MHD WAVES AND GLOBAL ALFVÉN EIGENMODES

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OUTLINE OF LECTURE 4

- Linearised MHD equations
- MHD waves
- Compressional waves
- Shear Alfvén waves
- Kinetic Alfvén waves
- Global Alfvén Eigenmodes
- Summary

ALFVÉN INSTABILITIES DRIVEN BY ALPHA-PARTICLES

- Alpha-particles (He^4 ions) are born in deuterium-tritium nuclear reactions with birth energy 3.52 MeV. These fusion-born ions are *super-Alfvénic*,

$$V_{Ti} \sim 10^6 \text{ m/s} \ll V_A \sim 5\cdot 10^6 \text{ m/s} < V_\alpha = 1.3\cdot 10^7 \text{ m/s} \ll V_{Te} \sim 8\cdot 10^7 \text{ m/s},$$

where $V_A = B_0/(\mu_0 n_i m_i)^{1/2}$, and the estimate is for D:T=50:50 ITER plasmas

- During slowing-down of alpha-particles, they pass the resonance condition $V_A = V_{\parallel\alpha}$ and may excite Alfvén waves with $\omega = k_{\parallel} V_A$ dispersion relation
- Free energy source: radial gradient of alpha-particle pressure. The instability results in radial re-distribution of alpha-particles
- Re-distribution may also cause losses of highly energetic alphas → damage to the first wall
- We have to assess possible wave-particle interaction in burning plasmas

STARTING MHD EQUATIONS

- For describing *plasma particles*, we take velocity moments of the kinetic equations for electron and thermal ion distribution functions and obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0;$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \frac{1}{c}\mathbf{J} \times \mathbf{B};$$

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \gamma \; p \nabla \cdot \mathbf{V} = 0;$$

$$\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B} = 0;$$

- For describing *electromagnetic fields* in the plasma, Maxwell's equations are used

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial \mathbf{t}}$$

$$\nabla \cdot \mathbf{B} = 0$$

- Here, scale lengths larger than Debye length are considered with $n_e = \sum_i Z_i \cdot n_i$

THE LINEARISATION PROCEDURE

- All the field and plasma variables are represented as sums of equilibrium (denoted by subscript 0) and perturbed (denoted by δ) quantities:

$$\mathbf{J} = \mathbf{J}_0 + \delta\mathbf{J}\,,\;\; \mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}\,,\;\; \mathbf{V} = \delta\mathbf{V}\,,\;\; p = p_0 + \delta p\,,\;\; \rho = \rho_0 + \delta\rho\,,\;\; \mathbf{E} = \delta\mathbf{E}\,, \qquad (*)$$

  where all the perturbed quantities satisfy $\delta << 1$, i.e. $|\delta\mathbf{J} / \mathbf{J}_0| << 1$ etc.

- Substitute the expressions (*) in the starting set of equations and obtain equations with terms: a) not having δ at all; b) having δ; c) having δ^2 etc.

- The terms NOT having δ are balanced thanks to the plasma equilibrium

$$\nabla p_0 = \frac{1}{c}\mathbf{J}_0 \times \mathbf{B}_0$$

  The relation between equilibrium quantities $\mathbf{J}_0$, p_0, B_0 MUST be kept in all equations with δ

- All the equations are linearised then, i.e. only linear terms in δ are kept and terms with δ^2 etc. are dropped off as small (since $\delta << 1$)

