



NEAR-THRESHOLD NONLINEAR EVOLUTION OF ENERGETIC PARTICLE DRIVEN WAVES

S.E. Sharapov

*Euratom/CCFE Fusion Association,
Culham Science Centre, Abingdon, Oxfordshire OX14 3DB, UK*



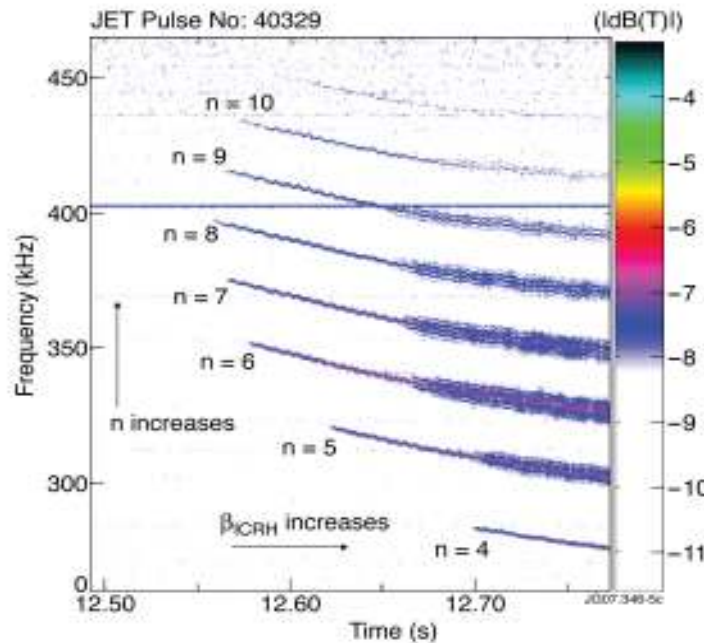
S.E.Sharapov, Lecture 5A, Australian National University, Canberra, 7-9 July 2010



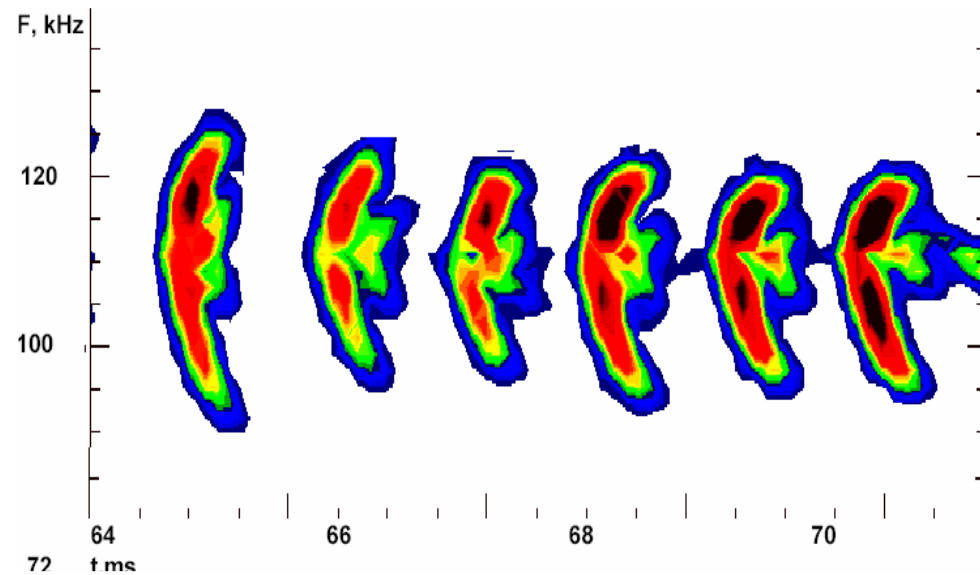
OUTLINE OF LECTURE 5A

- Nonlinear evolution near the threshold: qualitative analysis
- Nonlinear Berk-Breizman equation
- Experimental observation of the pitchfork splitting
- Experimental observation of the chaotic evolution
- Experimental observation of the explosive instability
- Summary

DIFFERENT REGIMES OF MODE EVOLUTION ARE OBSERVED



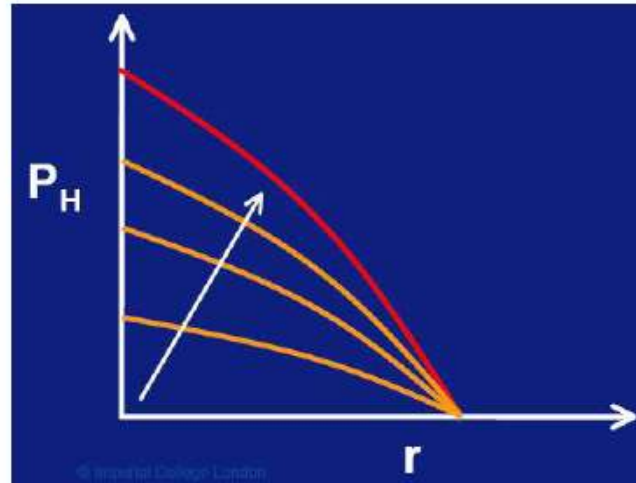
ICRH-driven TAEs during ICRH power ramp-up on JET



NBI-driven bursting TAEs on MAST

THE NEAR-THRESHOLD CONDITION

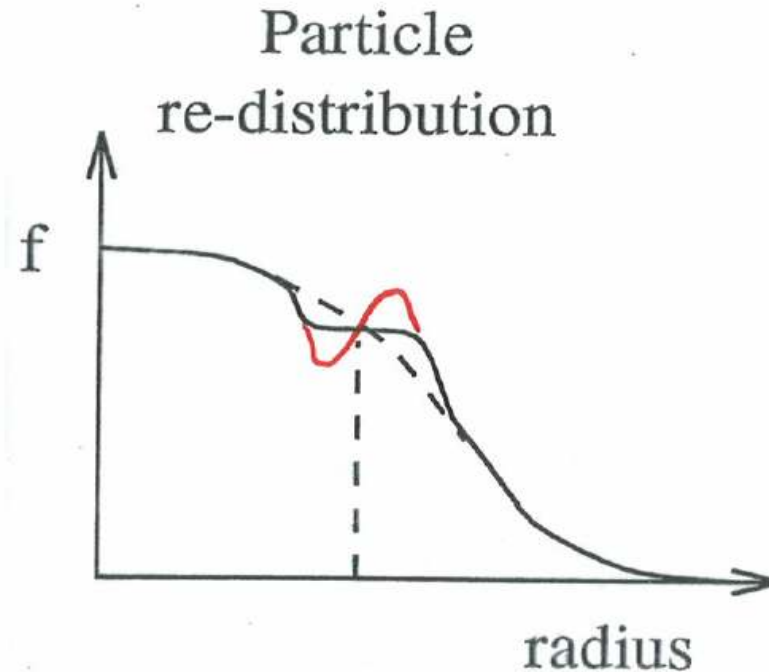
- Consider the scenario with a *gradual build-up of fast ion pressure* so that the fast ion drive of TAE, $\gamma_\alpha(t) \propto -\beta'_\alpha(t)$, increases in time at unchanged TAE damping γ_d



- TAE *instability threshold*: exact balance between TAE drive and damping, $\gamma_\alpha = \gamma_d$
- The *near-threshold* condition:

$$|\gamma_\alpha - \gamma_d| \ll \gamma_d \leq \gamma_\alpha$$

HOW TAE INSTABILITY SATURATES?



