RIESZ TRANSFORMS OF SOME PARABOLIC OPERATORS

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Abstract. We study boundedness on $L^p([0,T] \times \mathbb{R}^N)$ of Riesz transforms $\nabla(\mathcal{A})^{-1/2}$ for class of parabolic operators such as $\mathcal{A} = \frac{\partial}{\partial t} - \Delta + V(t,x)$. Here $V(t,x)$ is a non-negative potential depending on time t and space variable x. As a consequence, we obtain $W_x^{1,p}$-solutions for the non-homogeneous problem

$$\partial_t u - \Delta u + V(t,.)u = f(t,.),\ u(0) = 0$$

for initial data $f \in L^p([0,T] \times \mathbb{R}^N)$.

1. Introduction

Harmonic analysis of Schrödinger operators $A = -\Delta + V(x)$ has attracted attention in recent years. For example, the theory of Hardy and BMO spaces associated to such operators (see [9], [14] and the references there), L^p-boundedness of the associated Riesz transforms $\nabla A^{-1/2}$ (see e.g. [17] or [20]), spectral multipliers ([8]) have been developed. Related operators to Riesz transforms such as $D^2(-\Delta+V)^{-1}$, $V(-\Delta+V)^{-1}$, $V^{\frac{1}{2}}(-\Delta+V)^{-\frac{1}{2}}$, $\nabla(-\Delta+V)^{-\frac{1}{2}}$ have been studied on L^p-spaces under suitable assumptions on the potential V (see [19] or [4]). Less investigated is the L^p-boundedness of the analogous operators associated with parabolic Schrödinger operators $\mathcal{A} = \partial_t - \Delta + V(t,x)$. We refer for example to [5] where the L^p-boundedness of $\nabla^2(\partial_t - \Delta + V(x,t))^{-1}$, or equivalently $V(\partial_t - \Delta + V(t,x))^{-1}$ is proved for a special class of potentials. See also [12] for the case where V is time independent. To our best knowledge, Riesz transforms of $\mathcal{A}$ have not been studied. The aim of this note is to close this gap and prove under some assumptions on the potential $V = V(t,.)$ that $\nabla \mathcal{A}^{-1/2}$ is bounded on $L^p([0,T] \times \mathbb{R}^N)$ for suitable p.

Boundedness of the operator $\nabla(-\Delta + V(x))^{-1/2}$ on $L^p(\mathbb{R}^N)$ relies heavily on heat kernel bounds, i.e., bounds for the integral kernel of the semigroup $e^{-t(-\Delta+V)}$. For non-negative V such bounds are Gaussian and follow easily from the domination by the Gaussian semigroup. When dealing with Riesz transforms of parabolic operators, $\mathcal{A}$ looks like a degenerate operator in $N+1$ variables $(t, x_1, \cdots, x_N)$ (we do not have ∂_t^2 in the expression of $\mathcal{A}$). Therefore the methods to study $L^p(\mathbb{R}^N)$-boundedness of $\nabla(-\Delta + V(x))^{-1/2}$ do not work for $\nabla(\partial_t - \Delta + V(t,x))^{-1/2}$ on $L^p([0,T] \times \mathbb{R}^N)$. Even in the case $p = 2$ it is not clear (at least to us) whether the latter operator is always

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bounded. In the case where $V(t,x) = V(x)$ we shall see that the operator $\nabla(\partial_t - \Delta + V(x))^{-1/2}$ is bounded on $L^2([0,T] \times \mathbb{R}^n)$.

