CONJUGATION WEIGHTS AND WEIGHTED CONVOLUTION ALGEBRAS ON TOTALLY DISCONNECTED, LOCALLY COMPACT GROUPS

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Abstract. A family of equivalent submultiplicative weights on the totally disconnected, locally compact group G is defined in terms of the conjugation action of G on itself. These weights therefore reflect the structure of G, and the corresponding weighted convolution algebra is intrinsic to G in the same way that $L^1(G)$ is.

1. Introduction

The group convolution algebra $L^1(G)$ is key to the functional analytic approach to harmonic analysis on G, see [16, 17, 24, 25] for example, and the algebraic properties of $L^1(G)$ are intimately related to the structure of G. Indeed, $L^1(G)$ is only able to be defined because locally compact groups support a left-invariant Haar measure, and the normed algebra $L^1(G)$ carries complete information about G in the sense that, if $L^1(G)$ and $L^1(H)$ are isometrically isomorphic, then $G \cong H$, see [31].

When G is totally disconnected another group convolution algebra, a subalgebra of $L^1(G)$, may be associated with G. This algebra, which is denoted by $L^1_{\mathsf{cw}}(G)$, is defined below in Section 3 in terms of the action of G on itself by conjugation. It can therefore be expected that the structure if $L^1_{\mathsf{cw}}(G)$ will reflect properties of this action and, as a first step in this direction, it is shown that $L^1_{\mathsf{cw}}(G)$ is equal to $L^1(G)$ if and only if G is an [IN]-group.

The algebra $L^1_{\mathsf{cw}}(G)$ is a weighted convolution algebra on G, and basic definitions and properties of these algebras are recalled in next section.

2. Weighted Convolution Algebras

2.1. Beurling algebras and spectral synthesis.

Definition 2.1. A *submultiplicative weight* on a locally compact group G is a measurable function $w : G \to \mathbb{R}^+$ such that

$$w(xy) \leq w(x)w(y), \quad (x,\, y \in G).$$

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Given a submultiplicative weight w, a norm may be defined on $C_{00}(G)$, the space of continuous functions on G with compact support, by

$$\|\phi\|_w := \int_G |\phi(g)|w(g)\,\mathrm{dg}, \quad (\phi \in \mathrm{C_{00}(G)}),$$

where the integration is with respect to the Haar measure on G. Submultiplicativity of the weight w implies that $\|\cdot\|_w$ is an algebra norm on the convolution algebra $C_{00}(G)$, that is,

$$\|\phi * \psi\|_w \leq \|\phi\|_w \|\psi\|_w, \quad (\phi, \psi \in C_{00}(G)).$$

Denote the completion of $C_{00}(G)$ under this norm by $L^1(G, w)$. Then $L^1(G, w)$ may be identified with

$$\left\{ f \in L^1_{loc}(G) \mid \|f\|_w := \int_G |f(x)|w(x)\,\mathrm{d}x < \infty \right\}.$$

(As usual, functions that are equal almost everywhere are identified.) The algebra $L^1(G, w)$ is called a *weighted convolution algebra* on G or a *Beurling algebra*. Note that, when w is bounded below (necessarily by 1), $L^1(G, w)$ is a subalgebra of $L^1(G)$.

According to Y. Domar, [7], weighted convolution algebras were originally studied in [3] by A. Beurling with $G = \mathbb{R}$. The aim of these papers was to understand spectral synthesis on $\mathbb{R}$, and it was only during this period that the example of L. Schwartz showing the failure of spectral synthesis on $\mathbb{R}^3$ was discovered, [28]. Domar emphasized the Banach algebraic approach and systematized much of the earlier work. An important property considered in relation to spectral synthesis is that of being *regular*, which means that functions with compact support are dense in the Gel'fand transform, and Domar showed that $L^1(G, w)$ is regular if and only if w is *non-quasianalytic*, that is, $\sum_{n=1}^{\infty} n^{-2} \log w(x^n) < \infty$ for all $x \in G$.

The weighted convolution algebras to be studied in Section 3 have more in common with algebras that are not regular, such as the following.