- Non-linear TAE behaviour: competition between *the field of the mode* that tends to *flatten* distribution function near the resonance (effect proportional to the net growth rate $\gamma \equiv \gamma_L - \gamma_d$) and *the collision-like processes* that constantly *replenish* it (proportional to v_{eff})

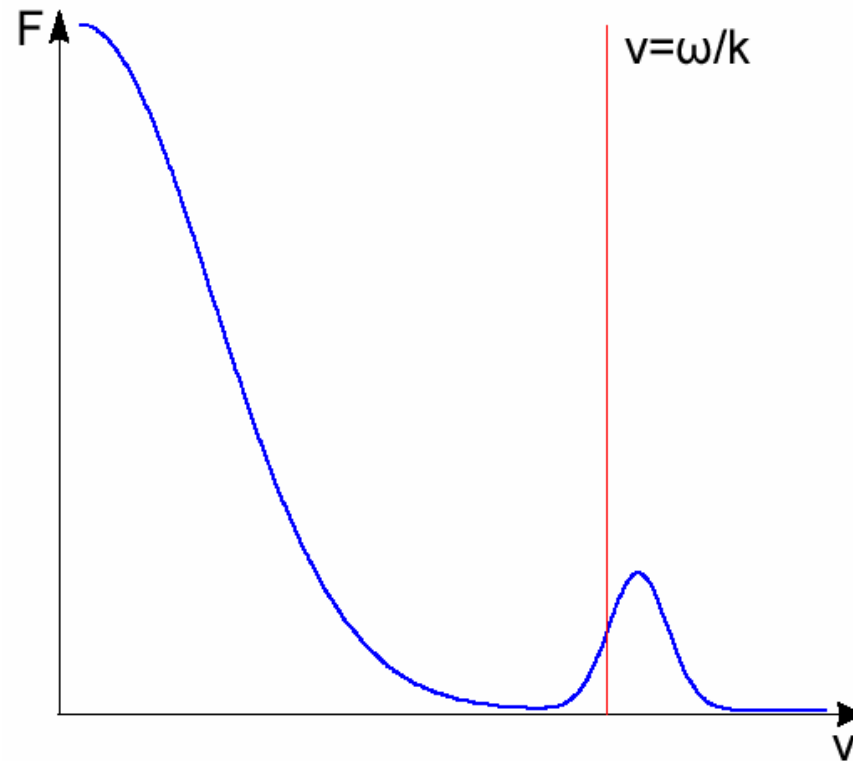
COLLISIONALITY

- The near-threshold regime *allows the “collisions”* restoring the unstable distribution function of fast ions *to compete with the mode growth*

$$|\gamma_{\alpha} - \gamma_d| \approx \nu_{eff}$$

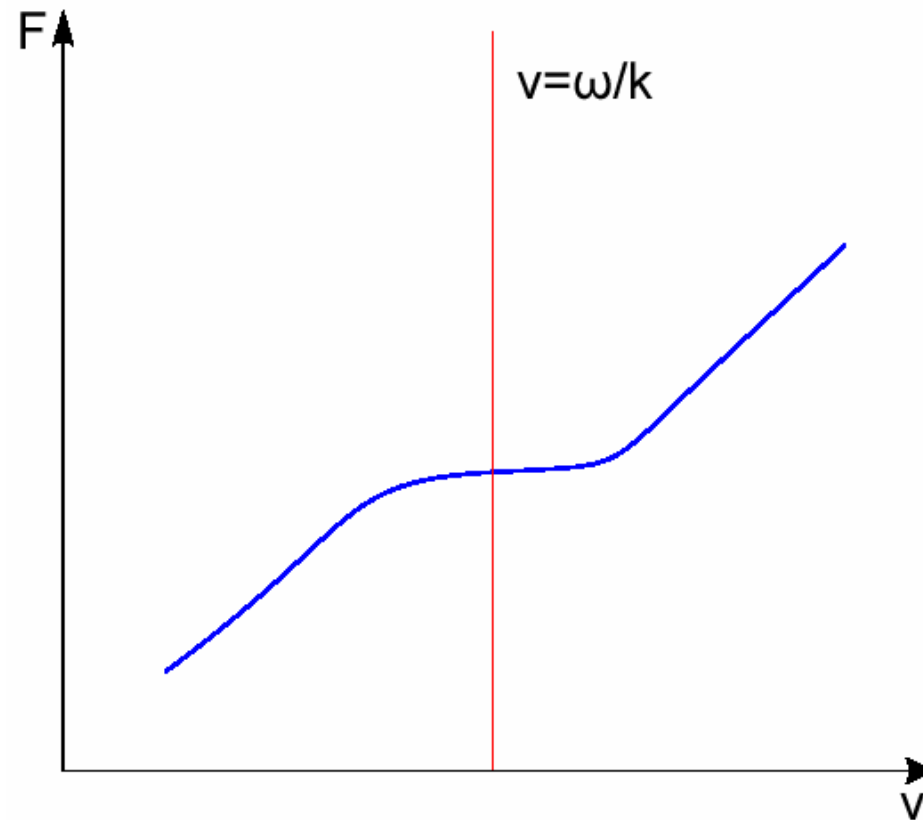
- Demonstrate this effect *analytically for the “bump-on-tail”* problem in a *1D velocity* space. This problem has physics similar to TAE, but is 1D.
- The “bump on tail” problem: consider the *nonlinear* evolution of a *marginally unstable* electrostatic wave with frequency $\omega = \omega_{pe} = \sqrt{4\pi n_e e^2 / m_e}$ in the presence of an unstable beam distribution function $F(x, v, t)$ with *collisional operator* (Berk et al., PRL 76 1256 (1996))

THE 1D BUMP ON TAIL INSTABILITY CAUSED BY $dF/dv > 0$



$$\left. \frac{\partial F_0}{\partial v} \right|_{\frac{\omega}{k}} > 0$$

QUASI-LINEAR PLATEAU IS FORMED IN *NONLINEAR* PHASE



STARTING EQUATIONS FOR THE BUMP ON TAIL PROBLEM

- Consider $F(x, v, t)$ in the presence of a wave with electric field

$$E = \frac{1}{2} \left[\hat{E}(t) e^{i(kx - \omega t)} + \text{c.c} \right]$$

- In order to accommodate the collisions, a Fokker-Planck equation has to be solved:

$$\frac{\partial F}{\partial t} + v \frac{\partial F}{\partial x} + \frac{e}{2m} \left[\hat{E}(t) e^{i(kx - \omega t)} + \text{c.c} \right] \frac{\partial F}{\partial v} = \left. \frac{dF}{dt} \right|_{\text{coll}}$$

together with Maxwell's equations for electric field ($\delta B = 0$ for this problem):

$$\left[-i\omega_{pe} \frac{\partial \hat{E}(t)}{\partial t} e^{i(kx - \omega t)} + \text{c.c} \right] + 4\pi \frac{\partial j_f}{\partial t} = 0$$

where j_f is the fast particle contribution to perturbed current produced by the wave.

THE COLLISIONAL OPERATOR

- Only resonant particles contribute in the non-linear wave evolution, so the competing collisional operator can be taken in the vicinity of the resonance:

$$\left. \frac{dF}{dt} \right|_{\text{coll}} = \alpha^2 \left(\frac{\partial F}{\partial u} - \frac{\partial F_0}{\partial u} \right) + \nu^3 \left(\frac{\partial^2 F}{\partial u^2} - \frac{\partial^2 F_0}{\partial u^2} \right) - \beta (F - F_0)$$

where

$$u = kv - \omega$$

- Here, α , ν , β are coefficients of drag, diffusion, and Krook operator respectively. For the Krook operator the coefficient is constant, but for drag and diffusion these are taken at the resonant point.

