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Current voxel optimization

Simons retreat, 2023

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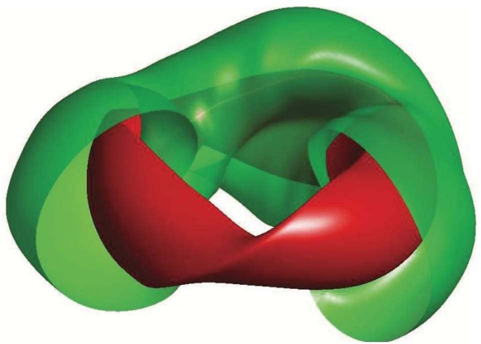
Overview of this talk

1. Review of three ways to do stage-2 optimization.
2. New method for coil optimization:
   a. No shape optimization.
   b. Makes no topology assumptions.
   c. Not quite “finite-build” coils.
3. Unifying the coil and permanent magnet problems as topology optimization.
4. Some fun results for QA, QH stellarators.

A brief review of stage-2 stellarator optimization

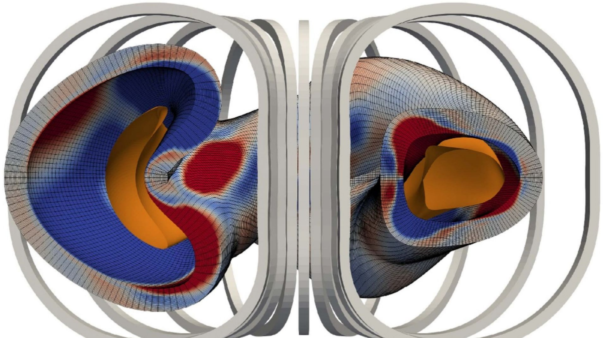
- Assume a good stage-1-optimized plasma is available
- Question is how to build coils such that normal component of **B** vanishes on the plasma surface.
  - Primary objective: $f_B = \min_{\boldsymbol{J}} \int_S \|(\boldsymbol{B}_{\text{coil}} - \boldsymbol{B}_{\text{target}}) \cdot \hat{\boldsymbol{n}}\|^2 d\boldsymbol{r}$
- Three methods: "winding surface" of currents, permanent magnets, or discrete filaments.

$$R = \sum R_{m,n} \cos(m\theta - n\zeta),$$
$$Z = \sum Z_{m,n} \sin(m\theta - n\zeta).$$



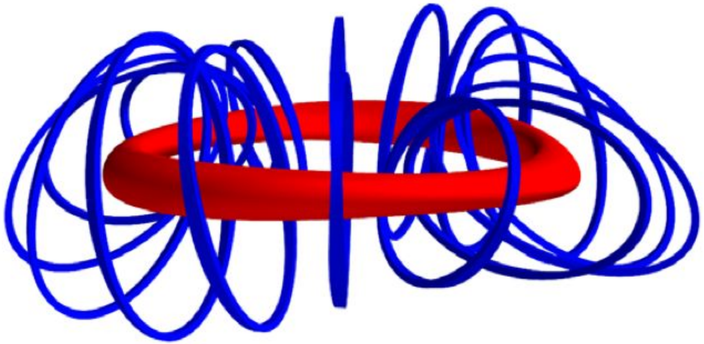
Landreman, *Nuclear Fusion*, 2017.

$$\boldsymbol{B}_M = \frac{\mu_0}{4\pi} \sum_{i=1}^{D} \left(\frac{3\boldsymbol{m}_i \cdot \boldsymbol{r}_i}{\|\boldsymbol{r}_i\|_2^5} \boldsymbol{r}_i - \frac{\boldsymbol{m}_i}{\|\boldsymbol{r}_i\|_2^3} \right)$$



Zhu, Zarnstorff, Gates, & Brooks, *Nuclear Fusion*, 2020.

$$x(t) = x_{c,0} + \sum_{n=1}^{N_F} [x_{c,n} \cos(nt) + x_{s,n} \sin(nt)],$$



Zhu, Hudson, Song, & Wan, *Nuclear Fusion*, 2017.

Stellarator coils can be designed with fixed grid and local basis functions

- New fourth method for coil optimization – allows for finite-builds & no topology assumptions.
- Initialize continuous 3D grid where current can flow, optimize for coils, but need to enforce current conservation.
- Divergence-free, linear basis functions for the currents: $\boldsymbol{J}_k \equiv \boldsymbol{\alpha}_k \cdot \boldsymbol{\phi}_k(\boldsymbol{r}_k') = \sum^{N} \alpha_{ki}\boldsymbol{\phi}_{ki}.$

centers of the cell at (x_k, y_k, z_k) and $\bar{\Delta} = (\Delta x_k \Delta y_k \Delta z_k)^{\frac{1}{3}}$,

- Still need constraints to match fluxes at interfaces:

$$X_k \equiv \frac{x - x_k}{\bar{\Delta}}, \quad Y_k \equiv \frac{y - y_k}{\bar{\Delta}}, \quad Z_k \equiv \frac{z - z_k}{\bar{\Delta}},$$

$$\int_{V_k' \cap V_l'} \hat{\boldsymbol{n}}' \cdot \left[\boldsymbol{J}_k(\boldsymbol{r}_k') - \boldsymbol{J}_l(\boldsymbol{r}_k')\right] d^2 r_k' = 0$$

$$\boldsymbol{\phi}_k(\boldsymbol{r}_k') = \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}X_k\\-Y_k\\0\end{bmatrix}, \begin{bmatrix}X_k\\0\\-Z_k\end{bmatrix}$$

  - N basis functions (N = 5) and D total voxels

$$\boldsymbol{B}_{\text{coil}}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}} = -\frac{\mu_0}{4\pi} \sum_{k=1}^{D} \int_{V_k'} \frac{\hat{\boldsymbol{n}} \times (\boldsymbol{r} - \boldsymbol{r}_k')}{\|\boldsymbol{r} - \boldsymbol{r}_k'\|^3} \cdot \boldsymbol{J}_k(\boldsymbol{r}_k') d\boldsymbol{r}_k'$$

