EQUILIBRIUM OF TOKAMAK PLASMA

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OUTLINE OF LECTURE 3

- **Governing equations**
- **Coordinate transforms**
- **Shafranov and flux coordinates**
- **Grad-Shafranov equation**
- **Summary**

GOVERNING IDEAL MHD EQUATIONS FOR PLASMA - 1

- For describing *plasma particles*, we take velocity moments of the kinetic equations for electron and thermal ion distribution functions and obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0;$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \frac{1}{c}\mathbf{J} \times \mathbf{B};$$

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \gamma \; p \nabla \cdot \mathbf{V} = 0;$$

$$\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B} = 0;$$

- For describing *electromagnetic fields* in the plasma, Maxwell's equations are used

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial \mathbf{t}}$$

$$\nabla \cdot \mathbf{B} = 0$$

- Here, scale lengths larger than Debye length are considered with $n_e = \sum_i Z_i \cdot n_i$

GOVERNING IDEAL MHD EQUATIONS FOR PLASMA - 2

- Express plasma current as sum of components parallel and perpendicular to magnetic field:

$$\mathbf{J} = \frac{(\mathbf{J}\cdot\mathbf{B})}{B^2}\mathbf{B} + \frac{\mathbf{B}\times(\mathbf{J}\times\mathbf{B})}{B^2}$$

- The second term is taken from the equation of motion

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla p + \frac{1}{c}\mathbf{J}\times\mathbf{B} \quad \rightarrow \quad \frac{1}{c}\mathbf{B}\times(\mathbf{J}\times\mathbf{B}) = \left(\mathbf{B}\times\rho\frac{d\mathbf{V}}{dt}\right) + (\mathbf{B}\times\nabla p)$$

- From Amperes law we have that

$$\nabla\cdot\mathbf{J} = \nabla\cdot\left(\frac{c}{4\pi}\nabla\times\mathbf{B}\right) = 0$$

- Substituting first two equations in the third and using a vector identity and $\nabla\cdot\mathbf{B} = 0$ we arrive at the main governing equation of Ideal MHD

$$\mathbf{B}\cdot\nabla\left(\frac{\mathbf{J}\cdot\mathbf{B}}{B^2}\right) + \nabla\cdot\left(4\pi\rho\frac{\mathbf{B}\times d\mathbf{V}/dt}{B^2}\right) + \nabla\cdot\left(\frac{\mathbf{B}\times\nabla p}{B^2}\right) = 0$$

GOVERNING IDEAL MHD EQUATIONS FOR PLASMA - 3

- We introduce equilibrium (denoted by subscript 0) and perturbed (denoted by δ) quantities and let $\mathbf{V}_0 = 0$, i.e.

$$\mathbf{J} = \mathbf{J}_0 + \delta\mathbf{J}\,,\ \ \mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}\,,\ \ \mathbf{V} = \delta\mathbf{V}\,,\ \ p = p_0 + \delta p$$

- Static equilibrium means $d/dt = 0$, so we get the following equation for the equilibrium

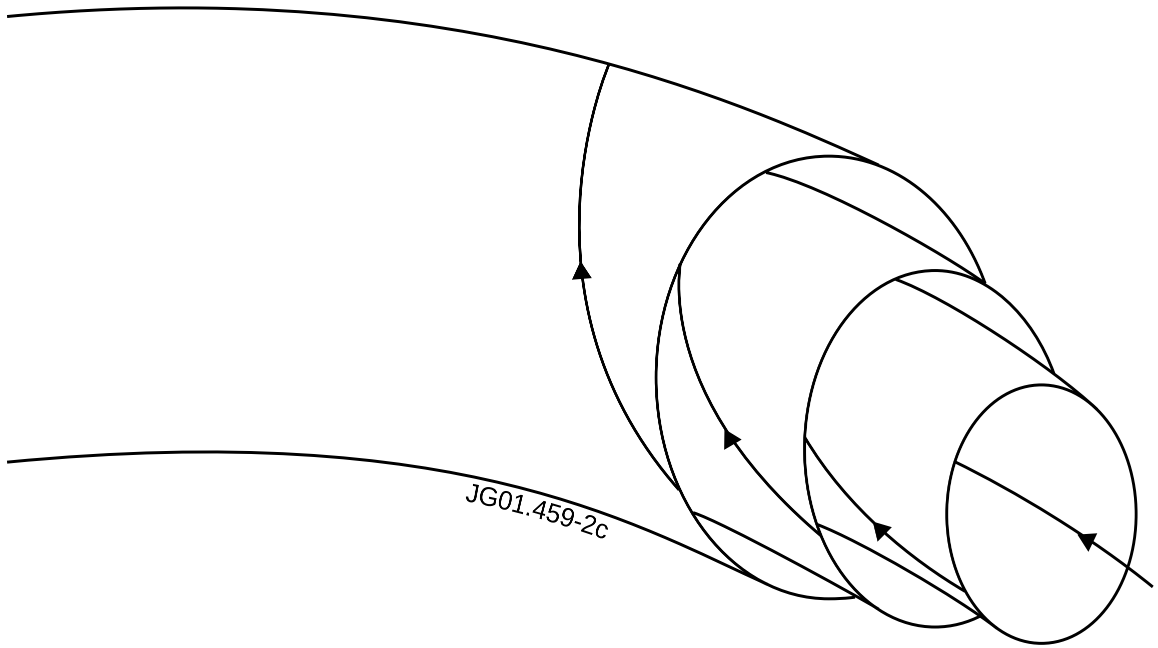
$$\nabla p_0 = \frac{1}{c}\mathbf{J}_0 \times \mathbf{B}_0$$