LINEARISED MHD EQUATIONS

- **The linearised ideal MHD equations take the form:**

$$\frac{\partial \delta\rho}{\partial t} + \nabla\cdot\left(\rho_0 \delta\mathbf{V}\right) = 0\,;$$

$$\rho_0 \frac{d\delta\mathbf{V}}{dt} = -\nabla\delta p + \frac{1}{4\pi}\left[\nabla\times\delta\mathbf{B}\right]\times\mathbf{B}_0\,;$$

$$\frac{\partial}{\partial t}\delta\mathbf{B} = \nabla\times\left[\delta\mathbf{V}\times\mathbf{B}_0\right];$$

$$\delta p = \gamma\ \frac{p_0}{\rho_0}\delta\rho\,;$$

- **Introduce plasma displacement from the equilibrium, ξ, related to $\delta\mathbf{V}$ via**

$$\delta\mathbf{V} = \partial\xi / \partial t$$

- **From the first and third equations we find then**

$$\delta\rho = -div\left(\rho_0\xi\right);\ \ \delta\mathbf{B} = \nabla\times\left[\xi\times\mathbf{B}_0\right] = -\mathbf{B}_0 div\xi_\perp + \mathbf{B}_0\frac{\partial\xi_\perp}{\partial z}$$

where we used $\nabla\times\left[\mathbf{a}\times\mathbf{b}\right] = \left(\mathbf{b}\nabla\right)\mathbf{a} - \left(\mathbf{a}\nabla\right)\mathbf{b} + \mathbf{a}\ \ div\mathbf{b} - \mathbf{b}\ \ div\mathbf{a}$**, and** $\mathbf{B}_0 \uparrow\uparrow \mathbf{e}_z$

EQUATION FOR IDEAL MHD WAVES

- **Substitute the expressions for $\delta\rho,\ \delta\mathbf{B}$ in the remaining two equations and obtain**

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = c_S^2 \nabla div\boldsymbol{\xi} + V_A^2 \nabla_\perp div\boldsymbol{\xi}_\perp + V_A^2 \frac{\partial^2 \boldsymbol{\xi}_\perp}{\partial z^2},$$

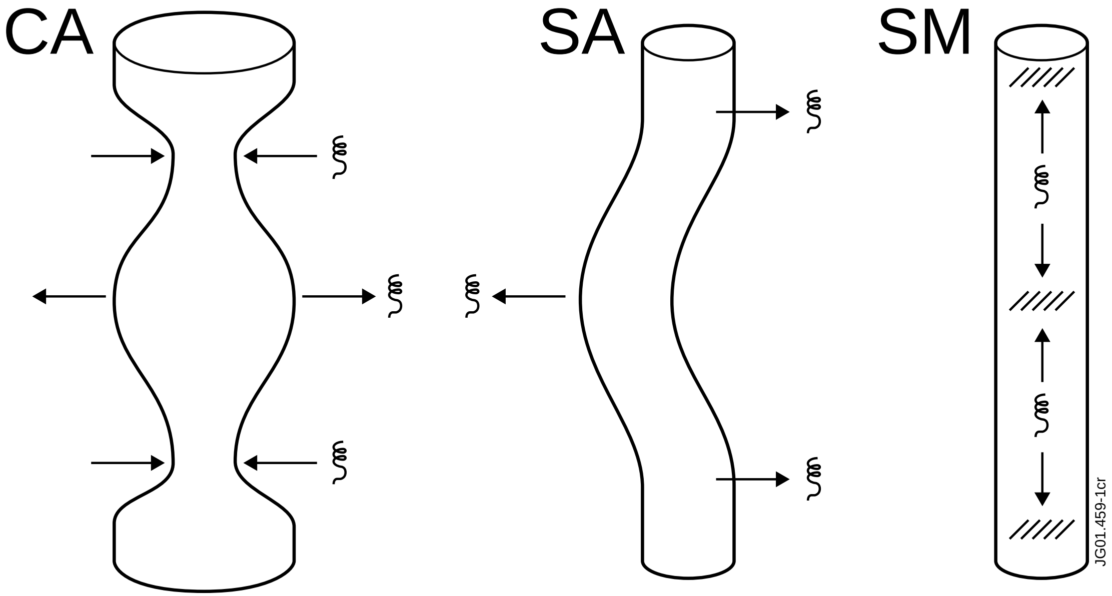
Where $c_S^2 = \gamma\ p_0 / \rho_0$ is the ion sound speed, $V_A^2 = B_0^2 / (4\pi\rho_0)$ is the Alfvén velocity