Example 2.2. Let $G = (\mathbb{Z}, +)$ and $w(n) = b^{|n|}$ $(n \in \mathbb{Z})$, where $b \geq 1$. Then $\ell^1(\mathbb{Z}, w)$ is a subalgebra of $\ell^1(\mathbb{Z})$. The annulus

$$\mathbf{A}_b = \{z \in \mathbb{C} \mid 1/b \leq |z| \leq b\}$$

is the carrier space of $\ell^1(\mathbb{Z}, w)$ and the Gel'fand transform maps

$$\delta_n \mapsto z^n.$$

The range is contained in the algebra of continuous functions on $\mathbf{A}_b$ that are analytic on the interior.

Work in more recent times has studied related questions for non-abelian G. It is shown in [15] for example that, if G is an $[FC]^-$-group and w is symmetric and non-quasianalytic, then $L^1(G, w)$ satisfies the Wiener property, that is, each proper closed ideal is annihilated by an irreducible *-representation. Compactly generated groups with polynomial growth are studied in [9, 10] and it is shown that $L^1(G, w)$ is symmetric if and only if $\lim_{n\to\infty} w(x^n)^{1/n} = 1$ for every $x \in G$. A functional calculus is developed in these papers and in [8] and is used to show that $L^1(G, w)$ has the Wiener property.

Certain weighted convolution algebras $L^1(\mathbb{R}^n, w)$ also appear naturally as part of the description of topologically simple modules over $L^1(G)$, where G is a simply connected, exponential, solvable Lie group, see [22] and [19].

2.2. **Equivalence of weights.**

Definition 2.3. Two submultiplicative weights w_1 and w_2 on G are *equivalent* if there are constants C and D with

$$Cw_1(x) \leq w_2(x) \leq Dw_1(x), \quad (x \in G).$$

The significance of equivalence of weights lies in the following, easily verified, assertion.

Proposition 2.4. *The weights w_1 and w_2 on G are equivalent if and only if $L^1(G, w_1) = L^1(G, w_2)$.*

Before proceeding to the definition of the particular weighted convolution algebras on totally disconnected groups that are the subject of this note, it is worth observing that Example 2.2 may be extended to produce many distinct weighted convolution algebras on any given locally compact group.

Example 2.5. Let $\ell : G \to \mathbb{R}^{\geq 0}$ be a *length function* on G, that is,

$$\ell(xy) \leq \ell(x) + \ell(y), \qquad (x, y \in G).$$

(One way to define a length function on a locally compact group G is to use a word metric: if K is a symmetric generating set for G, define $w(x) = \min\{n \in \mathbb{N} \mid x \in K^n\}$.) Then

$$w(x) := b^{\ell(x)} \text{ and } w(x) := (1 + \ell(x))^\alpha$$

are submultiplicative weights for every $b \geq 1$ and $\alpha \geq 0$. Provided that ℓ is unbounded, such weights with different bases b or different exponents α are not equivalent. Hence there are uncountably many inequivalent submultiplicative weights on G.

The submultiplicative weights (and corresponding convolution algebras) in the previous example are only loosely tied to the structure of the underlying group. The weights on totally disconnected groups that are about to be defined are tied to the structure of the groups much more closely than those defined in terms of length functions.

3. Totally Disconnected Groups

3.1. **Compact open subgroups and weights.**

Throughout this section, G will denote a totally disconnected, locally compact group. The following result, proved by van Dantzig in the 1930's [6], is fundamental in the structure theory of such groups. See [17, Theorem II.7.7] for a proof.

Theorem 3.1 (van Dantzig)**.** *Let G be a totally disconnected, locally compact group and $\mathcal{U}$ be a neighbourhood of the identity. Then there is a compact open subgroup $V \subset \mathcal{U}$.*

Denote the set of all compact, open subgroups of G by $\mathcal{B}(G)$. Any two compact open subgroups, U and V, of G are *commensurable*, that is, the index of $U \cap V$ in U, which will be denoted by $[U : U \cap V]$, is finite, as is $[V : U \cap V]$.

A submultiplicative weight on G may be defined for each $V \in \mathcal{B}(G)$.

Proposition 3.2. *Let $V \leq G$ be a compact open subgroup. Then the function*

$$w_V(x) := [xVx^{-1} : xVx^{-1} \cap V], \quad (x \in G),$$

is a continuous, submultiplicative weight on G.

Proof. Continuity of w_V at x follows from the fact that it is constant on the open neighbourhood xV.