- It follows then that $\mathbf{B}_0 \nabla p_0 = 0, \quad \mathbf{J}_0 \nabla p_0 = 0$, i.e. plasma pressure is constant along magnetic field lines and along the current lines; the plasma may expand freely along these lines

- Since ∇p_0 is perpendicular to the surface $p_0 = const$, the magnetic field lines and the current lines must lie on the $p_0 = const$ surfaces

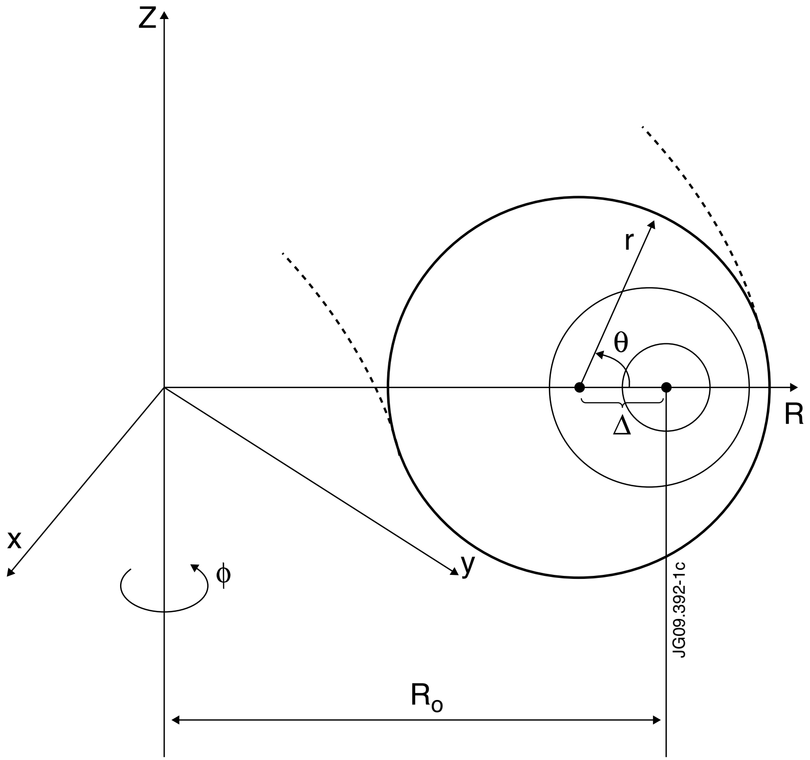
- Toroidal surfaces enclosed within each other perfectly fit all the conditions above → torus

GOVERNING IDEAL MHD EQUATIONS FOR PLASMA - 4



$$B = B_0\left(1-\left(r/R\right)\cos\vartheta\right)$$

WHAT COORDINATES ARE BEST FOR DESCRIBING EQUILIBRIUM?



$$\nabla p_0 = \frac{1}{c}\mathbf{J}_0 \times \mathbf{B}_0$$

COORDINATE TRANSFORMS - 1

- Begin with **Cartesian coordinates** (x, y, z)
- Basis of unit vectors = $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$
- Representation of a vector:

$$\mathbf{A} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$$

- Components of a vector:

$$x = (\mathbf{e}_y \times \mathbf{e}_z)\cdot \mathbf{A};\ y = (\mathbf{e}_z \times \mathbf{e}_x)\cdot \mathbf{A};\ z = (\mathbf{e}_x \times \mathbf{e}_y)\cdot \mathbf{A}$$

