibility if an opportunity came up, as my wife and I were living in our home town in Adelaide. However, a second chair in Mathematics at the University of Waikato was advertised, to which I was eventually appointed.

[When I told Peter Karmel (then the Flinders University Vice-Chancellor) I thought I was too young to apply, he said that he was much the same age when appointed to a chair – and not to go if one was not offered to me.]

On arrival at Waikato in mid-1971, I worked with Bruce as he wrote up a Physics Report alongside Hosking & Marinoff (1973), research begun a little earlier when I proceeded on leave from Flinders University to Culham Laboratory. That Report was referenced there as also intended for publication in the same journal (*Plasma Physics*), but Bruce did not manage that as he had assumed various time-consuming responsibilities in the development of the new School of Science and as a Professorial representative on the University of Waikato Council. It may be that rendition of the Liley form in Chapter 2 of Hosking & Dewar (2016) will remedy this historical omission at last, especially to provide a basis for discussion of magnetic nulls in magnetised plasma.

photograph.pdf



Bruce Sween Liley

Published in: Robert L. Dewar; Ian Hutchinson; R. S. “BAS” Pease; Alan Ware; *Physics Today* **55,** 80-81 (2002)
DOI: 10.1063/1.1485600

Figure: Bruce Sween Liley

- Braginskii. S.I. 1965 In M.A. Leontovich (Ed.) *Reviews of Plasma Physics*, vol. 1, pp. 205-311, Consultants Bureau, New York.
- Chapman, S. & Cowling, T.G. 1970 *The Mathematical Theory of Non-Uniform Gases*, Third Edition, Cambridge University Press.
- Hosking , R.J. & Dewar, R.L. 2016 *Fundamental Fluid Mechanics and Magnetohydrodynamics*, Springer.
- Hazeltine, R.D. & Meiss, J.D. 2003 *Plasma Confinement*, Dover.
- Hosking, R.J., & Marinoff, G.M. 1973 *Plasma Physics* **15**, 327-341.

- Callen, J.D., Qu, W.X., Siebert, K.D., Carreras, B.A., Shaing, K.C. & Spong, D.A. 1987 *Plasma Physics and Controlled Thermonuclear Research*, Volume 2, IAEA, Vienna.
- Chew, G.F., Goldberger, M.I. & Low, F.F. 1956 *Proc. Roy. Soc. London* A **236**, 112-118.
- Herdan, R. & Liley, B.S., 1960 *Rev. Mod. Phys.* **32**, 731-741.
- Ramos, J.J.
  2005 *Physics of Plasmas* **12**, 112301;
  2007 *Physics of Plasmas* **14**, 052506.

- Hosking, R.J. & Robinson, D.C. 1979
  Controlled Fusion and Plasma Physics (Proc. 9th Europ. Vonf. Oxford, 1979) BP6 61
- Hender,T.C. & Robinson, D.C. 1981
  Nuclear Fusion **21**, 755-758.
- Hollweg, J.V.
  1985 *J. Geophys. Res. (Space Phys.)* **90**, 7620-7622;
  1986 *Astrophys. J.* **306**, 730-739.
- Craig, I.J.D. 2010
  Astron. Astrophys. **515**, A96.
- Craig, I.J.D. & Litvinenko, Y. E.
  2009 *Astron. Astrophys.* **501**, 755-760;
  2010 *Astrophys. J.* **725**, 886-893.

Dewar, R.L., Hutchinson, I., Pease, R.S. & Ware. A. 2002 *Physics Today* **55**, doi: 10.1063/1.1485600, American Institute of Physics.