Sparse Grids

Mini-course on the application of computational mathematics to plasma physics

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25. June 2019

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- Highdimensional problems are hard
- regular grids
- sparse grids
- a combination born of necessity

Outline

- Highdimensional problems are hard
- regular grids
- sparse grids
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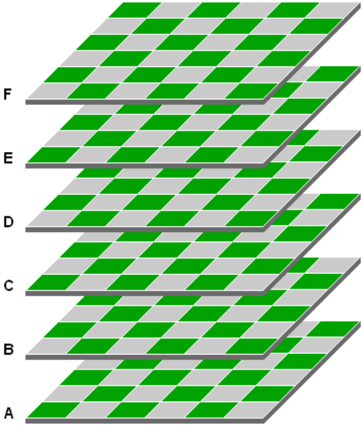
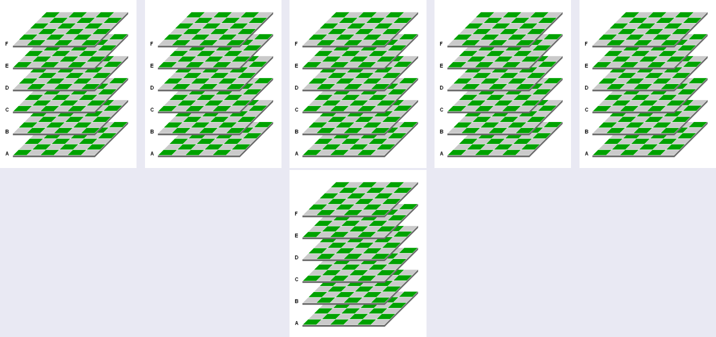
cubic chess or the curse of dimension for chess players



boards for V.R. Parton's cubic chess with $6^3 = 216$ cells (from Wikipedia)

a 4D chess board

a 4D chess board with $6^4 = 1296$ cells



- storage of multidimensional data possible but costs grow like $O(m^d)$
- moving data between the cells gets complicated in chess and in computing – movement of pawn corresponds to stencil – with 3^d cells
- data movement $O(3^d m^d)$ – and moving data is the weak point of new HPC

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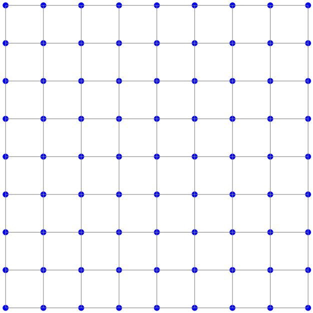
regular isotropic grid



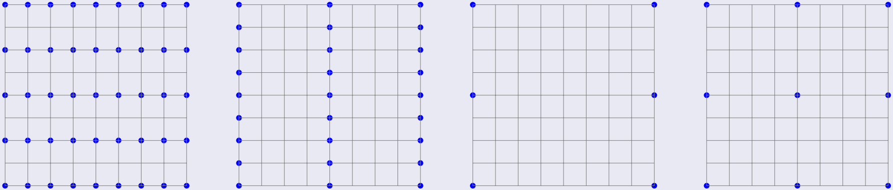
- approximate unknown function $u(x, y)$
- compute only values $u(x_i, y_j)$ on discrete grid points
- interpolate values $u(x, y)$ for other points (x, y)
- regular isotropic grid: $x_i = ih$ and $y_j = jh$

the challenge: curse of dimension

In two dimensions $1/h^2$ grid points, in d dimensions $1/h^d$ grid points but accuracy proportional to h^2

regular anisotropic grids

more general regular grids

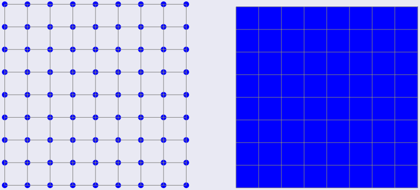


- choose fine grid when $u(x, y)$ has large gradients
- choose coarse grid when $u(x, y)$ is smooth
- gradients may be different in different directions
- choose anisotropic grid when $u(x, y)$ varies differently in different directions