- If transform to another coordinate system (α, β, γ), then

$$\mathbf{A} = \alpha\mathbf{e}_\alpha + \beta\mathbf{e}_\beta + \gamma\mathbf{e}_\gamma$$, with

$$\alpha = x\frac{\partial x}{\partial \alpha} + y\frac{\partial y}{\partial \alpha} + z\frac{\partial z}{\partial \alpha},\ \beta = x\frac{\partial x}{\partial \beta} + y\frac{\partial y}{\partial \beta} + z\frac{\partial z}{\partial \beta},\ \gamma = x\frac{\partial x}{\partial \gamma} + y\frac{\partial y}{\partial \gamma} + z\frac{\partial z}{\partial \gamma}.$$

COORDINATE TRANSFORMS - 2

- Define **curvilinear coordinates** $\left(\xi^1,\xi^2,\xi^3\right)$
- **Covariant basis** = $\left(\nabla\xi^1,\nabla\xi^2,\nabla\xi^3\right)$
- Covariant representation of a vector: $\mathbf{A}=A_1\nabla\xi^1+A_2\nabla\xi^2+A_3\nabla\xi^3$
- Covariant components of a vector:

$$A_1=J\left(\nabla\xi^2\times\nabla\xi^3\right)\cdot\mathbf{A};\ A_2=J\left(\nabla\xi^3\times\nabla\xi^1\right)\cdot\mathbf{A};\ A_3=J\left(\nabla\xi^1\times\nabla\xi^2\right)\cdot\mathbf{A}$$

where Jacobian J is given by $J=\left[\left(\nabla\xi^1\times\nabla\xi^2\right)\cdot\nabla\xi^3\right]^{-1}$

- Substitute:

$$\mathbf{A}=\frac{\left(\nabla\xi^2\times\nabla\xi^3\right)\cdot\mathbf{A}}{\left(\nabla\xi^1\times\nabla\xi^2\right)\cdot\nabla\xi^3}\nabla\xi^1+\frac{\left(\nabla\xi^3\times\nabla\xi^1\right)\cdot\mathbf{A}}{\left(\nabla\xi^1\times\nabla\xi^2\right)\cdot\nabla\xi^3}\nabla\xi^2+\frac{\left(\nabla\xi^1\times\nabla\xi^2\right)\cdot\mathbf{A}}{\left(\nabla\xi^1\times\nabla\xi^2\right)\cdot\nabla\xi^3}\nabla\xi^3$$

- If transform to another coordinate system $\left(\tilde{\xi}^1,\tilde{\xi}^2,\tilde{\xi}^3\right)$, then

$$\mathbf{A}=\tilde{A}_1\nabla\tilde{\xi}^1+\tilde{A}_2\nabla\tilde{\xi}^2+\tilde{A}_3\nabla\tilde{\xi}^3,$$ with

$$\tilde{A}_i=\sum_j A_j\frac{\partial\xi^j}{\partial\tilde{\xi}^i}$$

COORDINATE TRANSFORMS – 3

- **Contravariant basis =** $\left(J\left(\nabla\xi^2\times\nabla\xi^3\right), J\left(\nabla\xi^3\times\nabla\xi^1\right), J\left(\nabla\xi^1\times\nabla\xi^2\right)\right)$
- Contravariant components: $A^1=\mathbf{A}\cdot\nabla\xi^1,\ \ A^2=\mathbf{A}\cdot\nabla\xi^2,\ \ A^3=\mathbf{A}\cdot\nabla\xi^3$
- Substitute:

$$\mathbf{A}=\mathbf{A}\cdot\nabla\xi^1\frac{\left(\nabla\xi^2\times\nabla\xi^3\right)}{\left(\nabla\xi^1\times\nabla\xi^2\right)\cdot\nabla\xi^3}+\mathbf{A}\cdot\nabla\xi^2\frac{\left(\nabla\xi^3\times\nabla\xi^1\right)}{\left(\nabla\xi^1\times\nabla\xi^2\right)\cdot\nabla\xi^3}+\mathbf{A}\cdot\nabla\xi^3\frac{\left(\nabla\xi^1\times\nabla\xi^2\right)}{\left(\nabla\xi^1\times\nabla\xi^2\right)\cdot\nabla\xi^3}$$

- Metric tensor is relationship between covariant and contravariant components:

$$A^i=\sum_j A_j\nabla\xi^j\cdot\nabla\xi^i=\sum_j g^{ij}A_j,$$

where the contravariant components of tensor $\mathbf{g}$ are given by

$$g^{ij}=\nabla\xi^i\cdot\nabla\xi^j=\frac{\partial\xi^i}{\partial x}\cdot\frac{\partial\xi^j}{\partial x}+\frac{\partial\xi^i}{\partial y}\cdot\frac{\partial\xi^j}{\partial y}+\frac{\partial\xi^i}{\partial z}\cdot\frac{\partial\xi^j}{\partial z}$$