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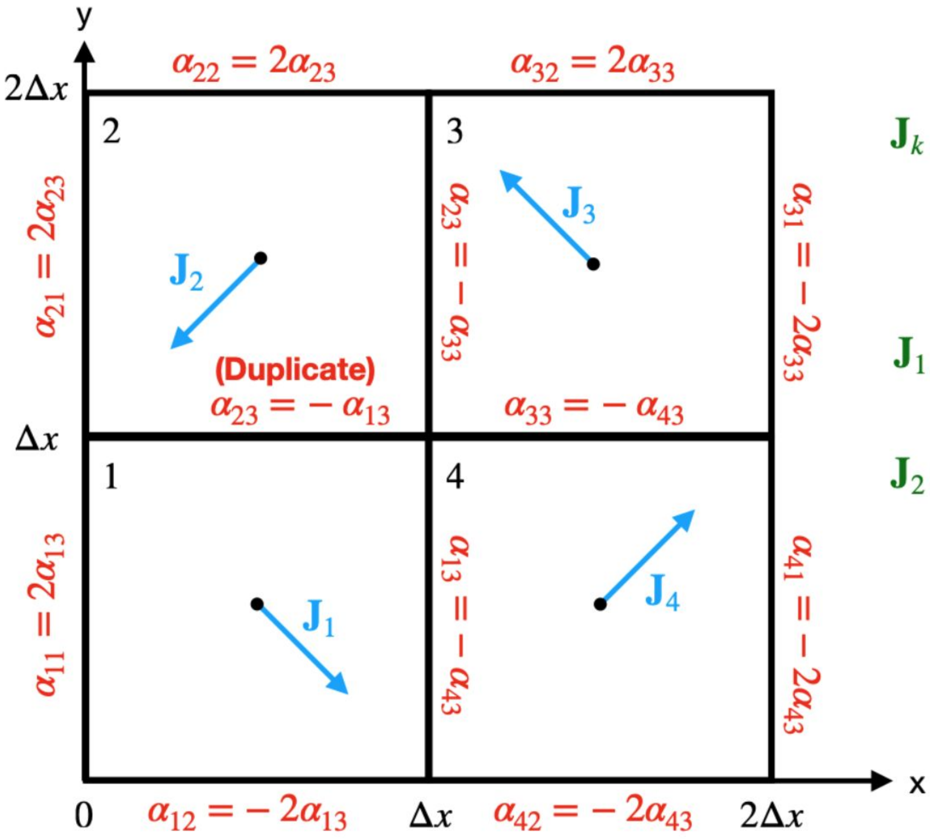
Simple 2D voxel solution illustrates circulating current solutions

***Solution** for four 2D square voxels. There are 12 free parameters and 11 unique constraints from flux matching at cell interfaces.*



Basis expansion

$$\mathbf{J}_k = \alpha_{k1}\begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_{k2}\begin{bmatrix}0\\1\\0\end{bmatrix} + \alpha_{k3}\begin{bmatrix}X_k\\-Y_k\\0\end{bmatrix}$$

Solution

$$\mathbf{J}_1 = \varphi\begin{bmatrix}2+X_k\\-2-Y_k\end{bmatrix} \quad \mathbf{J}_3 = \varphi\begin{bmatrix}-2+X_k\\2-Y_k\end{bmatrix}$$

$$\mathbf{J}_2 = -\varphi\begin{bmatrix}2+X_k\\2-Y_k\end{bmatrix} \quad \mathbf{J}_4 = -\varphi\begin{bmatrix}-2+X_k\\-2-Y_k\end{bmatrix}$$

Solution at midpoints $(X_k, Y_k) = (0,0)$

$$\mathbf{J}_1 = 2\varphi\begin{bmatrix}1\\-1\end{bmatrix} \quad \mathbf{J}_3 = 2\varphi\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\mathbf{J}_2 = 2\varphi\begin{bmatrix}-1\\-1\end{bmatrix} \quad \mathbf{J}_4 = 2\varphi\begin{bmatrix}1\\1\end{bmatrix}$$

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The voxel optimization problem is sparse regression

- Optimization in the voxel method reduces to optimizing the basis coefficients of the $\boldsymbol{\alpha}$:

$$f_B(\alpha) \equiv \frac{1}{2}\|\boldsymbol{A\alpha} - \boldsymbol{b}\|_2^2,$$

$$f_I(\alpha) \equiv \frac{1}{2}\|\boldsymbol{A}_I\boldsymbol{\alpha} - \boldsymbol{b}_I\|_2^2.$$

$$f_K(\alpha) \equiv \frac{1}{2D}\|\boldsymbol{\alpha}\|_2^2,$$

Match the target field | Tikhonov regularization | Avoid the trivial solution | Zero out most voxels

$$\min_{\boldsymbol{\alpha}} \left\{ f_B(\boldsymbol{\alpha}) + \kappa f_K(\boldsymbol{\alpha}) + \sigma f_I(\boldsymbol{\alpha}) + \lambda \|\boldsymbol{\alpha}\|_0^G \right\}$$

$$s.t. \quad \boldsymbol{C\alpha} = 0.$$

Enforces div(**J**) = 0 everywhere

$$\|\boldsymbol{\alpha}\|_0^G = \left\| \left[\|\boldsymbol{\alpha}_1\|_2, \ldots, \|\boldsymbol{\alpha}_D\|_2\right] \right\|_0 = \text{total \# of } \|\boldsymbol{\alpha}_k\|_2 \neq 0.$$

the l_0 norm of the two-norms of the $\boldsymbol{\alpha}_k$ in each cell

Permanent magnet optimization is the same problem

- Optimization in the **PM** method reduces to optimizing the **dipole moments** α:

Match the target field | Tikhonov regularization | **TF coils provide this so σ = 0** | Zero out most **magnets**

$$\min_{\alpha}\left\{f_B(\alpha)+\kappa f_K(\alpha)+\sigma f_I(\alpha)+\lambda\|\alpha\|_0^G\right\}$$

$$s.t. \quad C\alpha = 0.$$

Enforces **max dipole strengths**

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The quasi-equivalent fully discrete problem is called topology optimization

- Very important problem in structural mechanics and other fields!