To prove submultiplicativity, observe that $w_V(xy)$ is bounded above by

$$[xyV(xy)^{-1} : xyV(xy)^{-1} \cap xVx^{-1} \cap V],$$

which, since $xyV(xy)^{-1} \cap xVx^{-1} \cap V \leq xyV(xy)^{-1} \cap xVx^{-1} \leq xyV(xy)^{-1}$, is equal to

$$[xyV(xy)^{-1} : xyV(xy)^{-1} \cap xVx^{-1}] \\ \times [xyV(xy)^{-1} \cap xVx^{-1} : xyV(xy)^{-1} \cap xVx^{-1} \cap V].$$

The first factor is equal to $w_V(y)$ because conjugation by x is an automorphism of G. The second is less than or equal to $w_V(x)$ because the map

$$z\left(xyV(xy)^{-1} \cap xVx^{-1} \cap V\right) \mapsto z\left(xVx^{-1} \cap V\right)$$

is injective from

$$\left(xyV(xy)^{-1} \cap xVx^{-1}\right) / \left(xyV(xy)^{-1} \cap xVx^{-1} \cap V\right) \to \\ xVx^{-1} / \left(xVx^{-1} \cap V\right).$$

□

All weights as defined in the previous theorem are equivalent.

Proposition 3.3. *The weights w_U and w_V are equivalent for any two U, $V \in \mathcal{B}(V)$.*

Proof. Since $U \cap V$ is contained in both U and V, it suffices to treat the case when $U \leq V$. Let $x \in G$. Equivalence may then be seen by factoring the index $[xVx^{-1} : xUx^{-1} \cap U]$ in two ways. One factoring yields that $[xVx^{-1} : xUx^{-1} \cap U]$ is equal to

$$[xVx^{-1} : xUx^{-1}][xUx^{-1} : xUx^{-1} \cap U] = [V : U]w_U(x),$$

and the second that it is equal to

$$[xVx^{-1} : xVx^{-1} \cap V][xVx^{-1} \cap V : xUx^{-1} \cap U] \\ = w_V(x)[xVx^{-1} \cap V : xUx^{-1} \cap U].$$

Therefore

$$[V : U]w_U(x) = w_V(x)[xVx^{-1} \cap V : xUx^{-1} \cap U].$$

Since all indices are at least 1, it follows that

$$w_V(x) \leq [V : U]w_U(x) \text{ and } w_U(x) \leq [xVx^{-1} \cap V : xUx^{-1} \cap U]w_V(x).$$

Equivalence follows because

$$[xVx^{-1} \cap V : xUx^{-1} \cap U]$$
$$= [xVx^{-1} \cap V : xUx^{-1} \cap V][xUx^{-1} \cap V : xUx^{-1} \cap U] \leq [V : U]^2.$$

□

Definition 3.4. A weight $w(x) = [xVx^{-1} : xVx^{-1} \cap V]$ will be called a *conjugation weight.*

The algebra $L^1(G, w)$ is independent of the conjugation weight w by Proposition 3.3 and we denote it by $L^1_{\mathsf{cw}}(G)$. Since conjugation weights are bounded below by 1, $L^1_{\mathsf{cw}}(G)$ is a subalgebra of $L^1(G)$.

3.2. **Conjugation weights and the scale function.** As a consequence of its definition, $L^1_{\mathsf{cw}}(G)$ comes equipped with many natural algebra norms but, unlike $L^1(G)$ or the group C^*-algebra $C^*(G)$, there does not appear to be a single most natural norm. However, integration against the scale function, which is defined next, is a natural linear functional on $L^1_{\mathsf{cw}}(G)$.

Definition 3.5. Let $x \in G$. The *scale of* x is the positive integer

$$s(x) := \min\left\{[xVx^{-1} : xVx^{-1} \cap V] : V \in \mathcal{B}(G)\right\}.$$

The compact open subgroup V of G is *minimizing for* x if the minimum is attained at V.

This standard statement of the definition of the scale may be restated in terms of the conjugation weights: $s(x) = \min\{w_V(x) \mid V \in \mathcal{B}(G)\}$. The concepts of scale and minimizing subgroup are partial substitutes in the structure theory of totally disconnected, locally compact groups for linear algebra techniques in the theory of Lie groups, as may be seen in proofs in [1, 5, 11, 12, 13, 18, 23, 29, 34] for example. The following structural characterisation of minimizing subgroups combines results from [32, 33].