Our strategy to prove boundedness of $\nabla(\partial_t - \Delta + V(t,x))^{-1/2}$ on $L^p([0,T] \times \mathbb{R}^N)$ is based on the maximal regularity property of the corresponding non-autonomous Cauchy problem

$$\partial_t u - \Delta u + V(t,.)u = f(t,.),\ u(0) = 0 \qquad \text{(NACP)}$$

for initial data $f \in L^p([0,T] \times \mathbb{R}^N)$. Indeed the maximal regularity of (NACP) implies that the domain of $\mathcal{A}$ is contained in the domain of $A = -\Delta + V(t,x)$ (but seen as an operator on $L^p([0,T] \times \mathbb{R}^N)$, see (2)). Combining this embedding with the isomorphism between interpolation spaces and domains of fractional powers will allow us to use the boundedness of Riesz transforms of $-\Delta + V$. This simple idea is quite effective but has a disadvantage in the sense that it gives boundedness of $\nabla(I + \mathcal{A})^{-1/2}$ rather than $\nabla\mathcal{A}^{-1/2}$. If we assume that $V(t,x) \geq c > 0$, then boundedness of $\nabla\mathcal{A}^{-1/2}$ is equivalent to boundedness of $\nabla(I + \mathcal{A})^{-1/2}$.
One of our results asserts the following: suppose that there exists $W \in L^\infty_{loc}(\mathbb{R}^N)$ such that

$$c_1 W(x) \leq V(t,x) \leq c_2 W(x)\ (a.e.\ x \in \mathbb{R}^N) \text{ and all } \ t \in [0,T],$$

and there exists $\beta > 1/2$ such that

$$|V(t,x) - V(s,x)| \leq c_2 W(x)|t-s|^\beta\ (a.e.\ x \in \mathbb{R}^N) \text{ and all } \ t,s \in [0,T],$$

then $\nabla(I + \mathcal{A})^{-1/2}$ is bounded on $L^p([0,T] \times \mathbb{R}^N)$ for all $p \in (1,2]$. If $N \geq 3$ and $W \in L^{N/2-\epsilon}(\mathbb{R}^N) \cap L^{N/2+\epsilon}(\mathbb{R}^N)$ for some $\epsilon > 0$, then $\nabla(I + \mathcal{A})^{-1/2}$ is bounded on $L^p([0,T] \times \mathbb{R}^N)$ for $p \in (2,N)$.

Note that the maximal regularity of (NACP) we need in order to prove this result was studied in [18].

The ideas presented here work also for other operators as elliptic operators with time dependent coefficients. We shall however concentrate on Schrödinger operators with time dependent potentials.

We finally mention that the boundedness of the Riesz transforms of $\mathcal{A}$ implies that the solution $u(t,x)$ of the Cauchy problem (NACP) satisfies $u \in W^{1,p}_x([0,T] \times \mathbb{R}^N)$. The maximal regularity says that $u \in W^{1,p}_t([0,T] \times \mathbb{R}^N)$. Here $W^{1,p}_y$ denotes the Sobolev space with respect to the variable $y = t$ or $y = x$.

2. Preliminaries and known results

We first start by recalling some known results on Riesz transforms of time independent Schrödinger operators. We consider $A = -\Delta + V$ and the Riesz transform $\nabla A^{-1/2}$. For every $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$, it is plain that $\nabla A^{-1/2}$ is bounded on $L^2(\mathbb{R}^N)$ (with values $(L^2(\mathbb{R}^N))^N$). The problem of the L^p-boundedness of this Riesz transform has been investigated by several authors. We quote the following results.

Theorem 2.1. *(*[17, Chapter 7], [20] *or* [10]*) Let $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$ and $1 < p \leq 2$. Then the Riesz transform of $A = -\Delta + V$ is bounded in $L^p(\mathbb{R}^N)$.*

In particular, there exists a positive constant C such that

$$\|\nabla u\|_p + \|V^{\frac{1}{2}} u\|_p \le C\|(-\Delta + V)^{\frac{1}{2}} u\|_p$$

for every $u \in C_c^\infty(\mathbb{R}^N)$.