Match the target field | Tikhonov regularization | Avoid the trivial solution

$$\min_{\alpha} \left\{ f_B(\alpha) + \kappa f_K(\alpha) + \sigma f_I(\alpha) \right\}$$

$$s.t. \quad C\alpha = 0.$$

Linear constraints

$$\alpha = \{0, 1, 2, \dots\}$$

Only discrete values for the optimization variables

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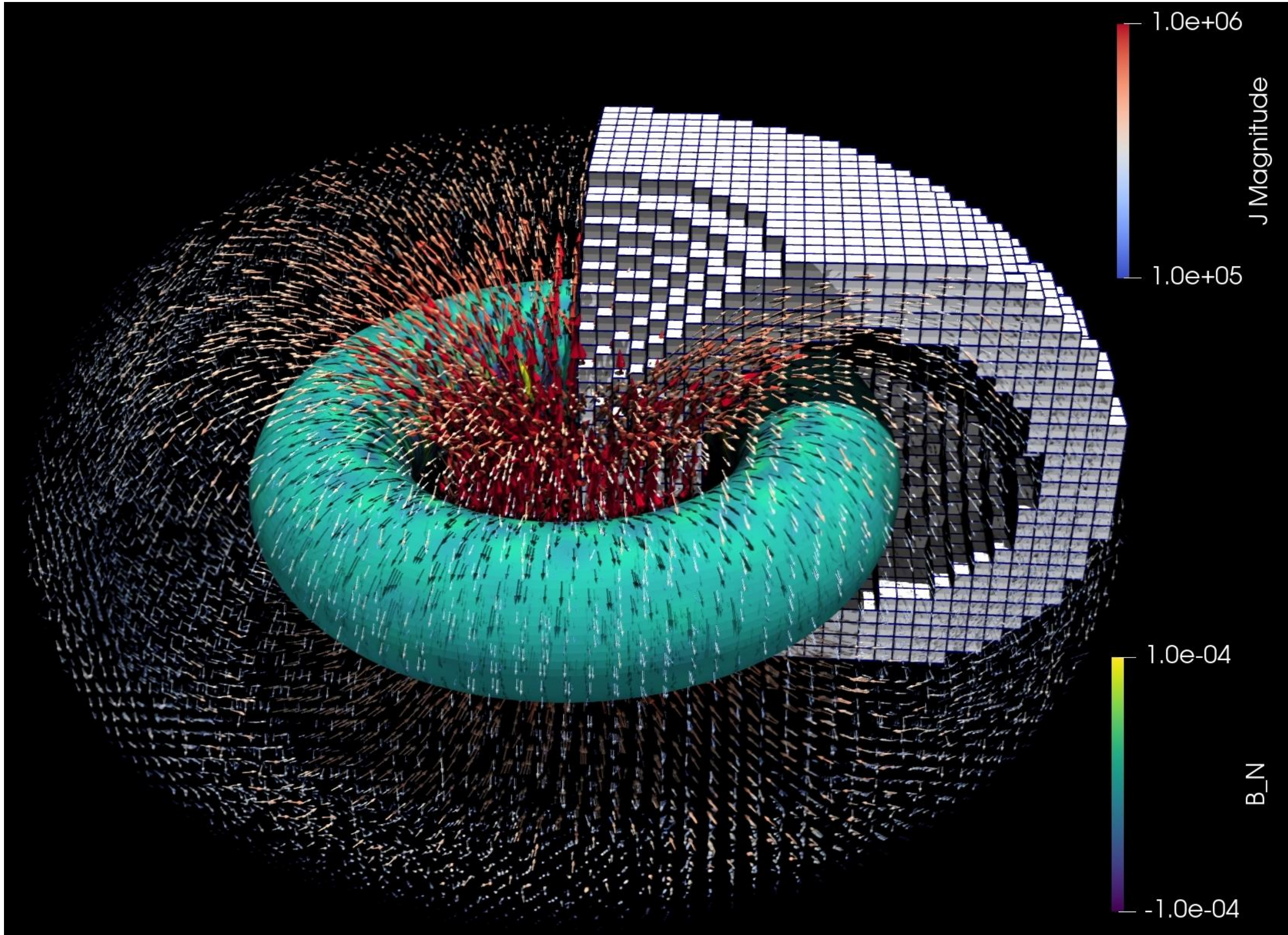
Results: axisymmetric torus with no sparsity

Full geometry & solution for an axisymmetric torus with 1 Tesla on-axis ($\lambda = 0$). The unique quarter of the voxel grid is pictured. ***B·n*** *errors are shown on the plasma surface S and the cell-averaged* ***J*** *solution vectors are color-coded by ||**J**||.*