- **This equation describes *linear MHD perturbations of homogeneous ideally conducting plasma*. Single vector equation gives three scalar equations for *three types of waves***

PLASMA DISPLACEMENT IN MHD WAVES

- *Compressional Alfvén and slow magnetosonic waves:* the "returning" force is the magnetic and the kinetic pressure

- *Shear Alfvén wave:* the "returning" force is the tension of magnetic field lines



COMPRESSIONAL WAVES - 1

- Coming back to the main equation

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = c_S^2 \nabla div \boldsymbol{\xi} + V_A^2 \nabla_\perp div \boldsymbol{\xi}_\perp + V_A^2 \frac{\partial^2 \boldsymbol{\xi}_\perp}{\partial z^2} \quad (*)$$

- Consider two "compressible" types of waves, in which $\xi_z \neq 0$ and $div \boldsymbol{\xi}_\perp \neq 0$
- The parallel displacement ξ_z is described by the parallel projection of equation (*):

$$\frac{\partial^2 \xi_z}{\partial t^2} = c_S^2 \frac{\partial^2 \xi_z}{\partial z^2} + c_S^2 \frac{\partial}{\partial z} div \boldsymbol{\xi}_\perp$$

- We obtain equation for $div \boldsymbol{\xi}_\perp$ by taking divergence of perpendicular projection of (*):

$$\frac{\partial^2 div \boldsymbol{\xi}_\perp}{\partial t^2} = c_S^2 \Delta_\perp div \boldsymbol{\xi}_\perp + V_A^2 \left(\Delta_\perp + \frac{\partial^2}{\partial z^2} \right) div \boldsymbol{\xi}_\perp + c_S^2 \Delta_\perp \frac{\partial \xi_z}{\partial z}$$

  Here, $\Delta_\perp = div \nabla_\perp$

- We see, that two equations for ξ_z and $div \boldsymbol{\xi}_\perp$ are ***coupled***

COMPRESSIONAL WAVES - 2

- Consider limit $\beta \approx c_S^2 / V_A^2 << 1$. In this case, equation for $div\boldsymbol{\xi}_\perp$ reduces to

$$\frac{\partial^2 div\boldsymbol{\xi}_\perp}{\partial t^2} = V_A^2 \Delta \; div\boldsymbol{\xi}_\perp$$

  which decouples from ξ_z and describes *Compressional Alfvén Wave*. The magnetic pressure $B_0^2 / 8\pi$ determines the "returning" force that acts *perpendicular* to $\mathbf{B}_0$

- The displacement ξ_z *parallel* to $\mathbf{B}_0$ is described by

$$\frac{\partial^2 \xi_z}{\partial t^2} = c_S^2 \frac{\partial^2 \xi_z}{\partial z^2}$$

  and such wave corresponds to *Ion Sound Wave* existing in plasma even without $\mathbf{B}_0$

- However, if β is *not* small, the decoupling of compressible waves does not work! The ion sound wave is modified then by the magnetic pressure and becomes *Slow Magnetosonic Wave*. The coupled equations give for ξ_z, $div\boldsymbol{\xi}_\perp \propto \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$ the following relation between wave frequency and wave vector (dispersion relation):

$$\omega^4 - \left(c_S^2 + V_A^2\right)k^2\omega^2 + V_A^2 c_S^2 k_z^2 k^2 = 0$$

SHEAR ALFVÉN WAVES

- In contrast to the compressional waves, the third type of MHD waves, so-called *Shear Alfvén wave*, is *incompressible*:

$$\xi_z = 0 \text{ and } div\, \boldsymbol{\xi}_\perp = 0$$

- For such waves the main MHD equation becomes simply

$$\frac{\partial^2 \boldsymbol{\xi}_\perp}{\partial t^2} = V_A^2 \frac{\partial^2 \boldsymbol{\xi}_\perp}{\partial z^2}$$