Theorem 3.6. *Let G be a totally disconnected, locally compact group. For $x \in G$ and $V \in \mathcal{B}(G)$ put*

$$V_+ := \bigcap_{k\geq 0} x^k V x^{-k} \quad \text{and} \quad V_- := \bigcap_{k\geq 0} x^{-k} V x^k.$$

Then V is minimizing for x if and only if

TA: *$V = V_+V_-$, and*

TB: *$V_{++} := \bigcup_{k\in\mathbb{Z}} x^k V_+ x^{-k}$ is closed.*

If V is minimizing for x, then $s(x) = [xV_+x^{-1} : V_+]$.

A subgroup satisfying the conditions **TA** and **TB** is said to be *tidy for x*. The scale has the following properties, which are established in [32].

Theorem 3.7. *The scale function $s : G \to \mathbb{Z}^+$ is a continuous for the given topology on G and the discrete topology on $\mathbb{Z}^+$. Furthermore:*

1. *$s(x) = 1 = s(x^{-1})$ if and only if there is $V \in \mathcal{B}(G)$ that is normalized by x;*

(2) $s(x^k) = s(x)^k$ *for every* $k \geq 0$*; and*
(3) $\Delta(x) = s(x)/s(x^{-1})$, *where* $\Delta : G \to \mathbb{R}^+$ *is the modular function on* G.

Definition 3.8. The group G is said to be *uniscalar* if $s(x) = 1$ for every $x \in G$.

If G is uniscalar, then every element of G normalizes some compact, open subgroup of G. That does not imply, however, that G has a compact, open, normal subgroup, as Example 3.13 below shows.

The scale function is not generally submultiplicative on G, as the example of the automorphism group of a tree examined in the last section of [32] shows. Nevertheless, the above remark, that the scale is the greatest lower bound of the conjugation weights, suggests that $L^1_{\mathsf{cw}}(G)$ is the largest convolution algebra of functions integrable against the scale. This bounded functional on $L^1_{\mathsf{cw}}(G)$ will be denoted

$$\varphi_{\mathsf{cw}}(f) := \int_G f(x)s(x)\,\mathrm{d}x, \quad (f \in L^1_{\mathsf{cw}}(G)).$$

Given this close link between $L^1_{\mathsf{cw}}(G)$ and the scale, the relationship between $L^1_{\mathsf{cw}}(G)$ and the functional φ_{cw} on one hand and the structure of G on the other is a promising area of investigation. This relationship is explored further in the remainder of this subsection.

A spectral radius formula. The following asymptotic formula for the scale was obtained by R. Möller in [21], in which he relied on results in [20].

Theorem 3.9. *Let G be a totally disconnected locally compact group and $x \in G$ and let V be any compact open subgroup of G. Then, for any $x \in G$,*

$$s(x) = \lim_{k\to\infty} [x^k V x^{-k} : x^k V x^{-k} \cap V]^{1/k}.$$

Möller's formula looks superficially like a spectral radius formula, and it becomes precisely that on the space $L^1_{\mathsf{cw}}(G)$: the translation operator $f \mapsto \delta_x * f$ on $L^1(G, w_V)$ has norm $w_V(x)$ and so the spectral radius of this map is $\lim_{k\to\infty} w_V(x^k)^{1/k}$ which, by Möller's formula, is the scale of x. Since the spectral radius is defined algebraically and is independent of the norm, $s(x)$ is equal to the spectral radius of the translation operator on $L^1_{\mathsf{cw}}(G)$. (Although not an element of the algebra $L^1_{\mathsf{cw}}(G)$, this translation operator is a multiplier of the algebra.)

Compact, open, normal subgroups. It is immediate from the definition of conjugation weights that, if $U \in \mathcal{B}(G)$ is normal, then $w_U(x) = 1$ for every $x \in G$. That the converse also holds is a theorem, an early version of which was proved by G. Schlichting in [27], and which, in the version equivalent to the following statement, was proved by G. Bergmann and H. Lenstra in [2].

Theorem 3.10. *Suppose that there is $V \in \mathcal{B}(V)$ such that the set of indices*

$$\{[xVx^{-1} : xVx^{-1} \cap V] \mid x \in G\}$$

is bounded. Then there is a compact, open subgroup $U \triangleleft G$.

It follows that $L^1_{\mathsf{cw}}(G) = L^1(G)$ if and only if G has a compact open normal subgroup, that is, G is an [IN]-group.