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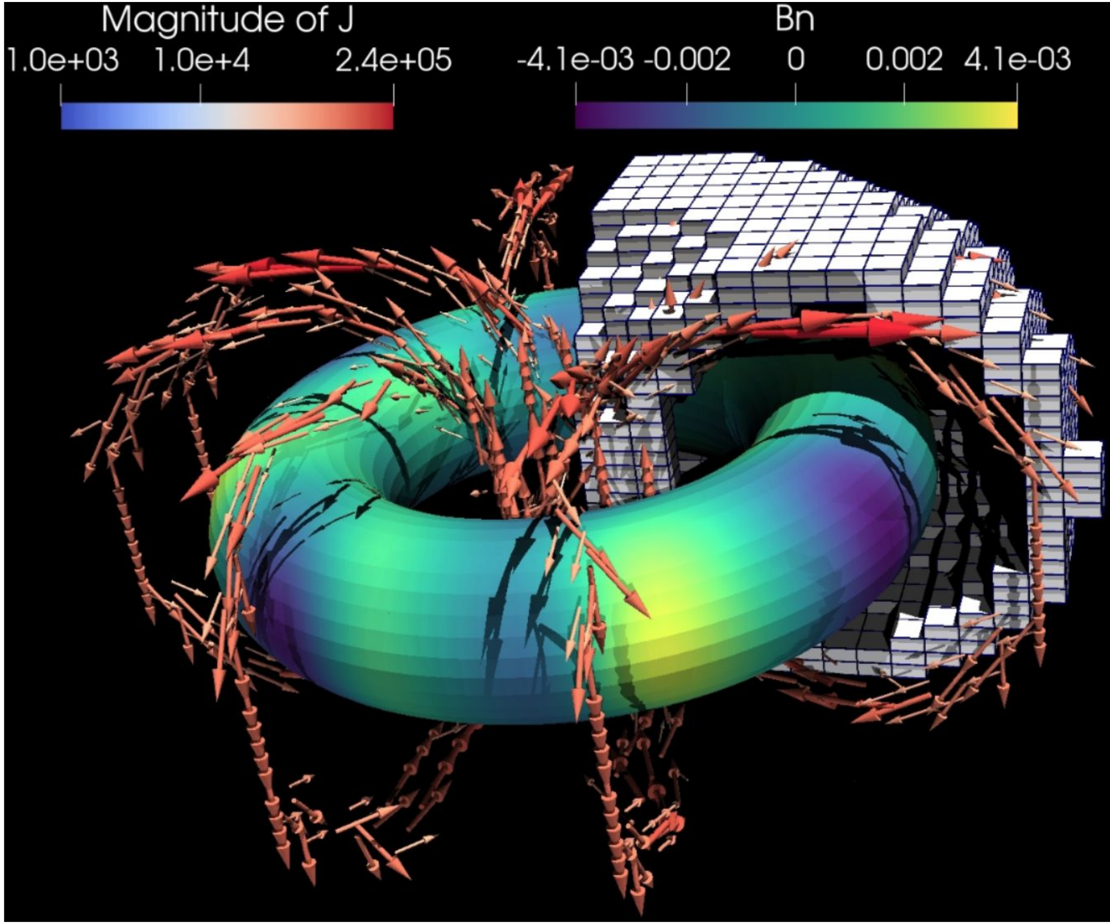
Results: axisymmetric torus with sparsity

Full geometry & solution for an axisymmetric torus with 1 Tesla on-axis $(\lambda > 0)$.



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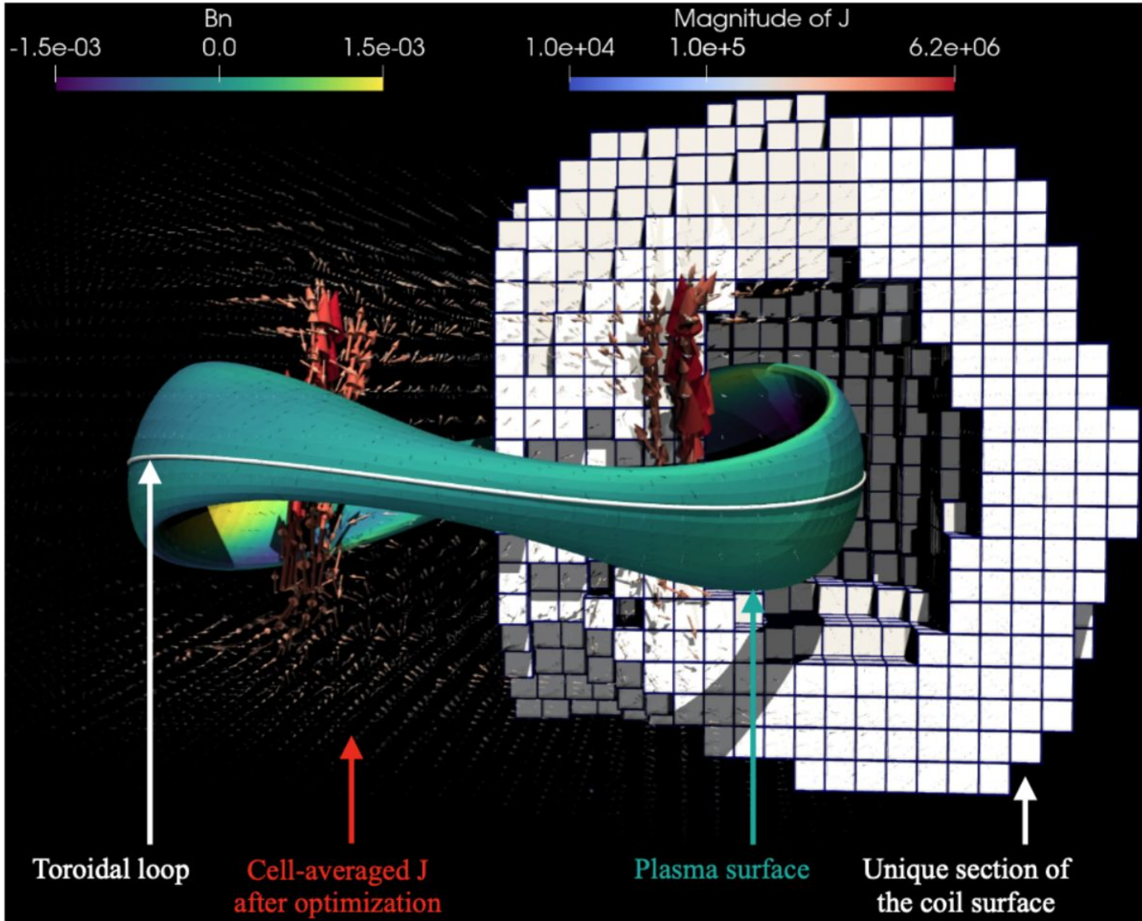
Results: Landreman/Paul QA stellarator without sparsity

Full geometry for the Landreman & Paul QA stellarator, ($\lambda = 0$). The unique quarter of the voxel grid is pictured.