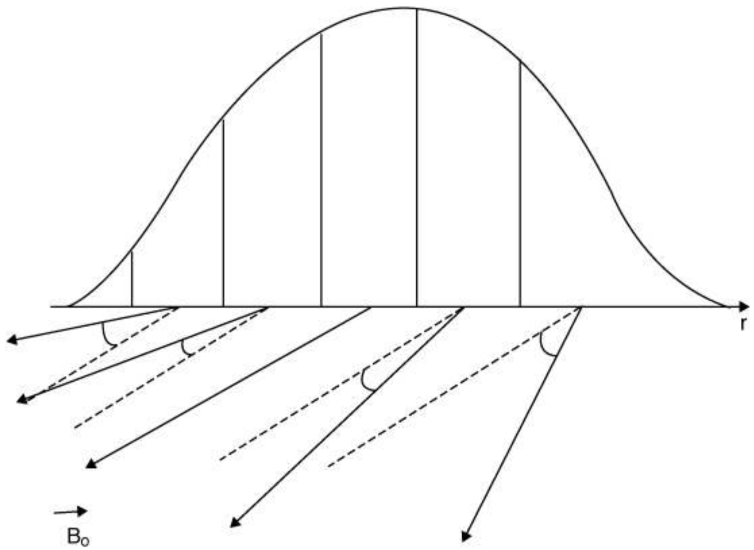
  which coincides with equation for string oscillations. The "returning" force is the tension of magnetic field lines, which act similarly to the strings

- In shear Alfvén wave the fluid displacement vector $\boldsymbol{\xi}$ and $\tilde{\mathbf{E}}$ are perpendicular to the magnetic field $\mathbf{B}_0$. The wave propagates along $\mathbf{B}_0$:

$$\omega = \pm k_\parallel V_A \text{ ; } V_A = \frac{B_0}{\sqrt{4\pi \sum_i n_i M_i}} \text{ ; } \quad k_\parallel = \mathbf{k} \cdot \mathbf{B}_0 / B_0$$

- Among all the waves in plasmas, the Alfvén wave (*H. Alfvén, Arkiv. Mat. Astron. Fysik 29B(2) (1942)*) constitutes the most significant part of the MHD spectrum and is probably the best studied

HOW THIS WAVE EVOLVES IN *INHOMOGENEOUS* PLASMA?



$$\omega = k_{\parallel}(r) \cdot V_A(r)$$

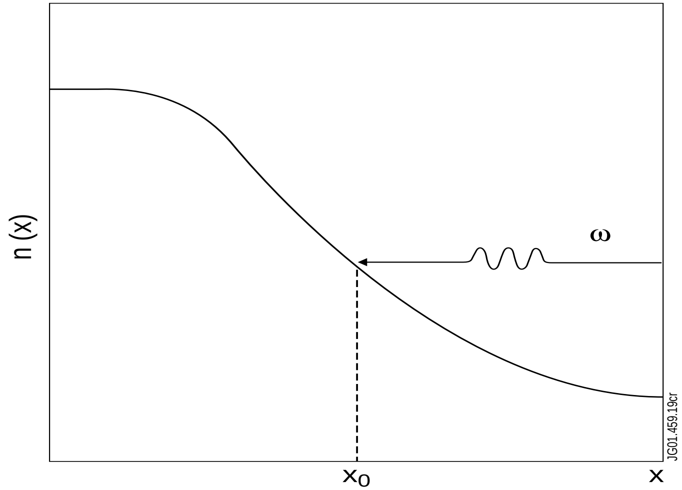
- **The life-time of a wave-packet of shear Alfvén waves is limited by the "phase mixing"**

$$\tau^{-1} \propto \frac{d}{dr}\left(k_{\parallel}(r) \cdot V_A(r)\right)$$

- **If SA wave packet cannot live long, why to care about SA interacting with fast ions?**
- **Radial gradients of plasma may modify such waves of comparable wavelength.**

INHOMOGENEOUS PLASMA - 1

- **Consider in detail the model: 1-D slab non-uniform cold plasma,** $n_0 = n_0(x)$, $P_0 = 0$, $\mathbf{B}_0 = B_0\mathbf{e}_Z$,