3.3. **Non-quasianalyticity of conjugation weights and amenability of $L^1_{\mathsf{cw}}(G)$.** Example 2.2 presents a submultiplicative weight on $\mathbb{Z}$ that is quasianalytic, and the Gel'fand transform of $\ell^1(\mathbb{Z}, w)$ for this weight consists of functions that are indeed analytic on an annulus. Algebras of analytic functions do not satisfy spectral synthesis, thus indicating why a condition like non-quasianalyticity is needed to establish regularity of weighted convolution algebras. A group G that has non-quasianalytic conjugation weights must also satisfy a quite restrictive condition.

Proposition 3.11. *If the conjugation weight w_V is non-quasianalytic for any $V \in \mathcal{B}(G)$, then it is non-quasianalytic for all and G is uniscalar. Conversely, every conjugation weight on a uniscalar group is non-quasianalytic.*

Proof. Since all conjugation weights are equivalent in the sense of Definition 2.3, if one weight is non-quasianalytic, all are.

If w_V is non-quasianalytic, then the condition that

$$\sum_{n=1}^{\infty} n^{-2} \log w_V(x^n) < \infty$$

implies that $\{n^{-1} \log w_V(x^n)\}$ has a subsequence that converges to 0, whence $\lim_{n\to\infty} w_V(x^n)^{1/n} = 1$ and x is uniscalar for every $x \in G$, by Theorem 3.9. For the converse, suppose that G is uniscalar and let V be a compact, open subgroup. Then, for any $x \in G$, there is $U \in \mathcal{B}(G)$ normalized by x and $w_V(x^n) = [x^nVx^{-n} : x^nVx^{-n} \cap V]$ is bounded by

$$[x^nVx^{-n} : x^nVx^{-n} \cap V \cap U] = \\ [x^nVx^{-n} : x^nVx^{-n} \cap U][x^nVx^{-n} \cap U : x^nVx^{-n} \cap V \cap U],$$

where $[x^nVx^{-n} : x^nVx^{-n} \cap U] = [V : V \cap U]$ because U is stable under x and

$$[x^nVx^{-n} \cap U : x^nVx^{-n} \cap V \cap U] \leq [U : U \cap V]$$

because the map

$$z\left(x^nVx^{-n} \cap V \cap U\right) \mapsto \\ z(U \cap V) : \left(x^nVx^{-n} \cap U\right) / \left(x^nVx^{-n} \cap V \cap U\right) \to U/(U \cap V)$$

is injective. Hence $w_V(x^n) \leq [V : U \cap V][U : U \cap V]$ for all $n \in \mathbb{Z}$ and w_V is non-quasianalytic. □

Uniscalar groups need not have compact, open normal subgroups, as Example 3.13 shows. Conjugation weights on such uniscalar groups will be unbounded, by Theorem 3.10.

Amenability. A Banach algebra, $\mathfrak{A}$ is *amenable* if the continuous cohomology group $\mathcal{H}^1(\mathfrak{A}, X^*)$ vanishes for every Banach $\mathfrak{A}$-bimodule X, see [26], and is *weakly amenable* if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*)$ vanishes. The algebra $\ell^1(\mathbb{Z}, w)$ in Example 2.2 is not weakly amenable because the map

$$f \mapsto \frac{d}{dz}\left(\sum_{n\in\mathbb{Z}} f(n)z^n\right)\Bigg|_{z=1}$$

is a derivation into $\mathbb{C}$, and composition with the map $\mathbb{C} \to \ell^\infty(\mathbb{Z}, 1/w)$ that sends a to the constant sequence $\{a\}$ is a derivation into $\ell^\infty(\mathbb{Z}, 1/w) \cong$

$\ell^1(\mathbb{Z},w)^*$ that is not inner. More generally, it was shown by N. Grønbæk in [14, Theorem 0] that, for submultiplicative weights w that are bounded below, $L^1(G,w)$ is amenable if and only if G is amenable and w is bounded. Combining with Theorem 3.10 yields the following[1].

Proposition 3.12. *$L^1_{\mathsf{cw}}(G)$ is amenable if and only if G is an amenable [IN]-group.*

Examples. This section concludes with some elementary examples of groups that have unbounded conjugation weights.