THE WEAK NONLINEARITY APPROACH – 1

- Consider the equation with both perturbed electric field of the wave and the collisions

$$\begin{aligned} \frac{\partial F}{\partial t} + \left(\frac{u + \omega}{k} \right) \frac{\partial F}{\partial x} + \frac{ek}{2m} \left[\hat{E}(t) e^{i(kx - \omega t)} + \text{c.c} \right] \frac{\partial F}{\partial u} - \nu^3 \frac{\partial^2 F}{\partial u^2} \\ - \alpha^2 \frac{\partial F}{\partial u} + \beta F = -\nu^3 \frac{\partial^2 F_0}{\partial u^2} - \alpha^2 \frac{\partial F_0}{\partial u} + \beta F_0 \end{aligned}$$

- For the sinusoidal wave we represent the distribution function as a Fourier series

$$F = F_0 + f_0 + \sum_{n=1}^{\infty} [f_n \exp(in\psi) + \text{c.c}]$$

$$\psi = kx - \omega t$$

- The wave equation relating the field and the fast particle current becomes then

$$\frac{\partial \hat{E}}{\partial t} + 4\pi e \frac{\omega}{k^2} \int f_1 du + \gamma_d \hat{E} = 0$$

THE WEAK NONLINEARITY APPROACH – 2

- Consider time scales shorter than nonlinear bounce period of the wave. With the distribution function being not too significantly perturbed, i.e. within the ordering

$$F_0 \gg f_1 \gg f_0, f_2$$

f_1 admits a power series in $\hat{E}(t)$

$$f_1 \approx C_1 \hat{E} + C_3 \hat{E}^3 + \dots$$

, which allows the first order (cubic) nonlinearity to be captured by the following truncated Fourier expansion:

$$\begin{aligned} \frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} - \alpha^2 \frac{\partial f_0}{\partial u} + \beta f_0 &= -\frac{ek}{2m} \left(\hat{E} \frac{\partial f_1^*}{\partial u} + \text{c.c} \right) \\ \frac{\partial f_1}{\partial t} + iuf_1 - \nu^3 \frac{\partial^2 f_1}{\partial u^2} - \alpha^2 \frac{\partial f_1}{\partial u} + \beta f_1 &= -\frac{ek}{2m} \hat{E} \frac{\partial}{\partial u} (F_0 + f_0 + f_2) \\ \frac{\partial f_2}{\partial t} + 2iuf_2 - \nu^3 \frac{\partial^2 f_2}{\partial u^2} - \alpha^2 \frac{\partial f_2}{\partial u} + \beta f_2 &= -\frac{ek}{2m} \hat{E}^* \frac{\partial f_1}{\partial u} + \mathcal{O}(\hat{E} f_3) \end{aligned}$$

THE WAVE AMPLITUDE EQUATION

$$\frac{dA}{d\tau} = A(\tau) - \frac{1}{2} \int_0^{\tau/2} dz z^2 A(\tau - z) \int_0^{\tau-2z} dx e^{-\hat{\nu}^3 z^2 (2z/3+x) - \hat{\beta}(2z+x) + i\hat{\alpha}^2 z(z+x)} \times A(\tau - z - x) A^*(\tau - 2z - x)$$

where $A = \left[ek\hat{E}(t) / m(\gamma_l - \gamma_d)^2 \right] [\gamma_l / (\gamma_l - \gamma_d)]^{1/2}$, $\tau = (\gamma_l - \gamma_d)t$, $\hat{\nu}^3 = \nu^3 / (\gamma_l - \gamma_d)^3$, $\hat{\alpha} = \alpha / (\gamma_l - \gamma_d)^2$, $\hat{\beta} = \beta / (\gamma_l - \gamma_d)$ and $\gamma_l = 2\pi^2 (e^2\omega / mk^2) \partial F_0(\omega/k) / \partial v$.

DIFFUSION ONLY CASE

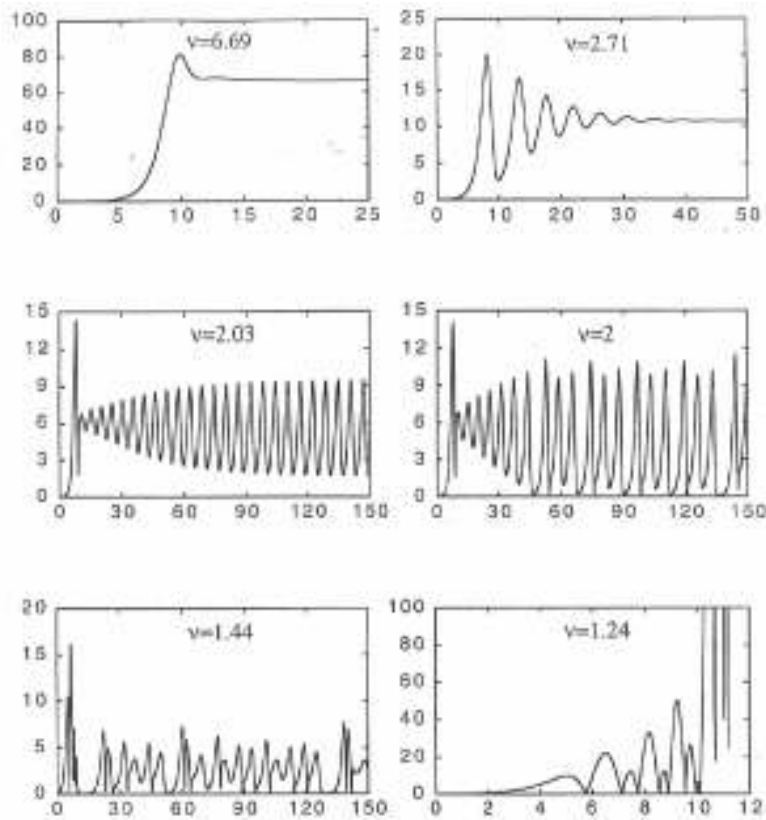
Nonlinear equation for the amplitude

$$\frac{dA}{dt} = A - \frac{1}{2} \int_0^{t/2} \tau^2 \int_0^{t-2\tau} \exp[-\nu^3 \tau^2 (2\tau/3 + \tau_1)] \\ \times A(t-\tau) A(t-\tau-\tau_1) A^*(t-2\tau-\tau_1) d\tau_1 d\tau$$

describes **four** regimes of mode evolution:

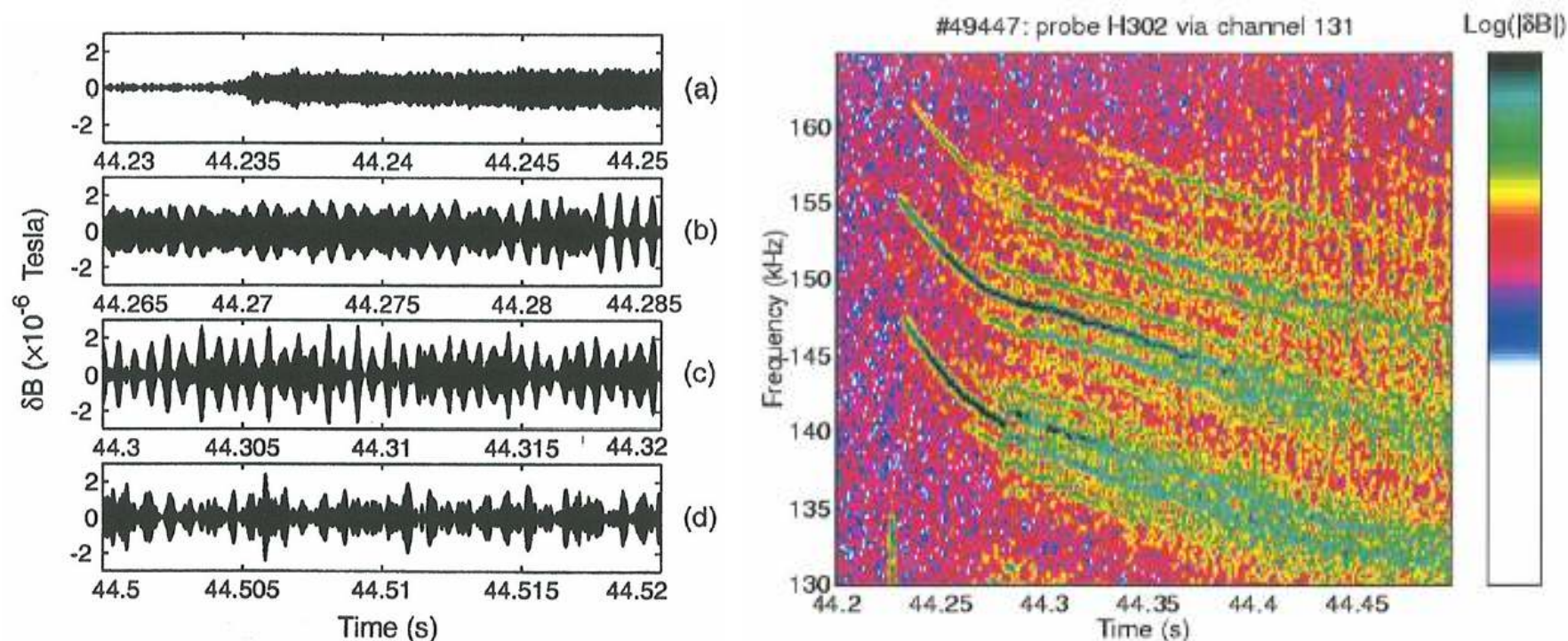
- a) Steady-state;
- b) Periodically modulated;
- c) Chaotic;
- d) Explosive

The explosive regime in a more complete non-linear model leads to **frequency-sweeping** 'holes' and 'clumps' on the perturbed distribution function (H.L.Berk, B.N.Breizman, and N.V.Petviashvili, Phys. Lett. A234 (1997) 213)



Evolution of $|A|$ in time, $t \equiv \gamma t$

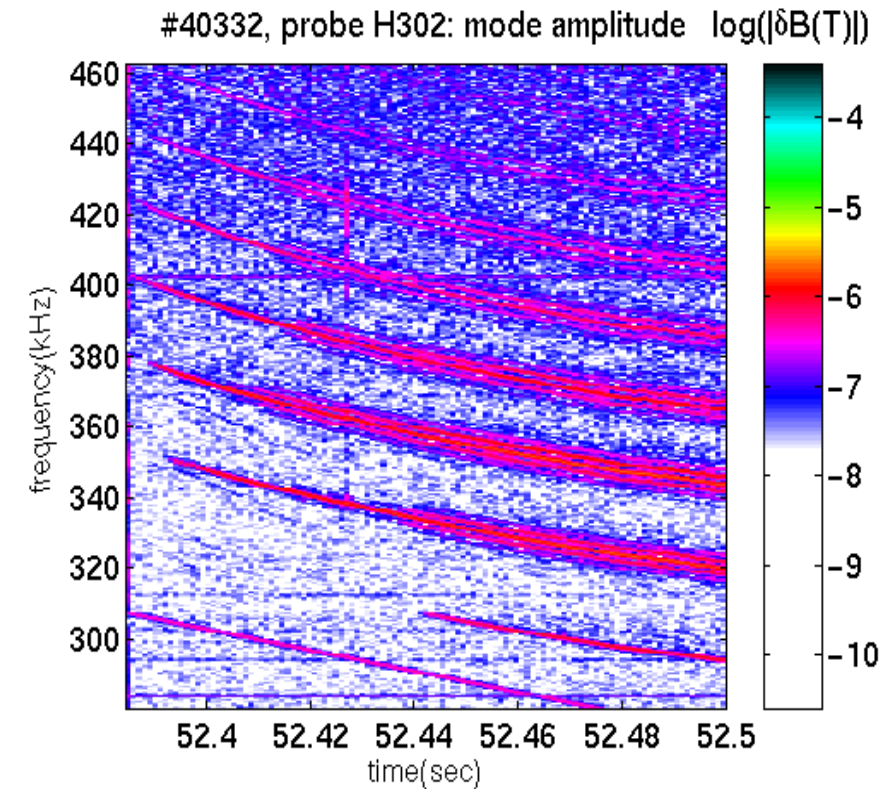
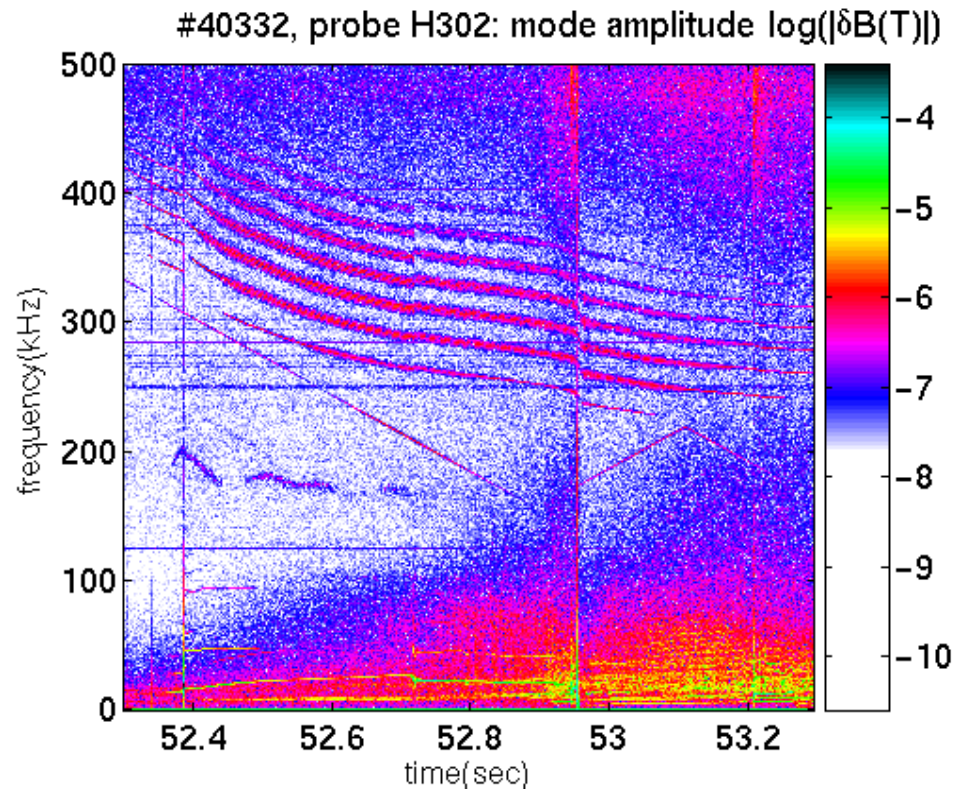
SUCH NONLINEAR SOLUTIONS ARE OBSERVED FOR ICRH-DRIVEN TAE TOO