- **The presence of *plasma gradients* modifies our MHD equation. An externally excited electromagnetic wave with perturbed** $\phi \propto \phi(x)\exp(ik_y y - i\omega t)$ **is described by**

$$\frac{d}{dx}\left(\omega^2 - \omega_A^2(x)\right)\frac{d\phi}{dx} - k_y^2\left(\omega^2 - \omega_A^2(x)\right)\phi = 0$$

$$\omega_A^2(x) \equiv k_\parallel^2 V_A^2(x)$$

INHOMOGENEOUS PLASMA - 2

- This equation has *zero coefficient at high order derivative at the point* $x = x_0$ of the local Alfvén resonance layer, where

$$\omega^2 = \omega_A^2(x_0)$$

- Investigating the equation in the vicinity of this point:

$$\frac{d}{dx}\left(\omega^2 - \omega_A^2(x)\right)\frac{d\phi}{dx} = 0 \quad \longrightarrow \quad \frac{d\phi}{dx} = \frac{const}{\omega^2 - \omega_A^2(x)}$$

- Expand the local Alfvén frequency in the vicinity of the point $x = x_0$:

$$\omega^2 = \omega_A^2(x_0) + d\omega_A^2(x)/dx\Big|_{x=x_0} \cdot (x - x_0)$$

and obtain

$$\phi \propto const \cdot \ln(x - x_0), \quad x > x_0$$

$$\phi \propto const \cdot \left(\ln|x - x_0| + i\pi\right), \quad x < x_0$$

The *wave energy peaks up at* $x = x_0$ and resonant absorption of the wave energy occurs at this point – *continuum damping*

LINEAR MODE CONVERSION TO KINETIC ALFVÉN WAVE

- **Finite ion Larmor radius & finite electron parallel conductivity incorporated in the layer $|x - x_0| \approx \rho_i$ remove the wave singularity**
- **Wave equation takes the form $(m_e / M_i << \beta_e << 1)$:**

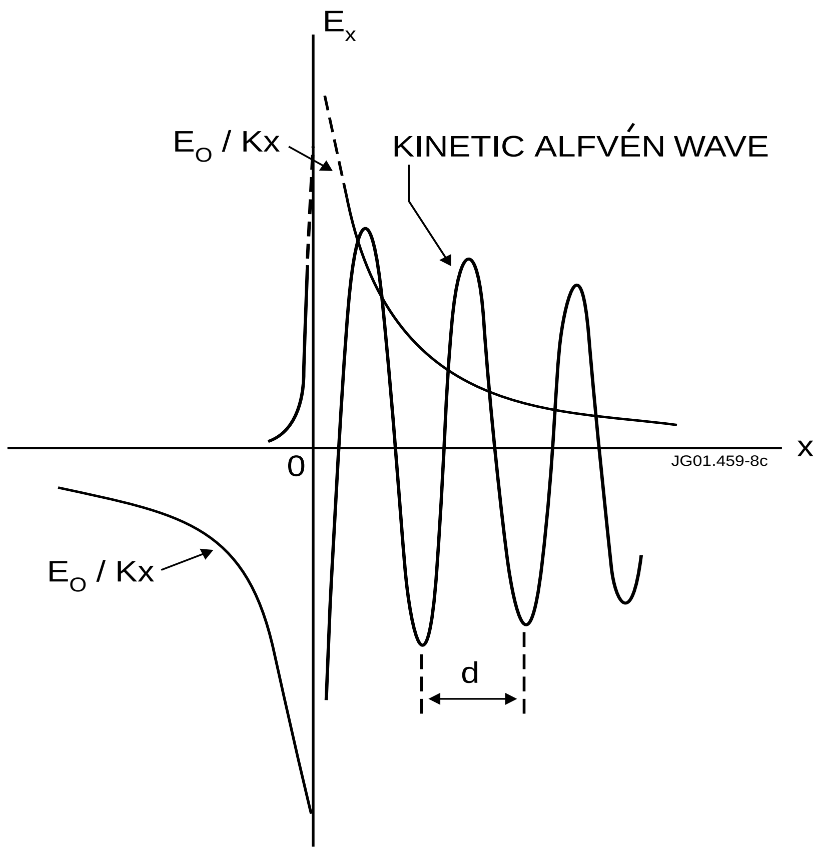
$$\omega^2 \rho_i^2 \nabla_\perp^2 \left(\frac{3}{4}(1 - i\delta_i) + \frac{T_e}{T_i}(1 - i\delta_e) \right) \nabla_\perp^2 \phi + + \frac{d}{dx}\left(\omega^2 - \omega_A^2(x)\right)\frac{d\phi}{dx} - k_y^2\left(\omega^2 - \omega_A^2(x)\right)\phi = 0$$