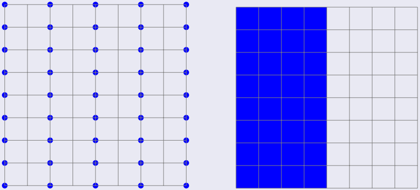
with anisotropic grids one can approximate multi-dimensional $u(x_1, \ldots, x_d)$ if u very smooth in most x_k

sampling and scale space

full grid captures all scales



subgrid captures less scales



- evaluation of $u(x, y)$ on the grid corresponds to sampling u on the grid points
- sampling on a fine grid captures high frequencies – small scale fluctuations (Nyqvist/Shannon)
- with anisotropic grids one can capture small scales in one dimension and different scales in another

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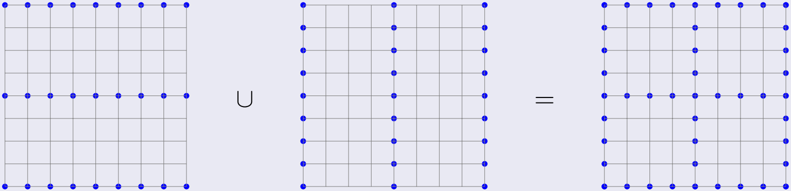
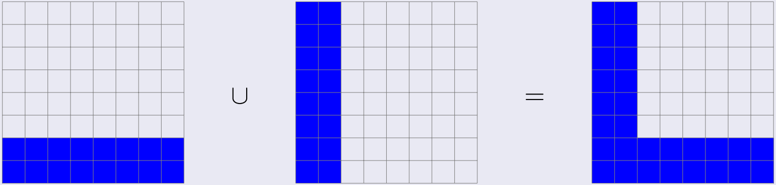
geometric definition of sparse grid

a simple sparse grid



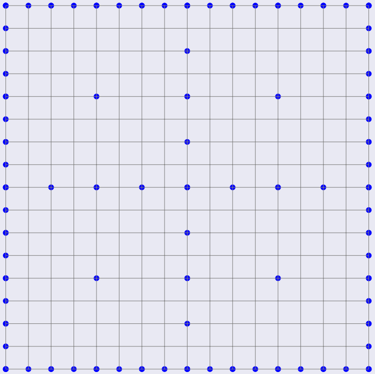
sparse grid in frequency / scale space



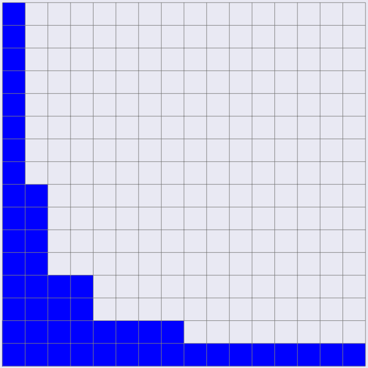
captures fine scales in both dimensions but not joint fine scales

hyperbolic cross

sparse grid points



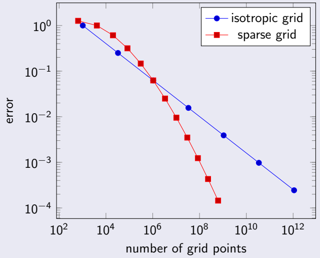
sparse grid scale diagram



the scale diagram displays (a quarater of) a hyperbolic cross

asymptotic error rates

five dimensional case



- only asymptotic error rates given here
- constants and preasymptotics also depend on dimension
- practical experience: with sparse grids up to 10 dimensions
- Zenger 1991

asymptotic rates	number of points	L_2 error
regular isotropic grids	h^{-d}	h^2
sparse grids	$h^{-1} \, \lvert \log_2 h \rvert^{d-1}$	$h^2 \, \lvert \log_2 h \rvert^{d-1}$