At gradually increasing ICRH power, TAEs exhibit steady state, periodically modulated, and chaotic regimes

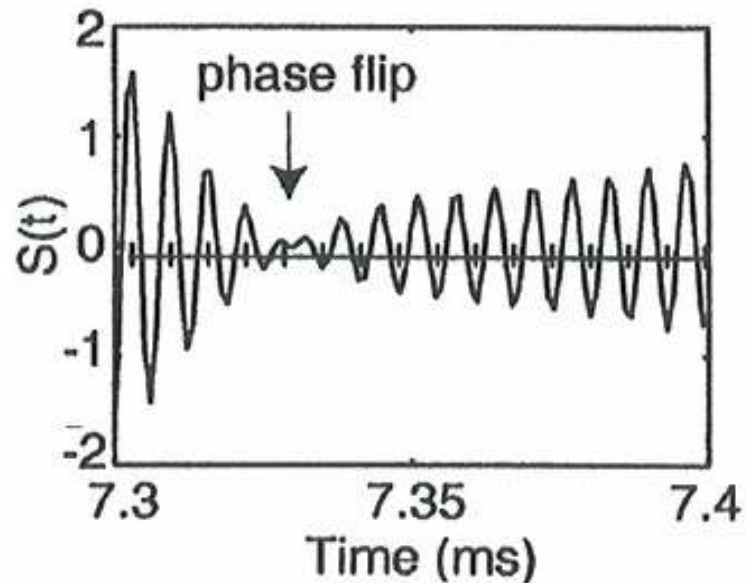
Magnetic spectrogram corresponding to the left Figure with raw data. Steady state, periodically modulated (pitchfork splitting), and chaotic regimes are seen

THE PERIODIC MODULATION REGIME FOR TAE

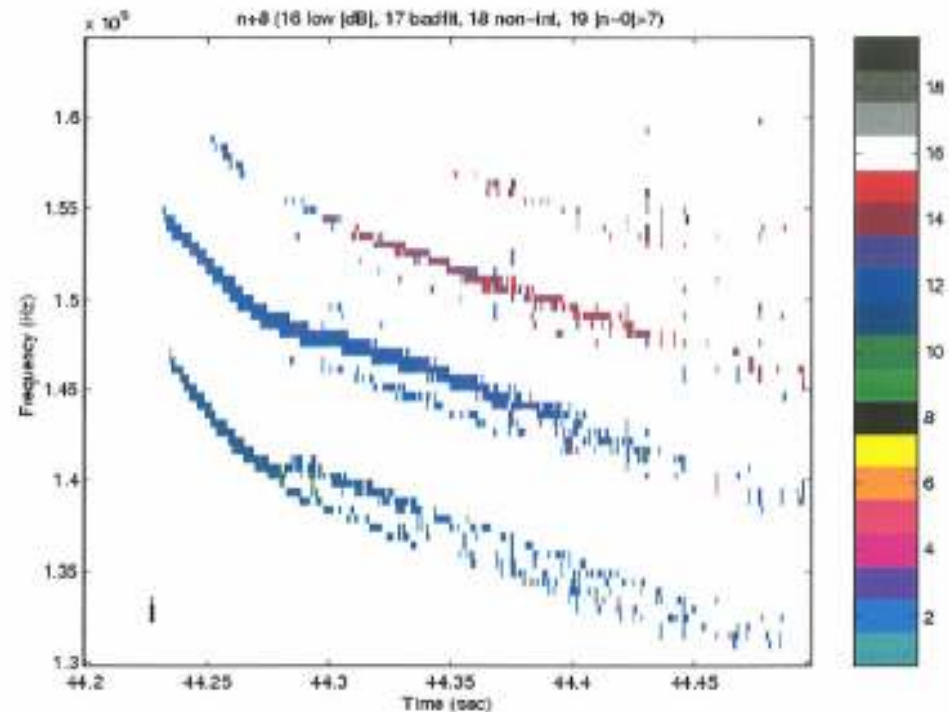


ICRH-driven TAEs during ICRH power ramp-up *Zoom of the left Figure showing the “pitchfork” splitting of TAEs*

THE CHAOTIC REGIME FOR TAE

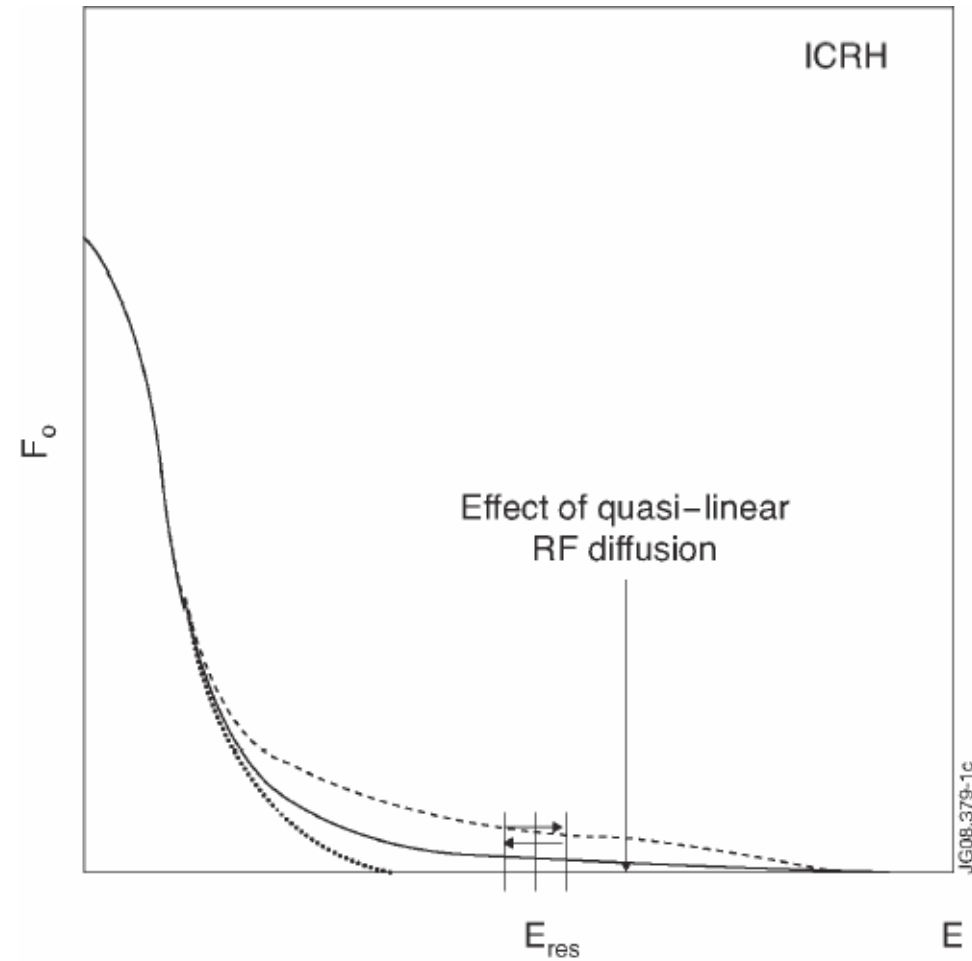


When TAE amplitude modulation becomes comparable to the amplitude, chaotic TAE evolution is observed



