# Algebraic approach to interfaces in the three-state Potts model

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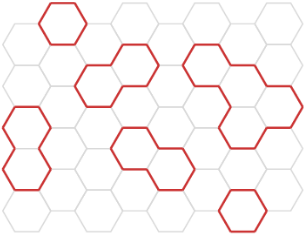
# From loops to webs

## Loop models: what, why, how?

- Self-avoiding (open or closed) simple curves in two dimensions
- Polymers, level lines, domain walls, electron gases
- Lattice: Integrability, knot theory, cellular algebras, category theory
- Continuum limit: CFT, CLE, SLE

## Definition and features

- Fix lattice of nodes and links
- Place bonds on some links so as to form set of loops
- Weight $x$ per bond (+ maybe further local weights) and $N$ per loop
- For $|N| \leq 2$, dense and dilute critical points $x_c^{\pm}$
- Continuum limit of compactified free bosonic field (Coulomb gas) [Nienhuis, Di Francesco-Saleur-Zuber, Duplantier, Cardy ...]



## Generalisation to webs

- Allow for branchings and bifurcations (with weights)
- Topological rules give weight to each connected web component
- Properties and possible critical behaviour?

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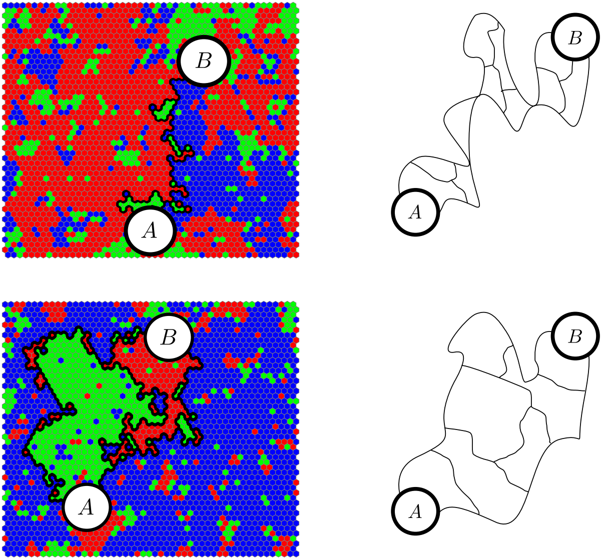
## Motivations for webs

- Domain walls in spin systems [Dubail-JJ-Saleur, Picco-Santachiara]
- Network models for topological phases [Kitaev, Levin-Wen, Fendley]
- Spiders in invariance theory [Kuperberg, Kim, Cautis-Kamnitzer-Morrison]

# Thin and thick domain walls ($Q = 3$ Potts model)



## Questions (physics)

- How to define a “good” model of webs on the lattice?
- Fractal dimension of such domain walls (bulk / boundary)?
- Fractal dimension of an entire web component?
- Topological weight of web versus chromatic polynomial in $Q = 3$?
- Web model away from this special point?

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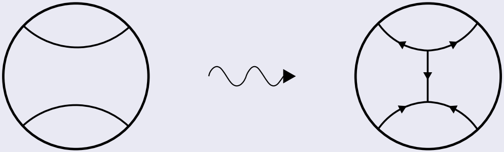
## Questions (mathematics)

- Algebraic construction accounting for bifurcations?
- Loop model has $U_{-q}(\mathfrak{sl}_2)$ symmetry, can we get $U_{-q}(\mathfrak{sl}_n)$?



# Web model from Kuperberg $A_2$ spider ($U_{-q}(\mathfrak{sl}_3)$ case)

## Lattice considerations

- Hexagonal (honeycomb) lattice $\mathbb{H}$ with nodes and links
- Configuration $c$ by drawing bonds on some links, with constraints:
  - Nodes have valence 0, 2 or 3: closed web with 3-valent vertices
  - Each bond is oriented. Orientations conserved at 2-valent nodes
  - Vertices are sources or sinks (all bonds point in or out)

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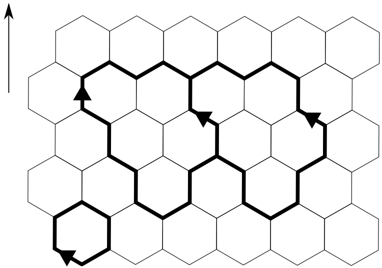
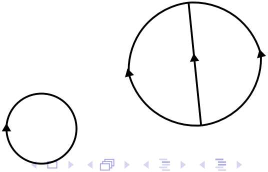
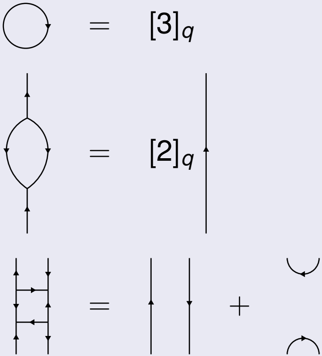
Each configuration can be seen as an abstract graph (vertices/edges). It is closed, planar, trivalent, bipartite. Fix an orientation (= 'up').