Example 3.13. Any group, $G = \bigcup_\lambda V_\lambda$, that is the union of compact, open subgroups is uniscalar. It is seen in the next two paragraphs that such groups may fail to have any compact, open, normal subgroups. Hence, by Theorem 3.10, conjugation weights on G are unbounded in this case and $L^1_{\mathsf{cw}}(G)$ is a proper subalgebra of $L^1(G)$.

Let F be a finite group and $H < F$ be a proper subgroup that is not normal in F. For instance, F might be the symmetric group S_3 and H might be an order 2 subgroup. Then

$$G := \left\{ g \in F^{\mathbb{Z}} \mid g(n) \in H \text{ for all but finitely many } n \right\}$$

is a group with the pointwise group operations. The subgroup

$$\{g \in G \mid g(n) \in H \text{ for all } n\}$$

may be identified with $H^{\mathbb{Z}}$ and G equipped with a group topology such that $H^{\mathbb{Z}}$ is a compact, open subgroup with the product topology. Then

$$G = \bigcup_{N\in\mathbb{N}} \{g \in G \mid g(n) \in H \text{ unless } |n| < N\}$$

and G has no compact, open normal subgroups.

Let X_q, with $q > 2$, be the regular tree in which every vertex has degree $q+1$, and let ∞ be an end of X_q. (The *ends* of X_q are defined to be equivalence classes of semi-infinite paths, $\varepsilon = [v_0, v_1, \dots)$, in X_q, where two such paths are equivalent if they have infinite intersection.) Let G be the *fixator* of ∞, that is,

$$G := \{g \in \mathsf{Aut}(X) \mid \exists \varepsilon \in \infty \text{ such that } g.v = v \text{ for every } v \in V(\varepsilon)\},$$

see [30] for this terminology. Then for any vertex $v \in V(X_q)$, the stabilizer $\mathrm{stab}_G(v)$ is a compact, open subgroup of G and

$$G = \bigcup_{v\in V(X_q)} \mathrm{stab}_G(v)$$

but G has no compact, open, normal subgroup.

Example 3.14. Let G be a totally disconnected, locally compact that has an element, x, and a compact, open subgroup, V, such that $xVx^{-1} \leq V$. Then V is clearly minimizing for x, $s(x) = 1$ and $s(x^{-1}) = [V : xVx^{-1}]$. If xVx^{-1} is strictly contained in V, then $s(x^{-1}) > 1$ and G is not uniscalar. Hence conjugation weights on G are unbounded in this case and $L^1_{\mathsf{cw}}(G)$ is

[1] I am grateful to Hung Le Pham for this remark.

a proper subalgebra of $L^1(G)$. The next two paragraphs give a couple of basic examples of such groups G.

Let p be a prime and $G = \mathbb{Q}_p \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on the additive group of p-adic numbers $\mathbb{Q}_p$ by $n.x = p^n x$. Then $V := \mathbb{Z}_p \rtimes \{0\}$ is compact and open in G and $xVx^{-1} < V$, where $x = (0,1) \in \mathbb{Q}_p \rtimes \mathbb{Z}$. We have $s(x) = 1$ and $s(x^{-1}) = p$ in this case. Moreover, since $\mathbb{Q}_p \rtimes \{0\}$ is abelian, V is minimizing for every $(y,n) \in G$ and

$$w_V(y,n) = s(y,n) = \begin{cases} 1 & \text{if } n \geq 0 \\ p^{-n} & \text{if } n < 0 \end{cases}.$$

As in Example 3.13, let X_q, with $q > 2$, be the regular tree in which every vertex has degree $q+1$, and let ∞ be an end of X_q. Let G be the *stabilizer* of ∞, that is,

$$G := \{g \in \mathsf{Aut}(X) \mid \varepsilon \in \infty \implies g.\varepsilon \in \infty\}.$$

Let $\lambda = (\dots, v_{-1}, v_0, v_2, v_2, \dots)$ be a doubly infinite path in X_q such that $[v_0, v_1, \dots) \in \infty$ and let $x \in \mathsf{Aut}(X_q)$ be a translation of X_q that has λ as its axis and $x.v_j = v_{j+1}$ for every $j \in \mathbb{Z}$. Let

$$H = \bigcup_{n \in \mathbb{Z}} \mathrm{stab}_G(v_n).$$

Then H is a normal subgroup of G and it may be shown that $G = H \rtimes \langle x \rangle$. Put $V = \mathrm{stab}_G(v_0)$. Then $xVx^{-1} < V$, so that V is minimizing for x, and $s(x) = 1$ and $s(x^{-1}) = q$. Furthermore, although it is not the case that V is minimizing for every element of G, we have

$$s(hx^n) = \begin{cases} 1 & \text{if } n \geq 0 \\ q^{-n} & \text{if } n < 0 \end{cases}$$

for any $g = hx^n \in G$. Note that $w_V(hx^n)$ is generally not equal to $s(hx^n)$ for this group.