The chaotic TAE evolution significantly complicates the phase analysis of TAE mode numbers

ICRH REPLENISHES FAST ION DISTRIBUTION VIA DIFFUSION



DRAG ONLY CASE - 1

Nonlinear equation for the amplitude

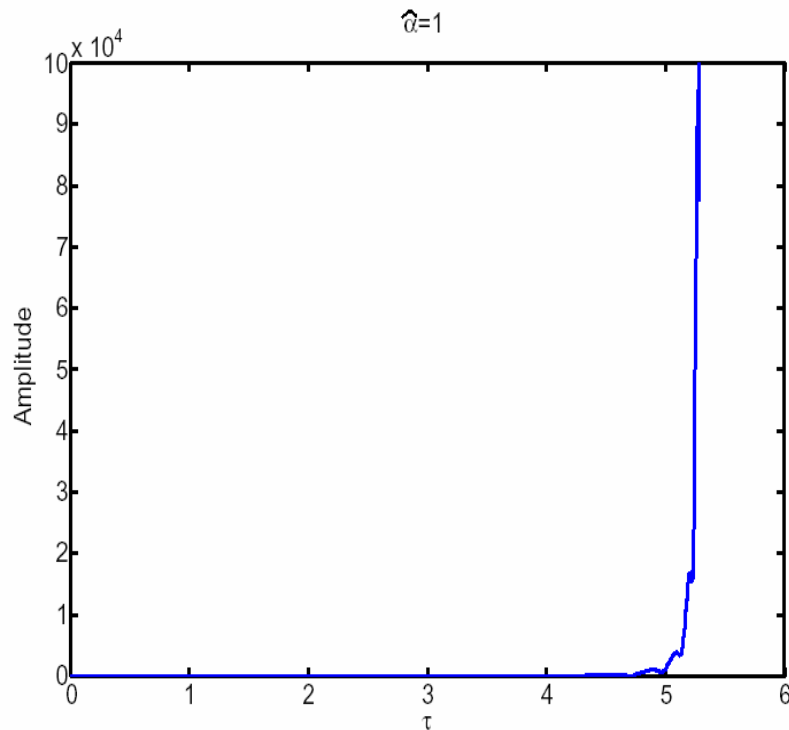
$$\frac{dA}{dt} = A - \frac{1}{2} \int_0^{t/2} \tau^2 \int_0^{t-2\tau} \exp[i\hat{\alpha}^2 \tau(\tau + \tau_1)] \times A(t-\tau) A(t-\tau-\tau_1) A^*(t-2\tau-\tau_1) d\tau_1 d\tau$$

In contrast to the diffusion case, drag gives **oscillatory behaviour in the kernel** leading to the **explosive evolution** of the amplitude blowing up in a finite time,

$$A \propto (t - t_0)^{-p}$$

This is **the only** scenario for the drag !

M.Lilley, B.N.Breizman, and S.E.Sharapov, Phys. Rev. Lett. (2009)



Evolution of $|A|$ in time, $t \equiv \gamma t$

DRAG ONLY CASE - 2

- Why there is such a difference? Consider qualitatively

$$\frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} - \alpha^2 \frac{\partial f_0}{\partial u} + \beta f_0 = -\frac{ek}{2m} \left(\hat{E} \frac{\partial f_1^*}{\partial u} + c.c. \right)$$

- The Krook and diffusion Homogeneous Differential Equations give the solutions not symmetric with respect to the change $u \rightarrow -u$, $t \rightarrow -t$, i.e.

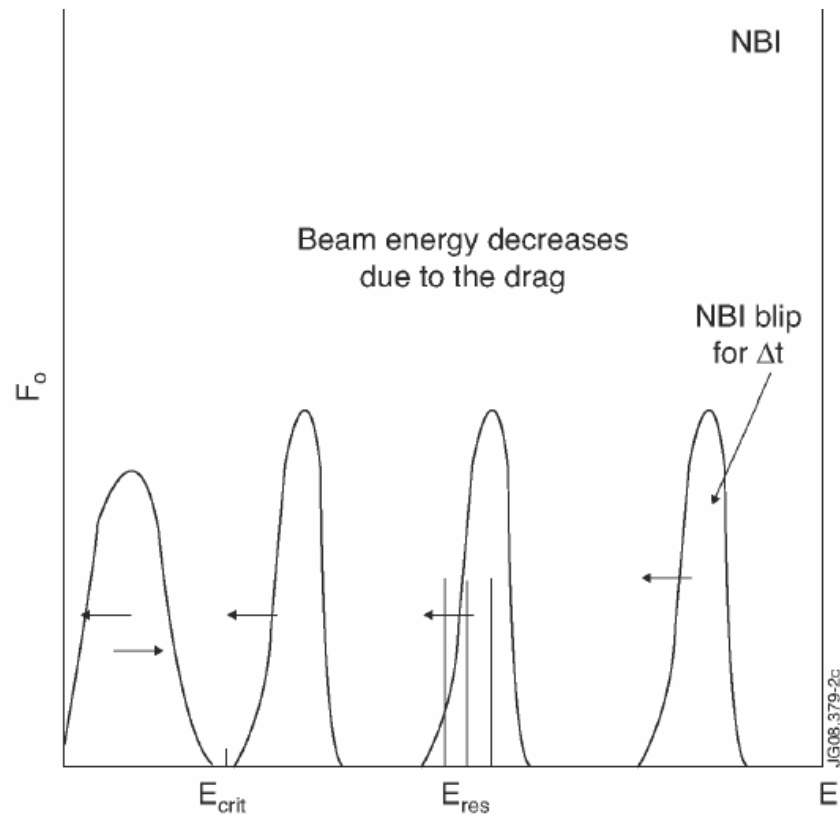
$$\frac{\partial f_0}{\partial t} + \beta f_0 = 0 \quad \rightarrow \quad f_0 \propto \exp(-\beta t)$$

$$\frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} = 0 \quad \rightarrow \quad f_0 \propto (\nu^3 t)^{-1/2} \exp\left(-\frac{u^2}{4\nu^3 t}\right)$$