- Covariant components of tensor $\mathbf{g}$ are given by the inverse matrix:

$$g_{ij}=\left[g^{ij}\right]^{-1}=\frac{\partial x}{\partial\xi^i}\cdot\frac{\partial x}{\partial\xi^j}+\frac{\partial y}{\partial\xi^i}\cdot\frac{\partial y}{\partial\xi^j}+\frac{\partial z}{\partial\xi^i}\cdot\frac{\partial z}{\partial\xi^j}$$

$$Det\left(g_{ij}\right)\equiv g=J^2$$

COORDINATE TRANSFORMS - 4

- Volume element:

$$d^3x = dxdydz = Jd\xi^1 d\xi^2 d\xi^3$$

- Length element:

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \sum_{i,j=1}^{3} g_{ij} d\xi^i d\xi^j$$

- Divergence:

$$\nabla \cdot \mathbf{A} = \frac{1}{J}\frac{\partial}{\partial \xi^i}\left(JA^i\right)$$

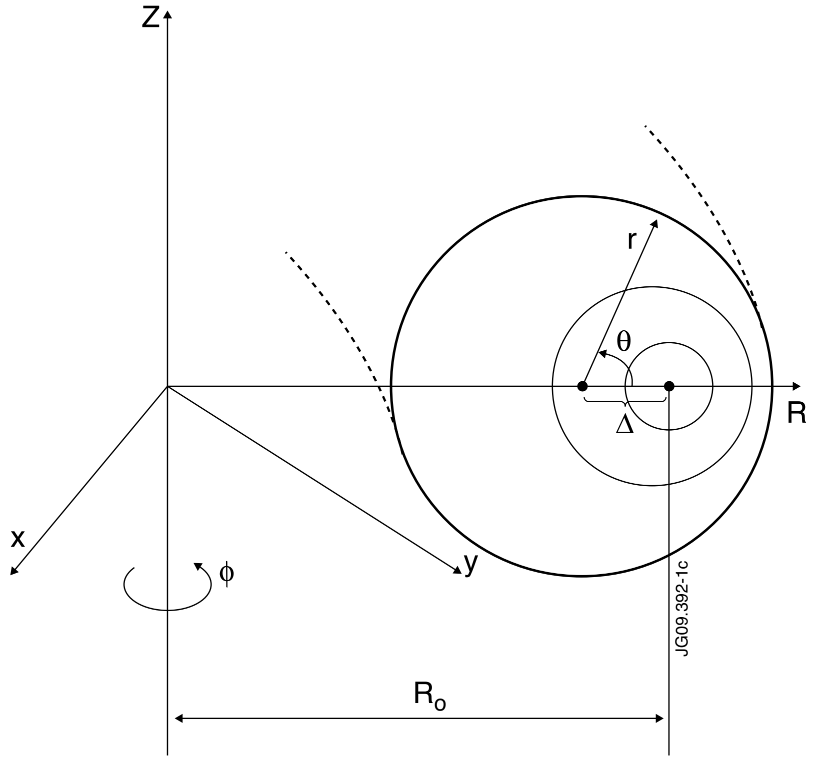
- Curl:

$$(\nabla \times \mathbf{A})^i = \nabla \xi^i \cdot \sum_{j,m} \nabla \xi^j \times \left(\frac{\partial}{\partial \xi^j} \nabla \xi^m A_m\right) = \frac{1}{J}\sum_{j,k} \varepsilon_{ijk} \frac{\partial A_k}{\partial \xi^j}$$

where ε_{ijk} is antisymmetric unit matrix whose only non-zero components are

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1; \ \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

SHAFRANOV COORDINATES – 1



$$R = R_0 + r\cos\vartheta - \Delta(r)$$
$$= \sqrt{x^2 + y^2};$$

$$\phi = -\zeta$$
$$= \tan^{-1}(y/x);$$

$$Z = r\sin\vartheta;$$

Inversed relations:

$$x = R\cos\phi = R\cos\zeta;$$

$$y = R\sin\phi = -R\sin\zeta$$

SHAFRANOV COORDINATES – 2

- Transformation matrix from $(x,\ y,\ Z)$ to $(\xi^1 = r,\ \xi^2 = \vartheta,\ \xi^3 = \zeta)$:

$$\frac{\partial x}{\partial r} = \left(\frac{\partial R}{\partial r}\right)\cos\zeta \qquad \frac{\partial x}{\partial \vartheta} = \left(\frac{\partial R}{\partial \vartheta}\right)\cos\zeta \qquad \frac{\partial x}{\partial \zeta} = -R\sin\zeta$$

$$\frac{\partial y}{\partial r} = -\left(\frac{\partial R}{\partial r}\right)\sin\zeta \qquad \frac{\partial y}{\partial \vartheta} = -\left(\frac{\partial R}{\partial \vartheta}\right)\sin\zeta \qquad \frac{\partial y}{\partial \zeta} = -R\cos\zeta$$