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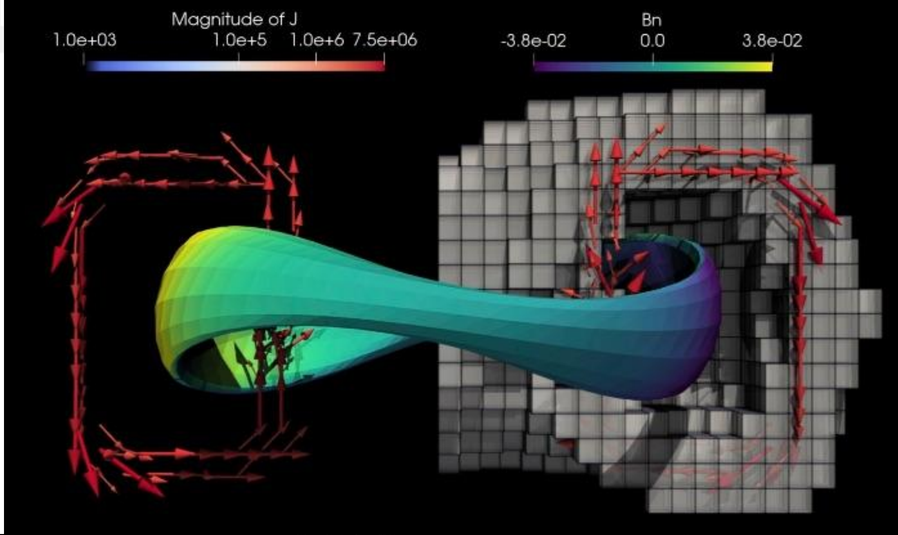
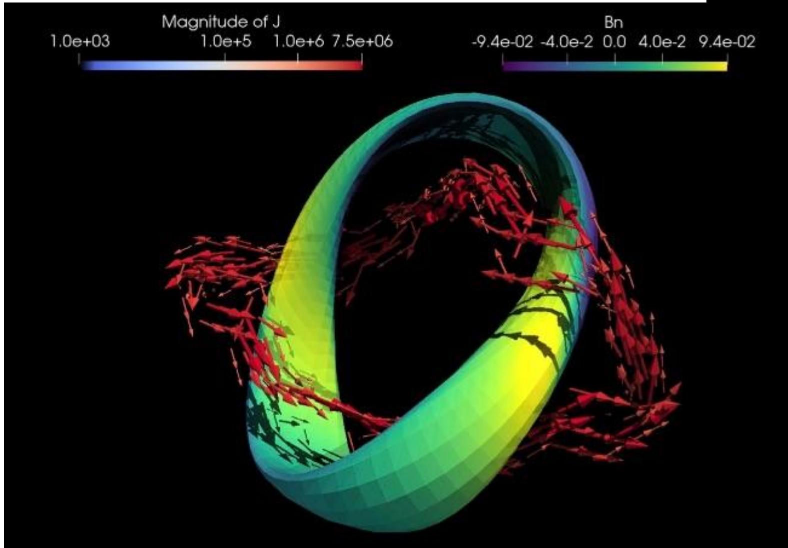
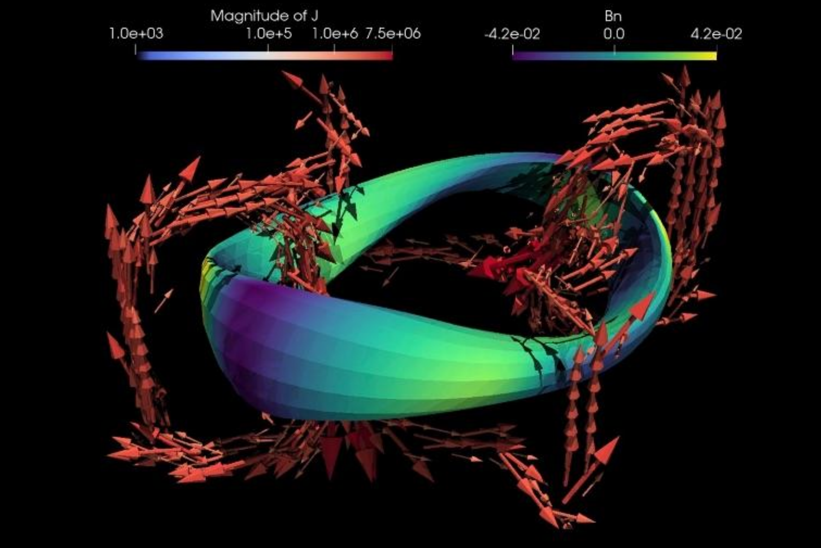
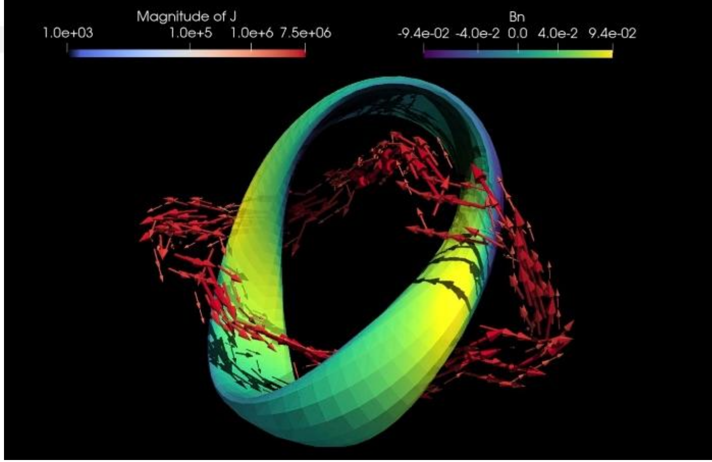
Results: Landreman/Paul QA stellarator with sparsity