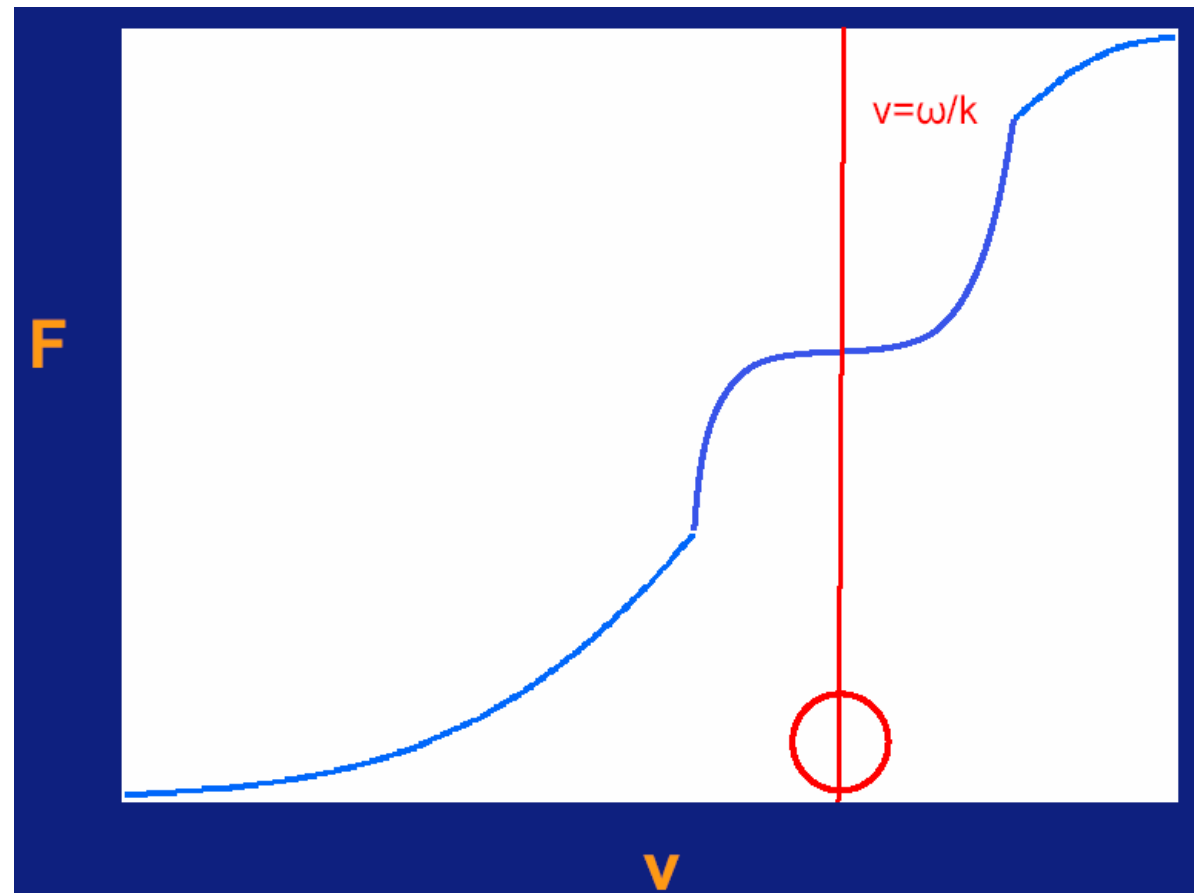
while the drag HDE gives a wave-type solution symmetric to $u \rightarrow -u$, $t \rightarrow -t$:

$$\frac{\partial f_0}{\partial t} - \alpha^2 \frac{\partial f_0}{\partial u} = 0 \quad \rightarrow \quad f_0 \propto f_0(u + \alpha^2 t)$$

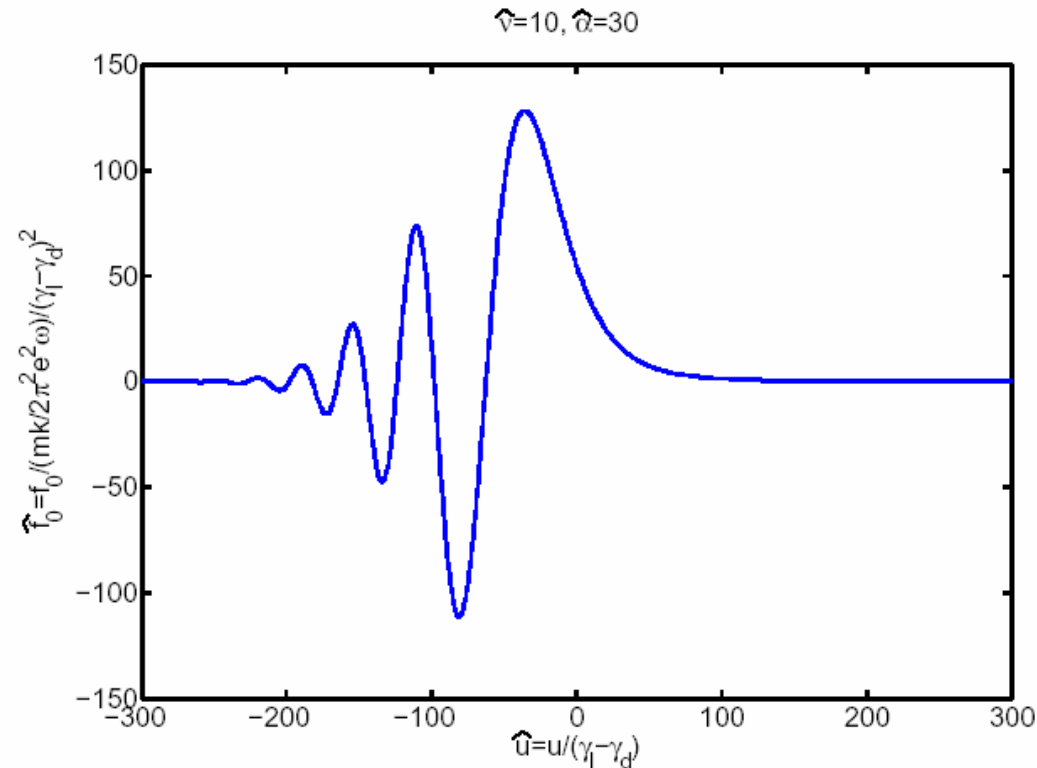
DRAG ONLY CASE - 3



DRAG ONLY CASE - 4



COMBINED DRAG + DIFFUSION SOLUTIONS



In the presence of both drag and diffusion, the solution f_0 becomes oscillatory and shifted from the resonance “downstream”