For $p = 1$, the Riesz transform is weak type $(1,1)$. For $p > 2$, the L^p boundedness requires additional assumptions on the potential V. If V satisfies some reverse Hölder inequalities, it is possible to prove that some values of $p > 2$ are allowed. Recall the following definition

Definition 2.2. *Let $1 < q \le \infty$. We say that $\omega \in B_q$, the class of the reverse Hölder weights of order q, if $\omega \in L^q_{loc}$, $\omega > 0$ a.e. and there exists a positive constant C such the inequality*

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^q \, dx\right)^{\frac{1}{q}} \le \frac{C}{|Q|}\int_Q \omega(x)\, dx \tag{1}$$

holds, for every cube Q of $\mathbb{R}^N$. If $q = \infty$, the left hand side of the inequality above has to be replaced by the essential supremum of ω on Q. The smallest positive constant C such that (1) holds is the B_q constant of ω.

Theorem 2.3. *([19]) Let $V \in B_q$, $\frac{N}{2} \le q \le N$. Then, set $\dfrac{1}{p_0} = \dfrac{1}{q} - \dfrac{1}{N}$, it holds*

$$\|\nabla(-\Delta + V)^{-\frac{1}{2}} f\|_p \le \|f\|_p$$

for every $1 < p \le p_0$ and $f \in L^p(\mathbb{R}^N)$.

This theorem has been extended in [4] as follows.

Theorem 2.4. *([4]) Let $V \in B_q$ for some $q > 1$. Then there exists $\varepsilon > 0$ such that $\nabla(-\Delta + V)^{-\frac{1}{2}}$ is bounded in L^p for $1 < p < 2(q + \varepsilon)$.*

Other results such as boundedness of $\nabla^2(-\Delta+V)^{-1}$ or $\nabla(-\Delta+V)^{-1}V^{1/2}$ can be found in [4] and [19]. We also mention the following result which does not require the Hölder reverse assumption.

Theorem 2.5. *([2], [3]) If $0 \le V \in L^{N/2-\epsilon} \cap L^{N/2+\epsilon}$ for some $\epsilon > 0$ then $\nabla(-\Delta + V)^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1, N)$.*

Now we move to the parabolic case. Fix $T > 0$ and set $Q = [0, T] \times \mathbb{R}^N$. We assume that $0 \le V \in L^p_{loc}(Q)$ and consider the parabolic operator

$$\mathcal{A} = \partial_t - \Delta + V(t, x)$$

on $L^p(Q)$, endowed with the maximal domain

$$D_p(\mathcal{A}) = \{u \in L^p(Q) \ : \ Vu \in L^1_{loc}(Q),\ \mathcal{A}u \in L^p(Q), u(0, \cdot) = 0\}.$$

Observe that, since $V \in L^p_{loc}$, C_c^∞ is contained in $D_p(\mathcal{A})$. We have

Theorem 2.6. *([5], [12]) Fix $p \in [1, \infty)$. For every $\lambda > 0$ the operator $\lambda + \mathcal{A}$ is invertible and $\|(\lambda + \mathcal{A})^{-1}\|_{p\to p} \le \dfrac{1}{\lambda}$. Moreover, C_c^∞ is a core for $\mathcal{A}$.*

When it is necessary to specify that $\mathcal{A}$ is acting on $L^p(Q)$ for some fixed p we use the notation $\mathcal{A}_p$. Because of the estimate of the resolvent of $\mathcal{A}$, the fractional power $\mathcal{A}^{-\frac{1}{2}}$ can be defined as follows:

$$\mathcal{A}^{-\frac{1}{2}} = c\int_0^\infty \frac{1}{\sqrt{s}}(s+\mathcal{A})^{-1}ds.$$

Note also that $\mathcal{A}$ is injective. Indeed, if $\mathcal{A}u = 0$ then $v(t,x) = e^t u(t,x)$ satisfies $v \in D(\mathcal{A})$ and $(\mathcal{A}+I)v = 0$. Hence $v = 0$ which implies $u = 0$.