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combining solutions from multiple grids

regular grid approximation

- regular grid $G_{\mathbf{h}}$
- function space $V_{\mathbf{h}}$
- Galerkin equations for $u_{\mathbf{h}}$

$$a(u_{\mathbf{h}}, v_{\mathbf{h}}) = \langle f, v_{\mathbf{h}} \rangle$$

for all $v_{\mathbf{h}} \in V_{\mathbf{h}}$

sparse grid approximation

- sparse grid $G_{SG} = \bigcup_{\mathbf{h}} G_{\mathbf{h}}$
- function space $V_{SG} = \sum_{\mathbf{h}} V_{\mathbf{h}}$
- Galerkin equations for u_{SG}

$$a(u_{SG}, v_{SG}) = \langle f, v_{SG} \rangle$$

for all $v_{SG} \in V_{SG}$

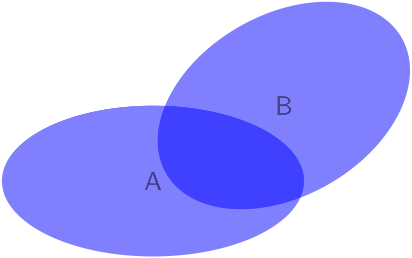
combination technique – where HPC comes in

compute all $u_{\mathbf{h}}$ in parallel and combine solutions using parallel reduction:

$$u_C = \sum_{\mathbf{h}} c_{\mathbf{h}} u_{\mathbf{h}}$$

Big question: when is $u_C \approx u_{SG}$?

Inclusion / exclusion principle in combinatorics



for the cardinality of sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

more general for additive α:

$$\alpha(A \cup B) = \alpha(A) + \alpha(B) - \alpha(A \cap B)$$

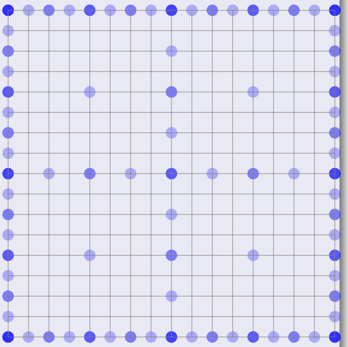
Theorem (de Moivre)

If $A_1, \ldots, A_m$ form intersection structure then

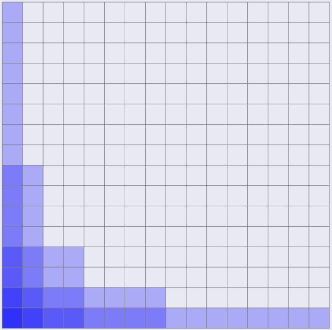
$$\alpha\left(\bigcup_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} c_i\, \alpha(A_i), \quad \text{for some } c_i \in \mathbb{Z}$$

overlap of grids and combination

sparse grid points



sparse grid scale diagram



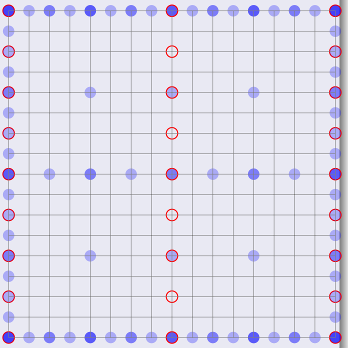
combination formula

$$u_C = u_{1,16} + u_{2,8} + u_{4,4} + u_{8,2} + u_{16,1} - u_{1,8} - u_{2,4} - u_{4,2} - u_{8,1}$$

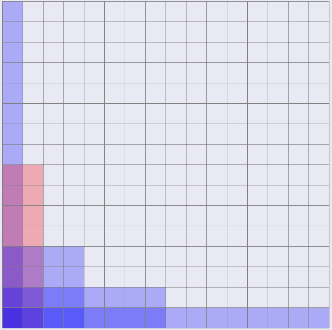
Griebel, Schneider, Zenger 1992

overlap = redundancy ⇒ (lossy) fault tolerance

sparse grid points



sparse grid scale diagram



revised combination formula

$$u_C = u_{1,16} + u_{4,4} + u_{8,2} + u_{16,1} - u_{4,2} - u_{8,1} - u_{1,4}$$

H. CTAC 2003, Harding 2012