THE SIMILARITY

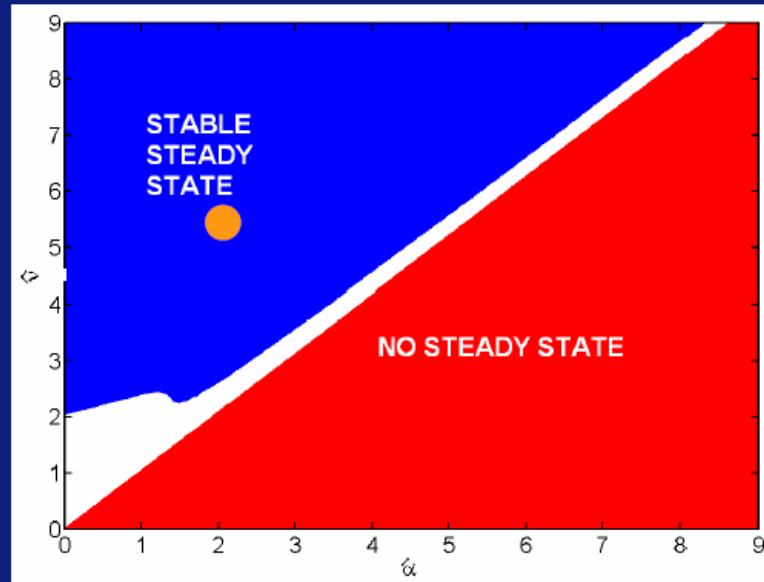


A pebble in a stream creates wave perturbation similar to the drag solution

NONLINEAR EVOLUTION SUMMARY

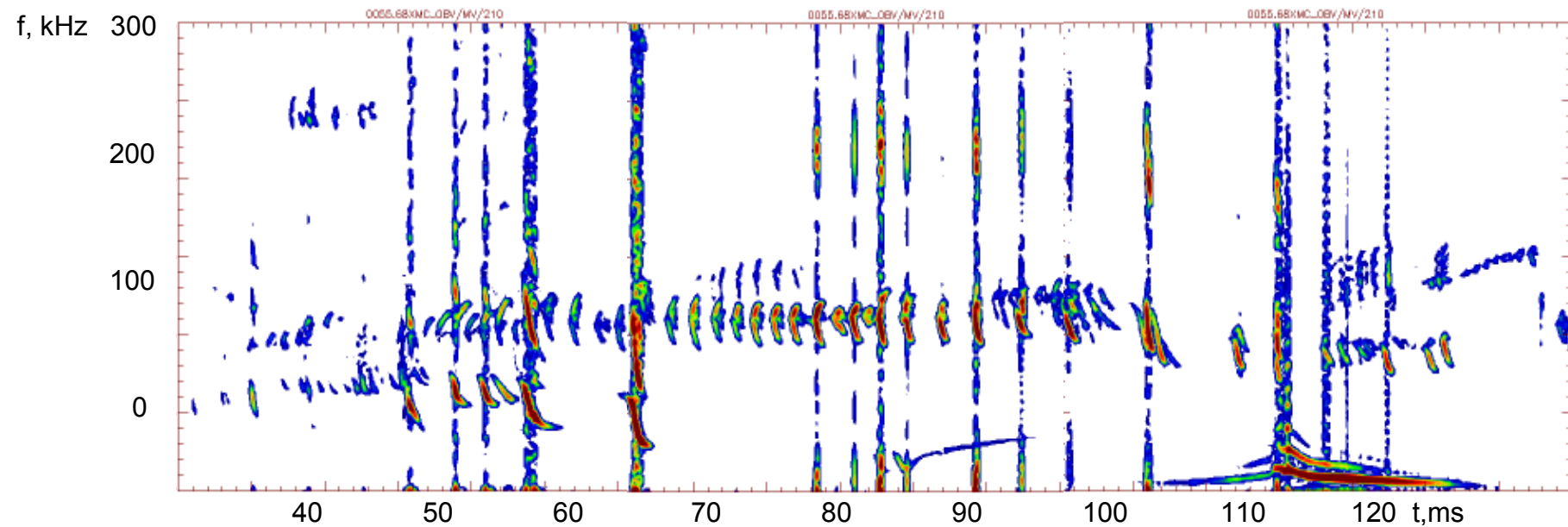
Diffusion + drag ($\hat{\beta} = 0$)

- For diffusion drag steady state solutions do exist
- For an appreciable amount of drag these solutions become unstable (pitch fork splitting etc.)
- Explosive solutions again when drag dominates



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BACK TO TAE: NBI-DRIVEN AEs ON MAST ARE DOMINATED BY THE DRAG IN THE V_A REGION



AEs are seen in bursts not as steady-state modes

HOW TO ASSESS DRAG VS DIFFUSION FOR NBI-DRIVEN TAE ?

- Introduce the resonance

$$\Omega_l \equiv \omega - n \langle \dot{\phi} \rangle - l \langle \dot{\mathcal{G}} \rangle = 0$$

where

$$\langle \dot{\phi} \rangle = \omega_{\phi}(E, P_{\phi}, \mu),$$

$$\langle \dot{\mathcal{G}} \rangle = \omega_{\mathcal{G}}(E, P_{\phi}, \mu)$$

- For NBI distribution function use the Fokker-Planck equation with Coulomb operator:

$$\frac{\partial f}{\partial t} = \frac{V_0^3}{\tau v^2} \left\{ \frac{\partial}{\partial v} \left[\frac{V_0^2 a(v)}{2v} \frac{\partial f}{\partial v} + b(v) f \right] + \frac{c(v)}{V_0} \cdot \frac{1}{\sin \mathcal{G}} \frac{\partial}{\partial \mathcal{G}} \left[\sin \mathcal{G} \frac{\partial f}{\partial \mathcal{G}} \right] \right\} - \nu f + pF$$

- We are interested in evolution of distribution function **across** the TAE resonance

$$\left. \frac{dF}{dt} \right|_{coll} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D} \cdot \frac{\partial f}{\partial \mathbf{v}} + \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{b} f = \left\langle \frac{\partial P_{\phi}}{\partial \mathbf{v}} \cdot \mathbf{D} \cdot \frac{\partial P_{\phi}}{\partial \mathbf{v}} \right\rangle \left(\frac{\partial \Omega}{\partial P_{\phi}} \right)^2 \frac{\partial^2 f}{\partial \Omega^2} +$$

$$\left\langle \frac{\partial P_{\phi}}{\partial \mathbf{v}} \cdot \mathbf{b} \right\rangle \left(\frac{\partial \Omega}{\partial P_{\phi}} \right) \frac{\partial f}{\partial \Omega}$$

DRAG VS DIFFUSION FOR NBI-DRIVEN TAE ON MAST

- The comparison of the drag and diffusion terms can be expressed via resonance width:

$$\frac{(\Delta\Omega_{\text{Diff}})^6}{(\Delta\Omega_{\text{Drag}})^6} \approx \left\langle \frac{\partial P_\phi}{\partial \mathbf{v}} \cdot \mathbf{D} \cdot \frac{\partial P_\phi}{\partial \mathbf{v}} \right\rangle^2 \left(\frac{\partial \Omega}{\partial P_\phi} \right) \left\langle \frac{\partial P_\phi}{\partial \mathbf{v}} \cdot \mathbf{b} \right\rangle^{-3}$$

- Substitute MAST parameters and obtain

$$\frac{(\Delta\Omega_{\text{Diff}})^6}{(\Delta\Omega_{\text{Drag}})^6} \approx m S \tau \frac{c}{e B_0} \frac{E_A}{r^2} \frac{\theta_b^4}{2} \frac{27}{64} \left(\frac{\pi m_b}{m_e} \right)^{3/2} \left(\frac{T_e}{E_A} \right)^{9/2}$$

$$\downarrow$$

$$\Delta\Omega_{\text{Diff}} / \Delta\Omega_{\text{Drag}} \approx 0.2 - 1.6$$

Since drag dominates over the diffusion in the TAE resonance region, the explosive solutions dominate

DRAG VS DIFFUSION FOR ALPHA-DRIVEN TAE ON ITER

- Considering slowing down isotropic distribution function of alpha-particles on ITER, we obtain

$$\Delta\Omega_{\text{Diff}}/\Delta\Omega_{\text{Drag}} \approx 1.4 \quad (\theta_b \sim 1)$$

- The Coulomb diffusion does dominate over the drag favouring the steady-state TAE scenarios. However, the drag is not negligibly small and since the ratio scales as

$$T_e^{3/4}$$

the drag may become dominant in some ITER regimes.

SUMMARY

- Near the marginal stability, a nonlinear equation is valid for the amplitude of the mode driven by fast particles

$$\frac{dA}{d\tau} = A(\tau) - \frac{1}{2} \int_0^{\tau/2} dz z^2 A(\tau - z) \int_0^{\tau-2z} dx e^{-\hat{\nu}^3 z^2 (2z/3+x) - \hat{\beta}(2z+x) + i\hat{\alpha}^2 z(z+x)} \times A(\tau - z - x) A^*(\tau - 2z - x)$$

- Diffusion and Krook collisional operators give 4 different nonlinear regimes: steady state, periodic modulation, chaotic, and explosive
- All these regimes are seen in ICRH-driven TAEs on JET, since ICRH quasi-linear diffusion well dominates over Coulomb effects
- Drag collisional operator gives only explosive solution. No steady-state is possible with drag
- The drag dominant regime is fulfilled in NBI-driven TAEs on MAST
- ITER will have diffusion somewhat exceeding drag due to fusion born alpha particles at the Alfvén resonance region