- **Solution of this equation satisfies the dispersion relation in the form of Kinetic Alfvén Wave:**

$$\omega^2 = k_\parallel^2 V_A^2(x)\left(1 + (k_x \rho_i)^2 \left(\frac{3}{4}(1 - i\delta_i) + \frac{T_e}{T_i}(1 - i\delta_e) \right)\right)$$

- **In contrast to the shear Alfvén wave, KAW propagates across $\mathbf{B}_0$, $\partial\omega_{KAW} / \partial k_x \neq 0$, and it has $\tilde{E}_\parallel \neq 0$**

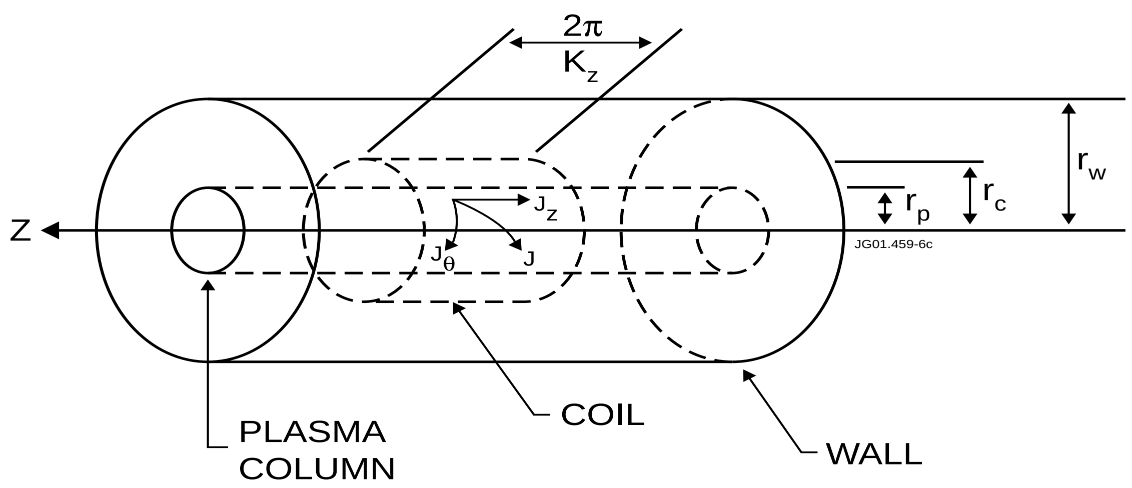
LINEAR MODE CONVERSION TO KINETIC ALFVÉN WAVE



DO ANY SHEAR ALFVÉN WAVES EXIST WHICH DO NOT SATISFY A LOCAL SA DISPERSION RELATION AND ARE FREE OF THE CONTINUUM DAMPING?