Now we can define the Riesz transform of $\mathcal{A}$ by $\nabla\mathcal{A}^{-1/2}$ (in the distributional sense). The question we are interested in is whether $\nabla\mathcal{A}^{-1/2}$ defines a bounded operator on $L^2(Q)$ or more generally on $L^p(Q)$ for some range of p. As explained in the introduction, our strategy to answer this question relies on the maximal regularity. In order to make this idea clear we need to define the following operators $\widetilde{A}$ and $\mathcal{D}$.

Fix $p \in (1,\infty)$ and $0 \le V \in L^p(Q)$. For fixed $t \in [0,T]$ we define on $L^p(\mathbb{R}^N)$ the operator $A(t) = -\Delta + V(t,.)$ as the Schrödinger operator with potential $V(t,.)$. For fixed t, $\Delta - V(t,.)$ is the generator of a sub-Markovian semigroup $S(s), s \ge 0$. Hence it acts on $L^p(\mathbb{R}^N)$ for all $p \in [1,\infty)$. The operator $A(t)$, when considered on $L^p(\mathbb{R}^N)$, will be seen as (minus) the generator of this semigroup on $L^p(\mathbb{R}^N)$. Now we define

$$D(\widetilde{A}) = \{u \in L^p(Q),\ u(t) \in D(A(t))\ a.e.\ A(.)u(.) \in L^p(Q)\} \tag{2}$$

where $(\widetilde{A}u)(t) = A(t)u(t)$. We define also

$$\begin{aligned} D(\mathcal{D}) &= W_0^{1,p}(0,T,L^p(\mathbb{R}^N)) = \{u \in W^{1,p}(0,T,L^p(\mathbb{R}^N)),\ u(0) = 0\}, \\ (\mathcal{D}u)(t) &= \partial_t u(t) = \frac{\partial}{\partial t}u(t). \end{aligned}$$

Note that the adjoint $\mathcal{D}^*$ of $\mathcal{D}$ is given by

$$D(\mathcal{D}^*) = \{u \in W^{1,p}(0,T,L^p(\mathbb{R}^N)),\ u(T) = 0\},\ (\mathcal{D}^*u)(t) = -\partial_t u(t).$$

Finally we recall the definition of L^p-maximal regularity for (NACP) considered on $L^p(\mathbb{R}^N)$ – for every $f \in L^p(Q)$ (we identify $L^p(Q)$ with $L^p([0,T],L^p(\mathbb{R}^N))$), there exists a unique solution $u \in W^{1,p}(0,T,L^p(\mathbb{R}^N))$ to (NACP) such that $u(t) \in D(A(t))$ a.e. and $A(.)u(.) \in L^p(Q)$. In other words, $\mathcal{D} + \widetilde{A} : D(\mathcal{D}) \cap D(\widetilde{A}) \mapsto L^p(Q)$ is closed and bijective as an operator on $L^p(Q)$. In particular,

$$D(\mathcal{A}) \subseteq D(\mathcal{D}) \cap D(\widetilde{A}) = W_0^{1,p}(0,T,L^p(\mathbb{R}^N)) \cap D(\widetilde{A}). \tag{3}$$

The literature on maximal regularity is so broad that it is impossible to provide a comprehensive bibliography here. The case of autonomous Cauchy problems (time independent operators) is mostly well understood and we refer the reader to [15], [11], [13], [7] and the references therein. Much less is known for non-autonomous problems and most of the known techniques use perturbation arguments. We refer to [1] and to [18] for an account. The latter paper contains a criterion in terms of sesquilinear forms for maximal regularity in Hilbert spaces.