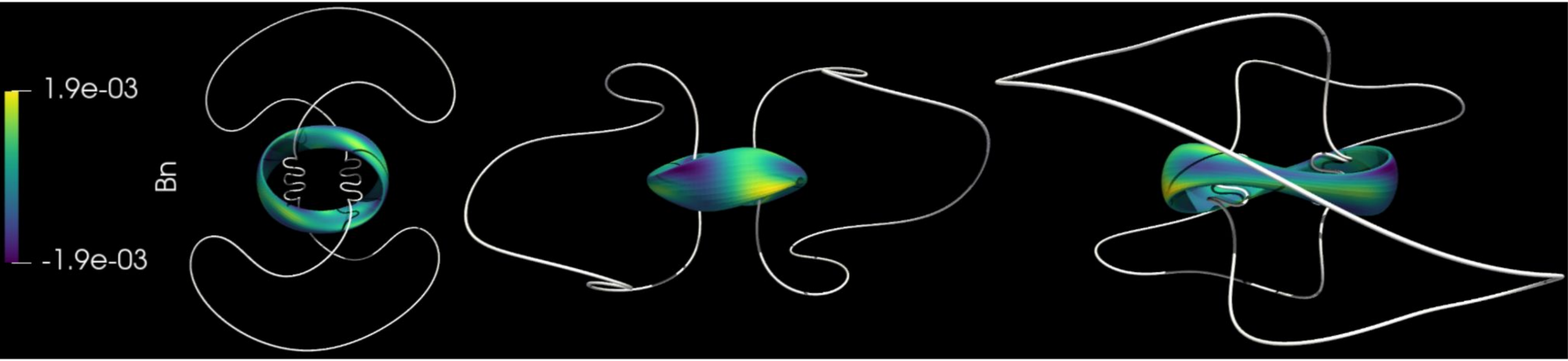






Voxel solutions can be used to initialize filament topology and then filament optimization performed





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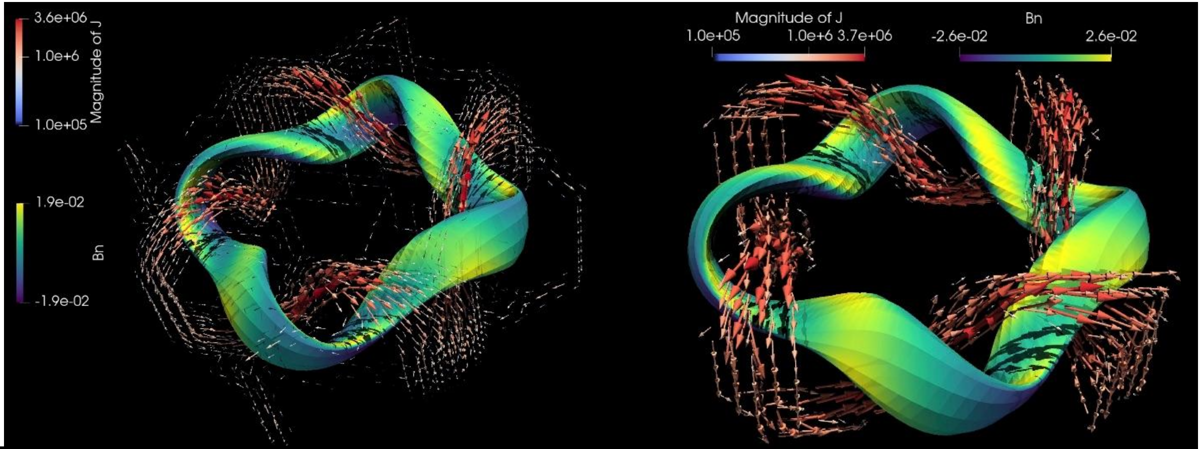
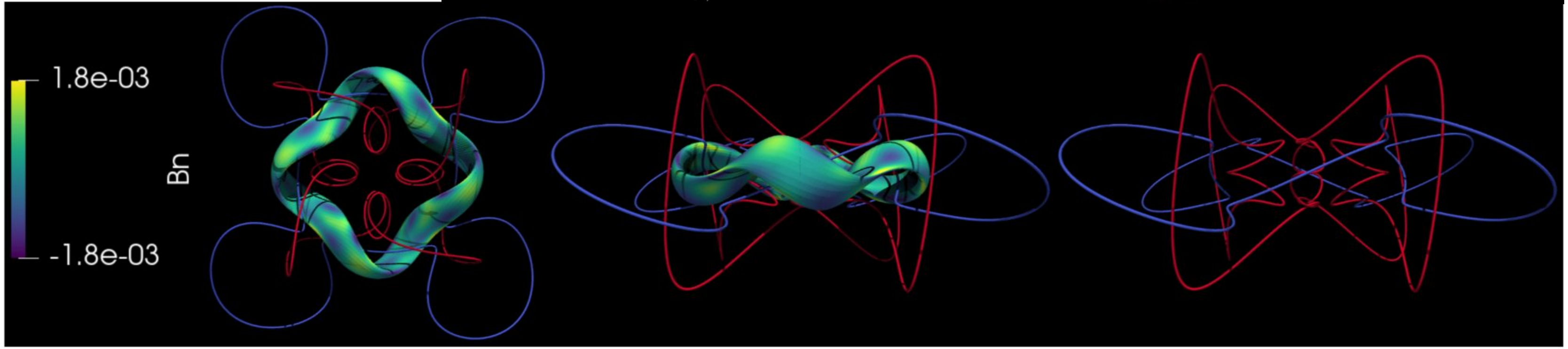
Three views of a 40 meter filament coil generated from a voxel solution.