$$\frac{\partial Z}{\partial r} = \sin\vartheta \qquad \frac{\partial Z}{\partial \vartheta} = r\cos\vartheta \qquad \frac{\partial Z}{\partial \zeta} = 0$$

so that the covariant components of the metric tensor

$$g_{ij} = \frac{\partial x}{\partial \xi^i}\cdot\frac{\partial x}{\partial \xi^j} + \frac{\partial y}{\partial \xi^i}\cdot\frac{\partial y}{\partial \xi^j} + \frac{\partial z}{\partial \xi^i}\cdot\frac{\partial z}{\partial \xi^j}$$

are given by, e.g. for g_{11}:

$$g_{11} = \left(\frac{\partial R}{\partial r}\right)^2 + \sin^2\vartheta = (\cos\vartheta - \Delta')^2 + \sin^2\vartheta = 1 - 2\Delta'\cos\vartheta + \Delta'^2$$

SHAFRANOV COORDINATES – 3

$$g_{ij} = \begin{pmatrix} 1 - 2\Delta'\cos\vartheta + \Delta'^2 & r\Delta'\sin\vartheta & 0 \\ r\Delta'\sin\vartheta & r^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix}$$

and we obtain *Shafranov's Jacobian* as

$$J_S = \sqrt{\det(g_{ij})} = rR \cdot (1 - \Delta'\cos\vartheta) \cong rR_0 \cdot \left(1 + \left(\frac{r}{R_0} - \Delta'\right) \cdot \cos\vartheta\right)$$

where we used the large aspect ratio ordering

$$\Delta / R_0 \cong \varepsilon^2 << \varepsilon = (r / R_0) << 1$$

SHAFRANOV COORDINATES – 4

- **We also need the contravariant metric elements. These can be found from**

$$g^{ij} = \left(g_{ij}\right)^{-1} = \frac{1}{J^2}\begin{pmatrix} g_{22}\cdot g_{33} & -g_{12}\cdot g_{33} & 0 \\ -g_{12}\cdot g_{33} & g_{11}\cdot g_{33} & 0 \\ 0 & 0 & g_{11}\cdot g_{22} - g_{12}^2 \end{pmatrix}$$

- **These contravaraiant elements are:**

$$g^{11} = \nabla r \cdot \nabla r = 1 + 2\Delta' \cos\vartheta$$

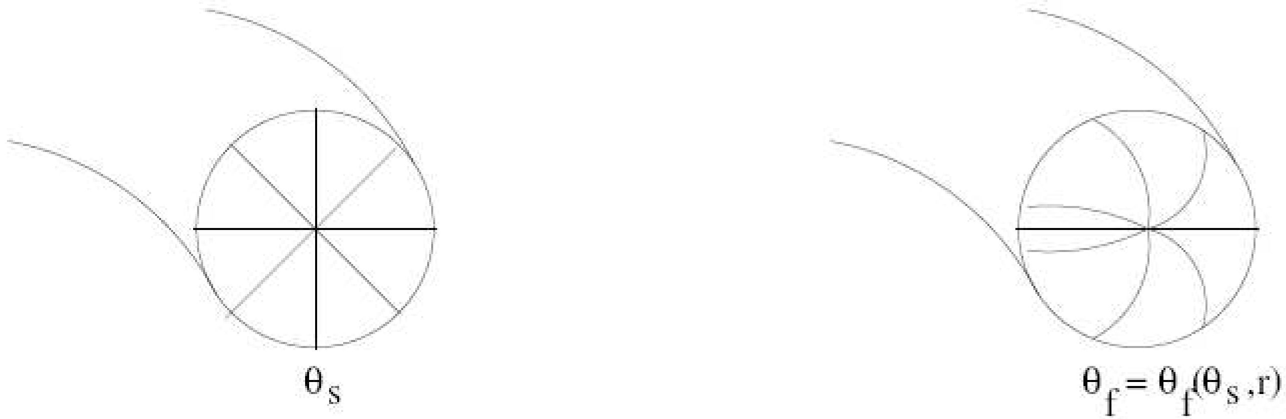
$$g^{12} = \nabla r \cdot \nabla \vartheta = -\frac{1}{r}\Delta' \sin\vartheta$$

$$g^{22} = \nabla \vartheta \cdot \nabla \vartheta = \frac{1}{r^2}$$

$$g^{33} = \nabla \zeta \cdot \nabla \zeta = \frac{1}{R^2}$$

FLUX-TYPE COORDINATES – 1

- **Now, we want to introduce safety factor** $q = \frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \vartheta}$ **which would be** ϑ ***-independent***
- **Since the 'density' of magnetic field lines is higher in the inner side of the torus, a different poloidal coordinate than the usual poloidal angle is required.**