3. Time independent potentials

Suppose that $0 \leq V = V(x) \in L^1_{loc}(\mathbb{R}^N)$. When dealing with the Riesz transform $\nabla(-\Delta + V)^{-1/2}$ the boundedness on $L^2(\mathbb{R}^N)$ is a trivial fact. Indeed, one has for every $u \in W^{1,2}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} V|u|^2 < \infty$,

$$\begin{aligned} \|\nabla u\|_2^2 &\leq \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V|u|^2 \\ &= \mathfrak{a}(u,u) \\ &= \|(-\Delta + V)^{1/2} u\|_2^2. \end{aligned}$$

Here $\mathfrak{a}$ denotes the symmetric bilinear form of the operator $-\Delta + V$. The keystone here lies in the standard fact that for any symmetric form $\mathfrak{a}$ with non-negative self-adjoint operator A,

$$D(\mathfrak{a}) = D(A^{1/2}) \text{ and } \mathfrak{a}(u,v) = (A^{1/2}u, A^{1/2}v). \tag{4}$$

This strategy does not work for $\nabla(\partial_t - \Delta + V(t,x))^{-1/2}$ on $L^2(Q)$ because the operator $\mathcal{A} = \partial_t - \Delta + V(t,x)$ is not self-adjoint. If $V = V(x)$ is independent of t we can prove the following result.

Proposition 3.1. *Let $0 \leq V \in L^2_{loc}(\mathbb{R}^N)$, $u \in D(\mathcal{A}^{\frac{1}{2}})$. Then*

$$\|\nabla u\|_{L^2(Q)} \leq \|\mathcal{A}^{\frac{1}{2}} u\|_{L^2(Q)}.$$

In particular, the Riesz transform $\nabla \mathcal{A}^{-1/2}$ is bounded on $L^2(Q)$.

Proof. Let $A = -\Delta + V$ on $L^2(\mathbb{R}^N)$ and denote by $\widetilde{A}$ the corresponding operator on $L^2(Q)$ (see (2)). It is easy to see that $\widetilde{A}$ is non-negative self-adjoint operator and $(\widetilde{A})^{1/2} = \widetilde{A^{1/2}}$. Therefore,

$$\begin{aligned} \|\nabla u\|_{L^2(Q)}^2 &= \int_{[0,T]} \int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dx dt \\ &\leq \int_{[0,T]} \int_{\mathbb{R}^N} |A^{1/2} u(t,x)|^2 dx dt \\ &= \|(\widetilde{A})^{1/2} u\|_{L^2(Q)}^2 = (\widetilde{A}u, u)_{L^2(Q)}. \end{aligned}$$

Thus

$$\|\nabla u\|_{L^2(Q)}^2 \leq \|\widetilde{A}^{1/2} u\|_{L^2(Q)}^2. \tag{5}$$

On the other hand, for $u \in D(\widetilde{A}^2) \cap D((\mathcal{D}^*\mathcal{D}))$,

$$(\widetilde{A}^2 u, u)_{L^2(Q)} \leq ((\widetilde{A} + \mathcal{D}^*)(\widetilde{A} + \mathcal{D})u, u)_{L^2(Q)}. \tag{6}$$

To see this, we use the fact $\widetilde{A}^2 \leq \widetilde{A}^2 + \mathcal{D}^*\mathcal{D}$ (in the quadratic form sense) and it suffices to prove that $\widetilde{A}^2 + \mathcal{D}^*\mathcal{D} \leq (\widetilde{A} + \mathcal{D}^*)(\widetilde{A} + \mathcal{D})$. Or equivalently, $\widetilde{A}\mathcal{D} + \mathcal{D}^*\widetilde{A} \geq 0$. Since $V = V(x)$ the operators A and $\mathcal{D}$ commute and hence

$$\begin{aligned} ((\widetilde{A}\mathcal{D} + \mathcal{D}^*\widetilde{A})u, u)_{L^2(Q)} &= (\mathcal{D}\widetilde{A}u, u)_{L^2(Q)} + (\widetilde{A}u, \mathcal{D}u)_{L^2(Q)} \\ = \int_{\mathbb{R}^N} \int_0^T \partial_t A u(t,x).u(t,x) dt dx &+ \int_{\mathbb{R}^N} \int_0^T A u(t,x).\partial_t u(t,x) dt dx \\ &= \int_{\mathbb{R}^N} A u(T,x).u(T,x) dx \geq 0. \end{aligned}$$