# Rules for ‘reducing’ a configuration [Kuperberg]




(1)

(2)

(3)

- Rotated and arrow-reversed diagrams not shown.
- A web component always has $\geq 1$ polygon of degree 0, 2 or 4.
- The three rules thus evaluate any web to a number (its weight)

Define $q$-deformed numbers: $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$

## Defining the web model

- Sum over configurations $c \in K$ on $\mathbb{H}$
- Local weights: $x_1$ (up bond), $x_2$ (down bond), $y$ (sink), $z$ (source)
- Partition function:

$$Z_{\mathrm{K}} = \sum_{c \in K} x_1^{N_1} x_2^{N_2} (yz)^{N_V} w_{\mathrm{K}}(c)$$

with $N_1$ up-bonds, $N_2$ down-bonds, and $N_V$ vertex pairs

# $\mathbb{Z}_3$ spin model

## Definition

- Spins $\sigma_i \in \mathbb{Z}_3 := \{0, 1, 2\}$ defined on triangular lattice $\mathbb{T} = \mathbb{H}^*$.
- Weight of link $(ij) \in \mathbb{T}$ defined as $x_{\sigma_j - \sigma_i}$, with $j$ to the right of $i$.
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## Partition function

$$Z_{\rm spin} = 3 \sum_{c \in K} x_1^{N_1} x_2^{N_2}$$

- Equivalent to web model if $w'_{\rm K}(c) := (yz)^{N_V} w_{\rm K}(c) = 1$ for any $c$.

# Equivalence at a special point:

$$q = e^{i\frac{\pi}{4}}\,,$$
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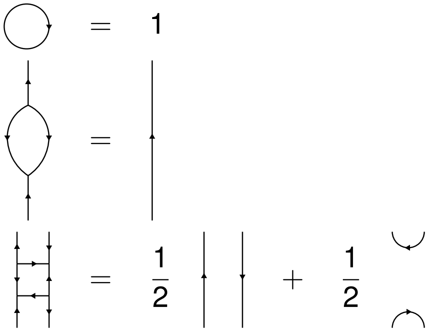
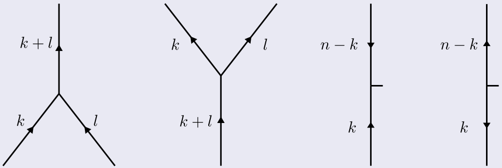
**Proof**: Absorb $y$ and $z$ into the vertices. Use $[3]_q = 1$ and $[2]_q = \sqrt{2}$. Then the rules become probabilistic:



# Generalisation to $U_{-q}(\mathfrak{sl}_n)$ symmetry

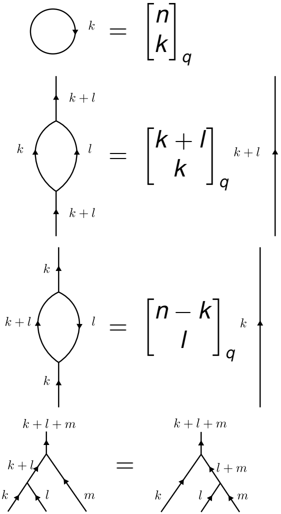
## Based on spider defined by [Cautis-Kamnitzer-Morrison]

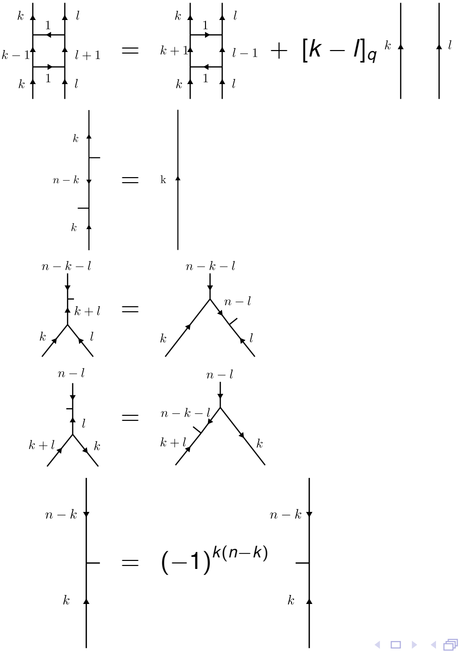
- Webs are still closed, oriented, planar, trivalent graphs. But not always bipartite as before.
- Edges carry an integer flow $i \in [\![1, n-1]\!]$.
- Generators conserve flow, or change by $n$ due to ‘tags’:




- Flow labels fundamental representations of $U_{-q}(\mathfrak{sl}_n)$. Orientation distinguishes between dual or not.