- **Searching for a good set of *flux-type* coordinates** $(r_f;\ \vartheta_f;\ \zeta_f)$ **related to *Shafranov* coordinates** $(r_S;\ \vartheta_S;\ \zeta_S)$ **via**

$$r_f = r_S;\quad \vartheta_f = \vartheta_f(r_f,\ \vartheta_S);\quad \zeta_f = \zeta_S$$

FLUX-TYPE COORDINATES – 2

- The expression for ϑ_f is found from the condition $\partial q / \partial \vartheta_f = 0$ as follows:
- Split $\mathbf{B}$ into polidal and toroidal parts, $\mathbf{B} = \nabla\zeta \times \nabla\psi + I(\psi)\nabla\zeta$ and see:

$$(\nabla\zeta)^2 = (1/R)^2 \quad \rightarrow \quad \mathbf{B}\cdot\nabla\zeta_f = I(\psi)/R^2$$

$$\nabla\psi = (d\psi/dr)\nabla r \quad \rightarrow \quad \mathbf{B}\cdot\nabla\vartheta_f = \frac{d\psi}{dr}\cdot\left(\frac{1}{\nabla r_f \times \nabla\vartheta_f \cdot \nabla\zeta_f}\right)^{-1} = \frac{\psi'}{J_f}\text{, where } J_f = 1/(\nabla r_f \times \nabla\vartheta_f \cdot \nabla\zeta_f)$$

- Require $q = \dfrac{\mathbf{B}\cdot\nabla\zeta}{\mathbf{B}\cdot\nabla\vartheta} = q(\psi)$, obtain $J_f = \dfrac{R^2(r,\ \vartheta)\cdot q(\psi)\psi'}{I(\psi)}$, with ϑ only involved in R^2
- On the other hand, the ratio of the Jacobians gives differential equation of the poloidal angle variables. So we can find the flux-type angle via Shafranov angle:

$$\frac{\partial\vartheta_f}{\partial\vartheta_S} = \frac{J_S}{J_f} \propto \frac{1 + ((r/R_0) - \Delta')\cos\vartheta_S}{1 + 2(r/R_0)\cos\vartheta_S} \cong 1 - \left(\frac{r}{R_0} + \Delta'\right)\cos\vartheta_S\text{,}$$

$$\downarrow$$

FLUX-TYPE COORDINATES – 3

$$\vartheta_f = \vartheta_S - \left(\frac{r}{R_0} + \Delta'\right)\sin\vartheta_S + C$$

Here, $C = 0$ since ϑ_f must change by 2π whenever ϑ_S does.

- Inverting the expression for $\vartheta_f(r_f,\ \vartheta_S)$ gives

$$\vartheta_S = \vartheta_f + \left(\frac{r}{R_0} + \Delta'\right)\sin\vartheta_f$$

and we can write using Taylor expansions:

$$\cos\vartheta_S \cong \cos\vartheta_f + \left((r/R_0) + \Delta'\right)\sin\vartheta_f \cdot \left(-\sin\vartheta_f\right) = \cos\vartheta_f - \eta(r)\left(1 - \cos 2\vartheta_f\right);$$

$$\sin\vartheta_S \cong \sin\vartheta_f + \left((r/R_0) + \Delta'\right)\sin\vartheta_f \cdot \left(+\cos\vartheta_f\right) = \sin\vartheta_f + \eta(r)\sin 2\vartheta_f;$$

where $\eta(r) \equiv \frac{1}{2}\left(\frac{r}{R_0} + \Delta'\right)$

- We can now re-write Shafranov coordinates to get flux-type coordinates:

$$R = R_0 + r\cos\vartheta_f - \Delta(r) - r\eta(r)\left(1 - \cos 2\vartheta_f\right);\ \phi = -\zeta;\ Z = r\sin\vartheta_f + r\eta(r)\sin 2\vartheta_f$$

- The flux-type Jacobian is $J_f = \sqrt{\det\left(g_{ij}^f\right)} \cong rR\sqrt{1 + 2\frac{r}{R_0}\cos\vartheta_f} \cong \frac{rR^2}{R_0}$

FLUX-TYPE COORDINATES – 4

- **With the flux-type coordinates we get the covariant matrix (put $\vartheta_f = \vartheta$ here):**

$$g_{ij}^f = \begin{pmatrix} 1-2\Delta'\cos\vartheta & r\left((r/R_0)\sin\vartheta + (r\Delta')'\sin\vartheta\right) & 0 \\ r\left((r/R_0)\sin\vartheta + (r\Delta')'\sin\vartheta\right) & r^2\left(1+4\eta\cos\vartheta+4\eta^2\right) & 0 \\ 0 & 0 & R^2 \end{pmatrix}$$