Helical coils for the Landreman-Paul QH stellarator

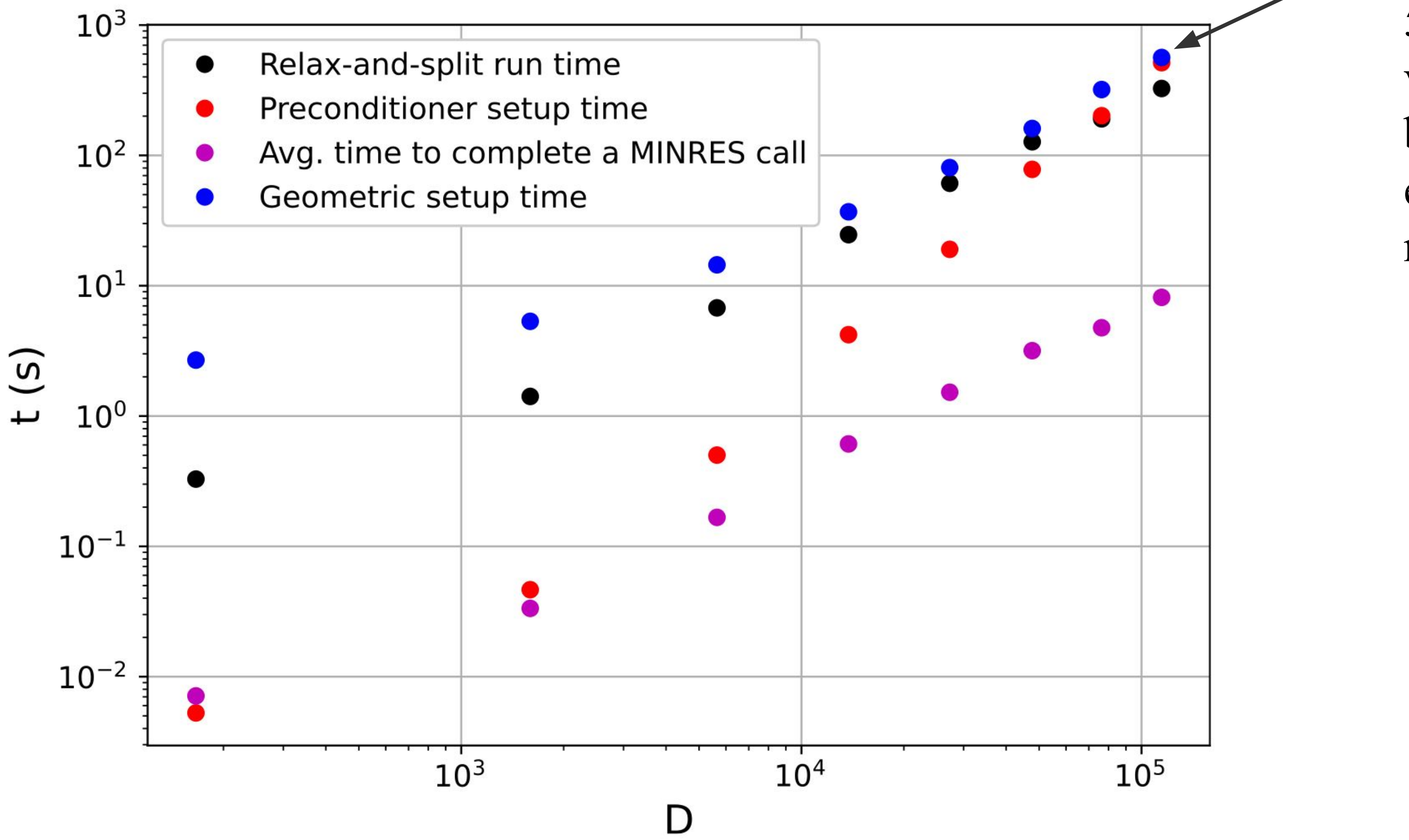
Often get solutions that aren't sparse enough to get coils out of





Three views of the helical coils with combined 24 + 29 = 53 meter length, generated from a voxel solution for the Landreman-Paul QH stellarator.

Some numerical speed tests



Grid of 114,208 unique voxels, 571,040 optimization variables, and 326 billion nonzero elements in the A^TA matrix from f_B.

Advertisement for permanent magnet work

Improved stellarator permanent magnet designs through combined discrete and continuous optimizations

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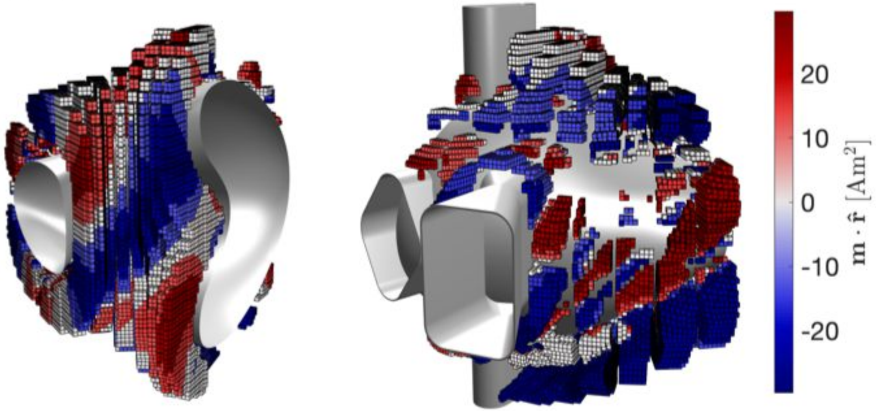
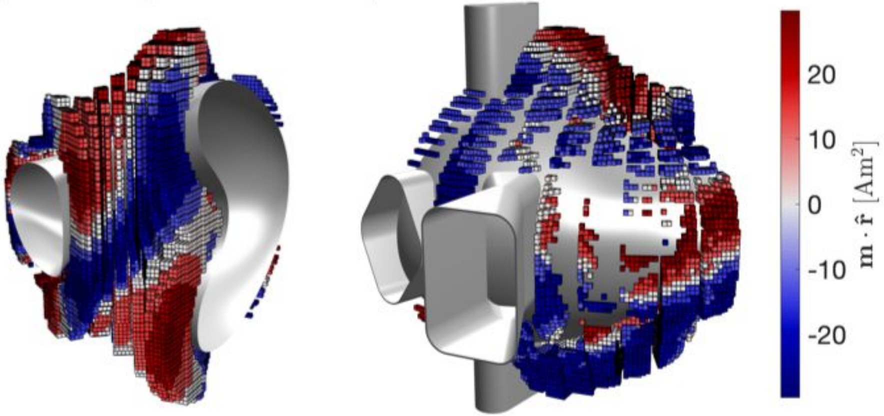
Original PM4Stell solution