From (6) and the maximal regularity of A on $L^2(\mathbb{R}^N)$ we obtain

$$\widetilde{A}^2 \le (\widetilde{A} + \mathcal{D}^*)(\widetilde{A} + \mathcal{D}) = \mathcal{A}^*\mathcal{A}.$$

Using the fact that $\widetilde{A}$ is self-adjoint and the fact that $\mathcal{A}$ and $\mathcal{A}^*$ commute (remember that $\mathcal{D}$ commute with A) we obtain

$$\widetilde{A} \le (\mathcal{A}^*)^{1/2}(\mathcal{A})^{1/2}.$$

This gives

$$\|\widetilde{A}^{1/2}u\|_{L^2(Q)}^2 \le ((\mathcal{A}^*)^{1/2}(\mathcal{A})^{1/2}u, u)_{L^2(Q)} = \|\mathcal{A}^{1/2}u\|_{L^2(Q)}^2.$$

We use now (5) and obtain the conclusion. □

It may be possible to extend the operator $\nabla \mathcal{A}^{-1/2}$ from $L^2(Q)$ to $L^p(Q)$ for all $p \in (1,2)$ by using the same method as in the case of $\nabla A^{-1/2}$. See [17] or [20]. This remains to be done since it is not clear what the approximation of identity one has to choose in order to apply the singular integral method there. We shall proceed as on L^2 by using maximal regularity. However, if A is not invertible, we will need to consider $I + \mathcal{A}$ instead of $\mathcal{A}$. We have

Theorem 3.2. *Let $1 < p \le 2$, $0 \le V \in L^p_{loc}(\mathbb{R}^N)$, $u \in D(\mathcal{A}^{\frac{1}{2}})$. Then*

$$\|\nabla u\|_{L^p(Q)} \le \|(I + \mathcal{A})^{1/2}u\|_{L^p(Q)}.$$

That is the non-homogeneous Riesz transform $\nabla(I + \mathcal{A})^{-1/2}$ is bounded on $L^p(Q)$.

Theorem 3.3. *Let $p > 2$, $0 \le V \in L^p_{loc}(\mathbb{R}^N)$. Suppose moreover that the Riesz transforms of $A = -\Delta + V$ is bounded on $L^p(\mathbb{R}^N)$. Then*

$$\|\nabla u\|_{L^p(Q)} \le \|(I + \mathcal{A})^{1/2}u\|_{L^p(Q)}$$

for every $u \in D(\mathcal{A}^{\frac{1}{2}})$. In other words, $\nabla(I + \mathcal{A})^{-1/2}$ is bounded on $L^p(Q)$.

By Theorems 2.3 or 2.4, we obtain boundedness of $\nabla(I + \mathcal{A})^{-1/2}$ on $L^p(Q)$ provided V is in an appropriate reverse Hölder class. By Theorem 2.5, $\nabla(I + \mathcal{A})^{-1/2}$ is bounded on $L^p(Q)$ for all $p \in (2, N)$ provided $V \in L^{N/2-\epsilon}(\mathbb{R}^N) \cap L^{N/2+\epsilon}(\mathbb{R}^N)$ for some $\epsilon > 0$.

The proof is based on the isomorphism between interpolation spaces and domains of fractional powers for operator having bounded imaginary powers. We first recall the following well known result.