- **The contravaraiant elements are:**

$$g^{11} = \nabla r \cdot \nabla r = 1 + 2\Delta'\cos\vartheta$$

$$g^{12} = \nabla r \cdot \nabla\vartheta = -\frac{1}{r}\left(\frac{r}{R_0} + (r\Delta')'\right)\sin\vartheta$$

$$g^{22} = \nabla\vartheta \cdot \nabla\vartheta = \frac{1}{r^2}\left(1 - 2\left(\frac{r}{R_0} + \Delta'\right)\cos\vartheta\right)$$

$$g^{33} = \nabla\zeta \cdot \nabla\zeta = \frac{1}{R^2} = \frac{1}{R_0^2}\left(1 - 2\frac{r}{R_0}\cos\vartheta\right)$$

GRAD-SHAFRANOV EQUATION – 1

- **For solving the equilibrium equation we introduce the poloidal magnetic field flux function ψ and the poloidal current density flux function I:**

$$\mathbf{B}_\vartheta = \frac{1}{R}\left[\nabla\psi \times \mathbf{e}_\phi\right]; \quad \mathbf{J}_\vartheta = \frac{2}{cR}\left[\nabla I \times \mathbf{e}_\phi\right]$$

- **The equilibrium equation $\mathbf{J}_\vartheta \times \mathbf{e}_\phi B_\phi + \mathbf{J}_\phi \times \mathbf{e}_\phi B_\vartheta = c\nabla p$ takes the form of *Grad-Shafranov equation* then for the poloidal flux ψ:**

$$R^2\nabla\cdot\left(\frac{1}{R^2}\nabla\psi\right) + I\frac{dI}{d\psi} + 4\pi R^2\frac{dp}{d\psi} = 0$$

, where (put $\vartheta_f = \vartheta$ here):

$$R^2\nabla\left(\frac{1}{R^2}\nabla\psi\right) = \frac{R^2}{J}\frac{\partial}{\partial r}\left(\frac{J}{R^2}\psi' g^{11}\right) + \frac{R^2}{J}\frac{\partial}{\partial\vartheta}\left(\frac{J}{R^2}\psi' g^{12}\right) = \frac{1}{r}\frac{\partial}{\partial r}(r\psi') + \cos\vartheta\left(\frac{1}{r}\frac{\partial}{\partial r}(2\Delta' r\psi') - \frac{1}{r}\left(\frac{r}{R_0} + (r\Delta')'\right)\psi'\right),$$

$$I\frac{dI}{d\psi} = I\frac{dI}{dr}\left(\frac{d\psi}{dr}\right)^{-1}, \qquad 4\pi R^2\frac{dp}{d\psi} = \left(\frac{4\pi R_0^2}{\psi'}\right)p'\left(1 + 2\frac{r}{R_0}\cos\vartheta\right)$$

GRAD-SHAFRANOV EQUATION - 2

- **Combining ϑ - independent terms first:**

$$\frac{1}{r}\left(r\psi'\right)' + \frac{II'}{\psi'} + \frac{4\pi R_0^2}{\psi'}p' = 0 \rightarrow \frac{1}{q}\left(\frac{r^2}{q}\right)' + \frac{II'}{B_0^2} + \frac{4\pi R_0^2}{B_0^2}p' = 0$$

where the safety factor q was introduced from $\psi' = rB_T / q \cong rB_0 / q$

- **Thus, only *TWO of the three flux quantities* q, I, and p may be specified *independently***

GRAD-SHAFRANOV EQUATION – 3

- **Combining terms $\propto \cos\vartheta$ then:**

$$\frac{1}{r}\left(2r\Delta'\psi'\right)' - \frac{1}{r}\psi'\left(\frac{r}{R_0} + \left(r\Delta'\right)'\right) + \frac{2r}{R_0}\frac{4\pi R_0^2}{\psi'}p' = 0 \qquad \rightarrow \qquad \left(r\Delta'\psi'^2\right)' = \frac{r}{R_0}\left(\psi'^2 - 8\pi\, rR_0^2 p'\right)$$