New PM4Stell solution with same f_B error but 30% fewer magnets!

(a) RC, $\mathbf{m} \cdot \hat{\mathbf{r}}$



(a) $\mathrm{GPMOb}_{45°,26}$$-RC-$$\mathrm{GPMOB}_{45°,26}$, $\mathbf{m} \cdot \hat{\mathbf{r}}$



Future directions for voxel optimization

- Future work includes:
  - Implementation of higher-order basis functions,
  - Tetrahedral meshes,
  - Algorithmic speedups through improved iterative solvers and preconditioners or improved sparse regression algorithms, additional loss terms in the optimization,
  - Reformulation as stochastic optimization to control for coil errors, and much more.
  - A reformulation may be possible that builds in the current conservation by construction.
  - Initial conditions for the optimization can bias the solutions towards producing a particular topological structure or a certain number of identifiable coils.
  - Loss terms to bunch up the currents better?

This work is so far most compelling for providing principled topology choices to initialize more complex filament optimization for stellarators.

Extra slides

Relax and split method for solving the optimization

Match the target field | Tikhonov regularization | Avoid the trivial solution | Zero out most voxels

$$\min_{\alpha} \left\{ f_B(\alpha) + \kappa f_K(\alpha) + \sigma f_I(\alpha) + \lambda \|\alpha\|_0^G \right\}$$

$$s.t. \quad C\alpha = 0.$$

Enforces div(**J**) = 0 everywhere

$$f_B(\alpha) \equiv \frac{1}{2}\|A\alpha - b\|_2^2,$$

$$f_I(\alpha) \equiv \frac{1}{2}\|A_I\alpha - b_I\|_2^2.$$

$$f_K(\alpha) \equiv \frac{1}{2D}\|\alpha\|_2^2,$$

Relax and split method for solving the optimization

$$\min_{\boldsymbol{\beta}} \left\{ \min_{\boldsymbol{\alpha}} \left\{ \frac{\|\boldsymbol{A\alpha} - \boldsymbol{b}\|_2^2}{2} + \frac{\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_2^2}{2\nu} \right\} + \lambda \|\boldsymbol{\beta}\|_0^G \right\}$$

$$s.t. \quad \boldsymbol{C\alpha} = \boldsymbol{0}.$$

$$\boldsymbol{\alpha}^{(j)} \equiv \arg\min_{\boldsymbol{\alpha}} \left\{ \frac{\|\boldsymbol{A\alpha} - \boldsymbol{b}\|_2^2}{2} + \frac{\|\boldsymbol{\alpha} - \boldsymbol{\beta}^{(j-1)}\|_2^2}{2\nu} \right\},$$

$$s.t. \quad \boldsymbol{C\alpha} = \boldsymbol{0},$$

$$\boldsymbol{\beta}^{(j)} \equiv \arg\min_{\boldsymbol{\beta}} \left\{ \frac{1}{2\nu} \|\boldsymbol{\alpha}^{(j)} - \boldsymbol{\beta}\|_2^2 + \lambda \|\boldsymbol{\beta}\|_0^G \right\}$$

$$\boldsymbol{\beta}_k^{(j)} = prox_{\nu\lambda\|(.)\|_0^G}(\boldsymbol{\alpha}_k^{(j)}) = \begin{cases} 0, & \|\boldsymbol{\alpha}_k^{(j)}\|_2 < M_{\nu\lambda} \\ \alpha_k^{(j)}, & \|\boldsymbol{\alpha}_k^{(j)}\|_2 \geq M_{\nu\lambda} \end{cases}.$$

Hyperparameters in the voxel method

Table B.1

Description of the hyperparameters for our proposed coil optimization. With reasonable values for the convex optimization, $\lambda = 0$, $\nu \to \infty$, $\sigma = 1$, and $\kappa = 10^{-15}$, the geometric parameters have converged by $D \approx 10,000$, $N_x \approx 6$, $n_\zeta n_\theta = 64^2$, and $n_\gamma = 8$. We find that these values are fairly robust to different stellarator configurations.

Hyperparameter	Type	Description	Default value
λ	Optimization	Specifies the strength of group sparsity-promotion.	0
ν	Optimization	How closely the $\boldsymbol{\alpha}^*$ and $\boldsymbol{\beta}^*$ solutions of Eq. (14) should match in L_2.	∞
κ	Optimization	Degree of Tikhonov regularization.	10^{-15}
σ	Optimization	How stringently to match the prescribed I_{target} through a toroidal loop.	1
D	Geometric	Number of grid cells.	$\sim 10^3 - 10^5$
N'	Geometric	Number of points used for each cell's Biot–Savart calculations.	6^3
$n_\zeta n_\theta$	Geometric	Number of uniformly-spaced quadrature points on the plasma surface.	16^2
n_γ	Geometric	Number of uniformly-spaced quadrature points on the toroidal loop.	8

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