DISCOVERY OF GLOBAL ALFVÉN EIGENMODE

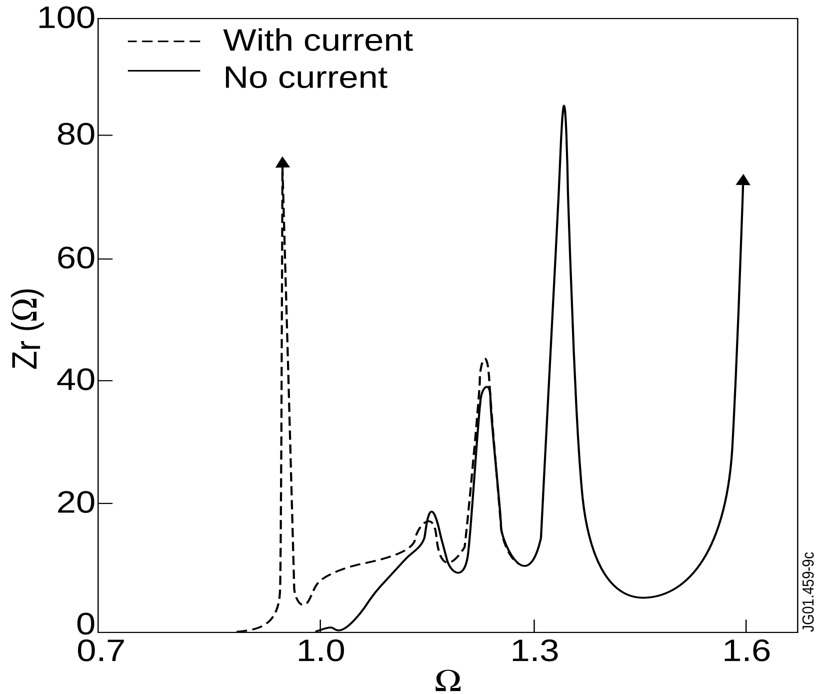
- **In cylindrical geometry, in addition to the continuous Alfvén spectrum, $\omega^2 = \omega_A^2(r) \equiv k_\parallel^2(r) V_A^2(r)$, a discrete Global Alfvén Eigenmode with frequency $\omega_{GAE} < \omega_A$ exists in plasma with current (***D.W.Ross et al. Phys. Fluids 25, 652 (1982);K.Appert et al. Plasma Phys. 24, 1147 (1982)*)**



The ideal plasma-coil-wall system used in the numerical investigation

DISCOVERY OF GLOBAL ALFVÉN EIGENMODE

- **A new high-quality, $Q \equiv \omega/\gamma \sim 10^3$, resonance was discovered during these Alfvén antenna studies, in plasmas with current**