- **Solving this equation for the Shafranov shift, we obtain:**

$$\Delta' = \frac{1}{r^2\psi'^2}\cdot\frac{r}{R_0}\int_0^r rdr\left(\psi'^2 - 8\pi rR_0^2 p'\right) = \frac{r}{R_0}\left(\frac{\langle\psi'^2\rangle}{2\psi'^2} - \frac{8\pi R_0^2}{\psi'^2}\left(p - \langle p\rangle\right)\right) = \frac{r}{R_0}\left(\frac{\langle B_p^2\rangle}{2B_p^2} + \frac{8\pi}{B_p^2}\left(\langle p\rangle - p\right)\right)$$

where $\langle p\rangle \equiv \frac{2}{r^2}\int_0^r rdr\cdot p(r)$

- **Introduce internal inductance** $l_i = \langle B_p^2\rangle / B_p^2$ **and poloidal beta** $\beta_p = \frac{8\pi}{B_p^2}\left(\langle p\rangle - p\right)$ **to obtain**

$$\Delta' = \frac{r}{R_0}\left(\frac{l_i}{2} + \beta_p\right)$$

TYPICAL JET EQUILIBRIUM

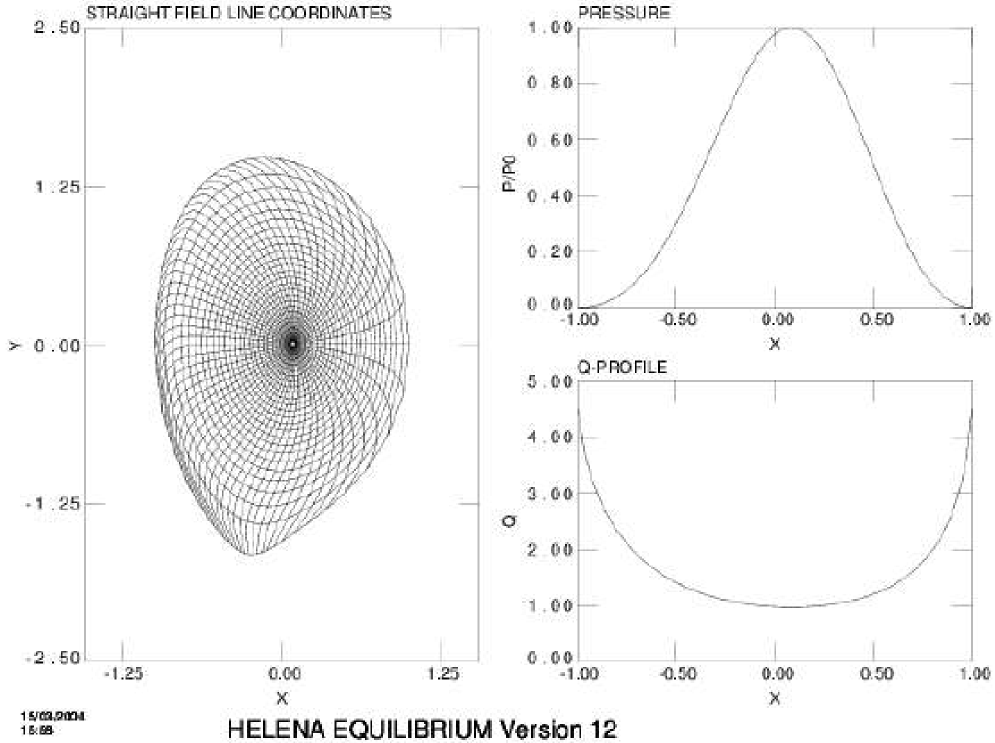


Figure 6: A typical output page from HELENA. $X = (R - R_0)/a$, $Y = z/a$. Left: Equilibrium magnetic flux structure. Upper right: Equilibrium pressure profile. Lower right: q-profile.

SUMMARY

- Plasma equilibrium is governed by equation $\nabla p_0 = \frac{1}{c}\mathbf{J}_0 \times \mathbf{B}_0$ which gives $\mathbf{B}_0 \nabla p_0 = 0, \quad \mathbf{J}_0 \nabla p_0 = 0$, i.e. plasma pressure is constant along magnetic field lines and along the current lines; the plasma may expand freely along these lines
- Since ∇p_0 is perpendicular to the surface $p_0 = const$, the magnetic field lines and the current lines must lie on the $p_0 = const$ surfaces
- Shafranov coordinates and flux-type coordinates differ by the poloidal angle variable. The approximate relation is $\vartheta_S = \vartheta_f + \left(\frac{r}{R_0} + \Delta'\right)\sin\vartheta_f$
- Grad-Shafranov equation has the form $R^2\nabla\cdot\left(\frac{1}{R^2}\nabla\psi\right) + I\frac{dI}{d\psi} + 4\pi R^2\frac{dp}{d\psi} = 0$
- Only two of the three flux quantities q, I, and p may be specified independently
- The internal inductance $l_i = \langle B_p^2\rangle / B_p^2$ and poloidal beta $\beta_p = \frac{8\pi}{B_p^2}(\langle p\rangle - p)$ determine Δ' as

$$\Delta' = \frac{r}{R_0}\left(\frac{l_i}{2} + \beta_p\right)$$