3.4. **Questions.** The rate of growth of a submultiplicative weight on a group G influences whether the weighted convolution algebra $L^1(G,w)$ carries an analytic structure or is amenable. At the same time, the rate of growth of the conjugation weights, w_V, $V \in \mathcal{B}(G)$, reflects the structure of G. It can be expected therefore that properties of the algebra $L^1_{\mathsf{cw}}(G)$ correlate with the structure of the totally disconnected, locally compact group G. Here are a few specific questions.

(1) For which groups is $L^1_{\mathsf{cw}}(G)$ *-invariant? When does it satisfy the Wiener property?
(2) Does the primitive ideal space of $L^1_{\mathsf{cw}}(G)$ reflect the structure of G? Does it carry analytic structure? Can this primitive ideal space be related to values of the scale?
(3) Describe the closed $L^1_{\mathsf{cw}}(G)$-submodule of $L^\infty_{\mathsf{cw}}(G)$ that is generated by φ_{cw}. When G is uniscalar, φ_{cw} is constant and this submodule is 1-dimensional. Can scale values on G be recovered from this submodule?

(4) Is $L^1_{\mathsf{cw}}(G)$ weakly amenable in some cases, in particular, when G is uniscalar? Should $\mathcal{H}^1(L^1_{\mathsf{cw}}(G), L^\infty_{\mathsf{cw}}(G))$ not vanish for some G, can it be computed and interpreted in terms of G?

4. General locally compact groups

The relationship between $L^1_{\mathsf{cw}}(G)$ and G could also be better understood by extending, if possible, the definition of the weighted convolution algebra from totally disconnected to general locally compact groups.

An elementary, and direct and comprehensive, way in which it might be thought that that could be done would be to modify Definition 3.4 by defining

$$w_K(x) = m(xKx^{-1})/m(xKx^{-1} \cap K),$$

where K is a compact neighbourhood of the identity. However, that does not seem to work as w_K, so defined, is not usually submultiplicative. The fact that V in Definition 3.4 is a compact, open *subgroup* appears to be essential for the proof that w_V is submultiplicative.

In a less elementary approach, submultiplicative weights may be defined separately for connected Lie groups. Let G be a connected Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\|\cdot\|$ be any algebra norm on $L(\mathfrak{g})$, the algebra of linear operators on $\mathfrak{g}$. Then

$$w_{\|\cdot\|}(x) = \|\mathrm{Ad}(x)\|, \quad (x \in G),$$

where $\mathrm{Ad} : G \to L(\mathfrak{g})$ is the adjoint representation, is a submultiplicative weight on G. Since all algebra norms on the finite-dimensional space $L(\mathfrak{g})$ are equivalent, all weights defined in this way on G are equivalent. Similarly to the conjugation weights in totally disconnected groups, it may be seen that:

- $\rho(x) = \inf \left\{ w_{\|\cdot\|}(x) \mid \|\cdot\| \text{ an algebra norm on } L(\mathfrak{g}) \right\}$, where $\rho(x)$ denotes that spectral radius of $\mathrm{Ad}(x)$;
- there is no natural, or best, choice of algebra norm on $L^1_{\mathsf{cw}}(G)$ (defined analogously as for totally disconnected groups); and
- integration against $\rho(x)$ is a natural bounded linear functional on $L^1_{\mathsf{cw}}(G)$;
- $w_{\|\cdot\|}$ is bounded if and only if G is an [IN]-group.

This definition of weight function can be extended to locally compact groups by approximating the connected component by Lie groups. It may be seen that the algebra obtained does not depend on which Lie group is used to approximate G. Finally, an algebra $L^1_{\mathsf{cw}}(G)$ may be defined for general locally compact groups by defining particular conjugation weights $w_{\|\cdot\|}$, for the conjugation action of G on its connected component G_0, and w_V, for the action of G on G/G_0, and setting $w(x) = w_{\|\cdot\|}(x)w_V(x)$.

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