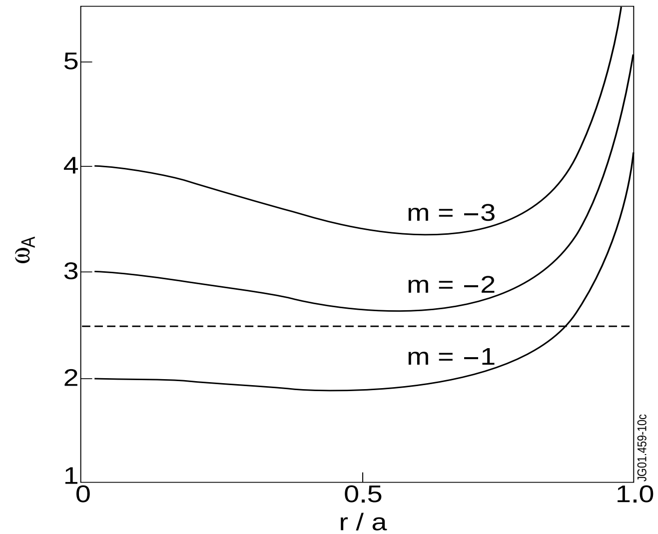


Real part of the coil impedance vs normalized frequency

GLOBAL ALFVÉN EIGENMODE

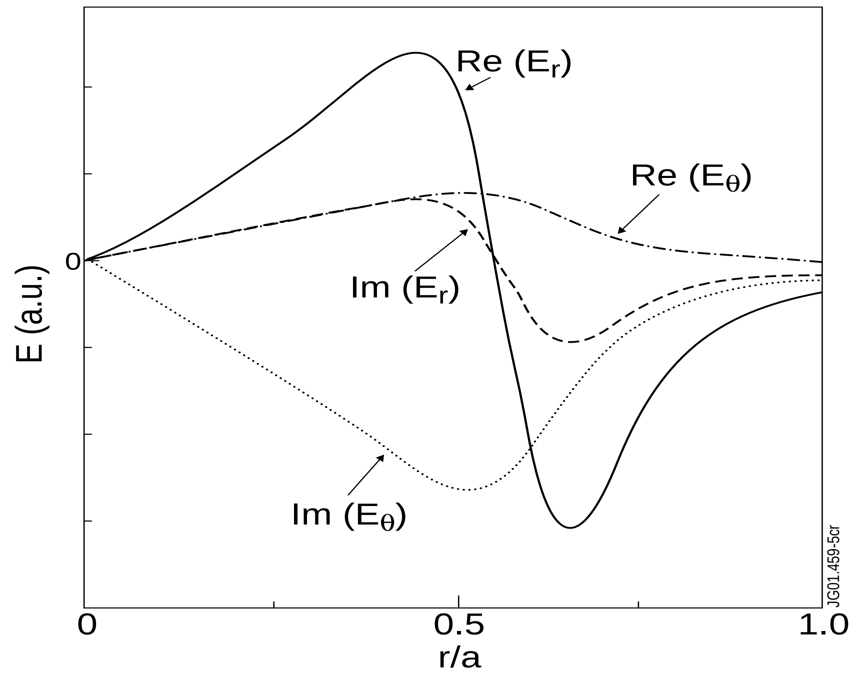
- **GAE with $\omega_{GAE} < \omega_A$ exists in Ideal MHD if the current profile determining dB_ϑ / dr provides a minimum in Alfvén continuum**

$$\left.\frac{d\omega_A(r)}{dr}\right|_{r=r_0} = 0, \text{ i.e. } \frac{1}{k_\parallel}\frac{dk_\parallel}{dr} = -\frac{1}{V_A}\frac{dV_A}{dr}$$



- **The local minimum of the Alfvén continuum provides a *maximum of the perpendicular refraction index* $N_r = ck_r / \omega$. Similarly to fiber optics, the *electromagnetic wave* has to propagate in a *"wave-guide" surrounding the region of the extremum refraction index.***

NO CONTINUUM DAMPING FOR GLOBAL ALFVÉN EIGENMODE



Ideal MHD GAE with m=-2

- **The eigenfrequency of GAE does *not* satisfy the local Alfvén resonance condition, $\omega_{GAE} \neq \omega_A(r)$, for $0 < r/a < 1$. Therefore, this SA mode has *no* singularity and does *not* experience *continuum damping***

SUMMARY

- **Linearised ideal MHD equations give the equation for plasma displacement**

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = c_S^2 \nabla div \boldsymbol{\xi} + V_A^2 \nabla_\perp div \boldsymbol{\xi}_\perp + V_A^2 \frac{\partial^2 \boldsymbol{\xi}_\perp}{\partial z^2}$$

which describes *compressional Alfvén, slow magnetosonic, and shear Alfvén waves*

- **The compressional waves can be decoupled at low-β, and they are coupled otherwise**
- **In shear Alfvén wave the fluid displacement vector $\boldsymbol{\xi}$ and $\tilde{\mathbf{E}}$ are perpendicular to the magnetic field $\mathbf{B}_0$. The wave propagates along $\mathbf{B}_0$: $\omega = \pm k_\parallel V_A$**
- **In inhomogeneous plasma, shear Alfvén wave experiences strong continuum damping**
- **The continuum damping in ideal MHD corresponds to linear mode conversion to *short waved-length Kinetic Alfvén Wave*, which has a radial group velocity**
- **In cylindrical plasma with current, a discrete eigenmode may exist, Global Alfvén Eigenmode, which has NO continuum damping**