Theorem 3.4. [6] *Let (Ω, μ) be a σ-finite measure space and let $1 < p < \infty$. Let L be a closed and densely defined operator on $L^p(\Omega, \mu)$. If the resolvent set of $-L$ contains $[0, +\infty)$, satisfies*

$$\|(\lambda + L)^{-1}\|_{p\to p} \le \frac{1}{\lambda}$$

and $(\lambda + L)^{-1}$ is positivity preserving for every $\lambda > 0$, then the operators L has bounded imaginary powers and there is $C > 0$ such that

$$\|L^{is}\|_{p\to p} \le c(1 + s^2)e^{\pi|s|/2}, \qquad s \in \mathbb{R}.$$

PROOF OF THEOREM 3.2 Since the semigroup generated by $-A = -(-\Delta + V)$ is a contraction semigroup on $L^p(\mathbb{R}^N)$, A has the maximal regularity (see [16]). Therefore, $D(\mathcal{A}) \subseteq D(\widetilde{A})$ by (3). On the other hand, $\widetilde{A}$ has bounded imaginary powers on $L^p(Q)$ (see Lemma 3.5 below, where the result is proved even when V depends on t). It is known that this implies that

$$[D(\widetilde{A}), L^p(Q)]_{\frac{1}{2}} = D(\widetilde{A}^{1/2})$$

where $[.,.]_\alpha$ denotes the complex interpolation space. Therefore,

$$[D(\mathcal{A}), L^p(Q)]_{\frac{1}{2}} \subseteq [D(\widetilde{A}), L^p(Q)]_{\frac{1}{2}} = D(\widetilde{A}^{1/2}). \tag{7}$$

By Theorems 2.6 and 3.4, $\mathcal{A}$ also has bounded imaginary powers on $L^p(Q)$. Thus,

$$[D(\mathcal{A}), L^p(Q)]_{\frac{1}{2}} = D(\mathcal{A}^{1/2}).$$

This and (7) show that $D(\mathcal{A}^{1/2}) \subseteq D(\widetilde{A}^{1/2})$ and

$$\|\widetilde{A}^{1/2}u\|_{L^p(Q)} \le C\|(I+\mathcal{A})^{1/2}u\|_{L^p(Q)}$$

for all $u \in D(\mathcal{A}^{1/2})$. By Theorem 2.1, we have

$$\begin{aligned}\int_0^T \|\nabla u(t,.)\|^p_{L^p(\mathbb{R}^N)}dt &\le C\int_0^T \|A^{1/2}u(t,.)\|^p_{L^p(\mathbb{R}^N)}dt \\ &= C\int_0^T \|\widetilde{A}^{1/2}u(t,.)\|^p_{L^p(\mathbb{R}^N)}dt.\end{aligned}$$

This together with the previous estimate imply the desired result. □

The proof of Theorem 3.3 is similar.

3.1. **Time dependent potentials.** In this section we consider the general case where $V = V(t,x)$. As before we assume that V is non-negative and locally integrable. Set $A(t) = -\Delta + V(t,x)$. As explained before, $-A(t)$ is the generator on $L^p(\mathbb{R}^N)$ of the sub-Markovian semigroup $S(s) = e^{-s(-\Delta+V(t,.))}$. We define again $\widetilde{A}$ and $\mathcal{A}$ as in the previous sections. We start with the following lemma.

Lemma 3.5. *Given $p \in (1,\infty)$, $\widetilde{A}$ has bounded imaginary powers on $L^p(Q)$.*

PROOF. Fix $\lambda > 0$. It is easy to see that $\widetilde{A}$ is a closed operator. We claim that $\lambda I + \widetilde{A}$ is invertible on $L^p(Q)$ and

$$(\lambda I + \widetilde{A})^{-1}u(t) = (\lambda I + A(t))^{-1}u(t). \tag{8}$$

In order to see this, fix $t \in [0,T]$ and define on $L^p(Q)$ the bounded operator S such that

$$(Su)(t) = (\lambda I + A(t))^{-1}u(t),$$

for a.e. t and and all $u \in L^p(Q)$. We obtain from the definition of $\widetilde{A}$

$$((\lambda I + \widetilde{A})S)u(t) = \lambda(\lambda I + A(t))^{-1}u(t) + A(t)(\lambda I + A(t))^{-1}u(t) = u(t).$$

Similarly, $S(\lambda I + \widetilde{A}) = I$. This shows (8).

The fact that $\lambda(\lambda I + A(t))^{-1}$ is a contraction on $L^p(\mathbb{R}^N)$ (for fixed t) gives

$$\|\lambda(\lambda I + \widetilde{A})^{-1}\|_{p\to p} \le 1.$$

In addition, (8) shows that $(\lambda I+\widetilde{A})^{-1}$ is positivity preserving. We can apply Theorem 3.4 and obtain the result. □

We recall the following result from [18].

Theorem 3.6. *Let $0 \le V(t,x) \in L^1_{loc}([0,T]\times\mathbb{R}^N)$. Suppose that there exists a non-negative potential $W \in L^1_{loc}(\mathbb{R}^N)$ such that V satisfies the following properties (in which c_1, c_2 are positive constants and $\beta > 1/2$)*

$$c_1 W(x) \le V(t,x) \le c_2 W(x) \ (a.e.\ x \in \mathbb{R}^N) \ and\ all\ \ t \in [0,T],$$

$$|V(t,x) - V(s,x)| \le c_2 W(x)|t-s|^\beta \ (a.e.\ x \in \mathbb{R}^N) \ and\ all\ \ t,s \in [0,T].$$

Then, for $1 < p < \infty$, the family $\{A(t) = -\Delta + V(t,\cdot),\ t \in [0,T]\}$ has maximal regularity.

The main result of this section is formulated as follows.

Theorem 3.7. *Suppose that $0 \le V(t,x) \in L^\infty_{loc}([0,T]\times\mathbb{R}^N)$ and satisfies the assumptions of Theorem 3.6 with some $W \in L^\infty_{loc}(\mathbb{R}^N)$.*
1) For every $p \in (1,2)$, $\nabla(I+\mathcal{A})^{-1/2}$ is bounded on $L^p(Q)$.
2) If $N \ge 3$ and $W \in L^{N/2-\epsilon}(\mathbb{R}^N) \cap L^{N/2+\epsilon}(\mathbb{R}^N)$ for some $\epsilon > 0$ then $\nabla(I+\mathcal{A})^{-1/2}$ is bounded on $L^p(Q)$ for $p \in (2,N)$.

The proof is very similar to the proof of Theorem 3.2. For assertion 2) one uses Theorem 2.5 and note that for every fixed t and $p \in (2,N)$

$$\|\nabla(-\Delta + V(t,.))^{-1/2}\|_{p\to p} \le C$$

with a constant C depending only on $\|V(t,.)\|_{N/2+\epsilon}$ and $\|V(t,.)\|_{N/2-\epsilon}$ (see [2], [3]). Hence C can be chosen depending only on $\|W\|_{N/2+\epsilon}$ and $\|W\|_{N/2-\epsilon}$ and so independent of t. The rest of the needed arguments are similar to proof of Theorem 3.2.

We can also formulate a result on boundedness on $L^p(Q)$ of $\nabla(I+\mathcal{A})^{-1/2}$ for time dependent potentials that are in a certain reverse class. For such potentials, the parabolic Cauchy problem has the maximal regularity. For this last property, we refer the reader to [5].

Finally as explained in the introduction, when the Riesz transform $\nabla(I+\mathcal{A})^{-\frac{1}{2}}$ is bounded on $L^p(Q)$, then for $u \in D(\mathcal{A})$

$$\|\nabla u\|_{L^p(Q)} \le C\|\nabla(I+\mathcal{A})^{-1/2}\|_{p\to p}\|(I+\mathcal{A})^{1/2}u\|_{L^p(Q)} \le C\|(I+\mathcal{A})^{1/2}u\|_{L^p(Q)}.$$

Hence, for $f \in L^p(Q)$ the Cauchy problem

$$\partial_t u + -\Delta u + V(t,.)u = f(t,.)$$

has a solution which is $W^{1,p}$ with respect to the space variable.

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