Rules (mirrored and the arrow-reversed versions omitted):




k l 1 k − 1 l + 1 1 k l
=
k l 1 k + 1 l − 1 1 k l
+ [k − l]_q k l
k n − k k = k
n − k − l k + l k l = n − k − l n − l k l
n − l l k + l k = n − l n − k − l k + l k
n − k k = (−1)^{k(n−k)} n − k k

## Short summary of results

- Case $n = 3$ gives back the Kuperberg web model.
- Case $n = 2$ gives the well-known Nienhuis loop model.
- Special point $q = e^{i\frac{\pi}{n+1}}$ equivalent to $\mathbb{Z}_n$ spin model.

# Outlook this far

- $\mathbb{Z}_n$ spin models known to be critical and integrable (with appropriate weights) [Fateev-Zamolodchikov]
- Therefore expect the special point to be critical for any $n$.
- Web models likely have larger critical manifold (vary $q$ and $x, y, z$).
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**To investigate criticality/integrability we wish a local formulation**

- Analogous to vertex models for Potts and O($N$) models.
- The locality enables us to define a transfer matrix / $R$-matrix.
  - Good for numerical study and makes contact with integrability.
  - Non-local TM also possible for loops, but seems difficult for webs.
- Vertex model defines equivalent ($n - 1$ component) height model.
  - Starting point for Coulomb gas construction and CFT identification.

# Local reformulation for $U_{-q}(\mathfrak{sl}_3)$ web model

## Basic idea

- Decorate bonds by extra degrees of freedom ($n = 3$ colours).
- They allow to redistribute the web weight locally.
- Summing over colours gives back the undecorated model.
- Each link can now be in 7 different states.

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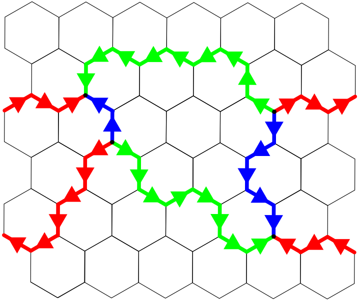

## Reminder for $n = 2$ loop case

- Write $N = q + q^{-1} = [2]_q$.
- Orient each loop in two ways (clockwise, anticlockwise).
- Give $q^{-\frac{\theta}{2\pi}}$ to a left-turn through angle $\theta$.

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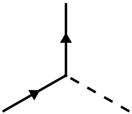
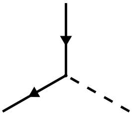


$= xq^{-\frac{1}{6}}$ ,



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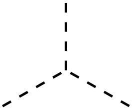

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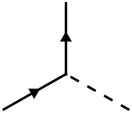

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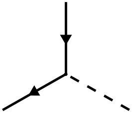

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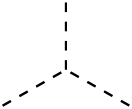


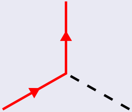
$= 1$

## Remark

Better to think of these two ‘orientations’ as colourings. The analogue for $n = 3$ is the three colours. The orientations distinguish (for $n \geq 3$) fundamental and dual fundamental, but for $n = 2$ the two coincide!

## Basic idea for $n = 3$

- Three colours RBG.
- Weight $q^2 + 1 + q^{-2} = [3]_q$ for sum over (say) clockwise loop.
  Opposite phases for an anticlockwise loop (same sum).
  Set $x_1 = x_2$ for convenience.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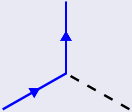
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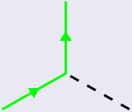
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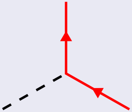
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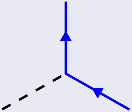
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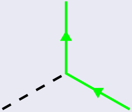
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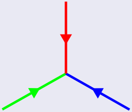
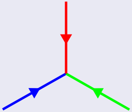
## The ‘tricky’ part involving vertices



$$= zx^{\frac{3}{2}} q^{-\frac{1}{6}} \,,$$



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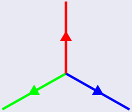

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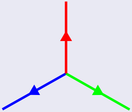

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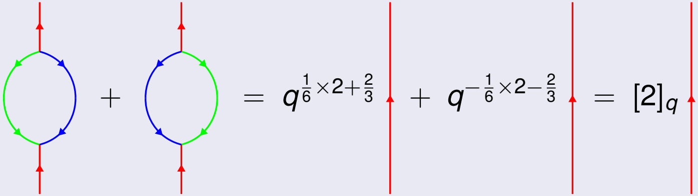
# Proof for the ‘digon’ rule (2)



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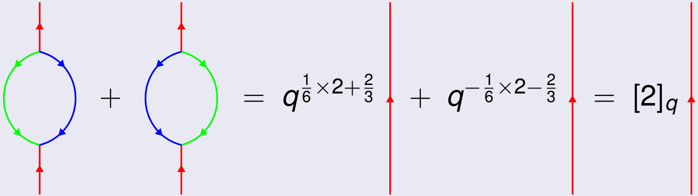
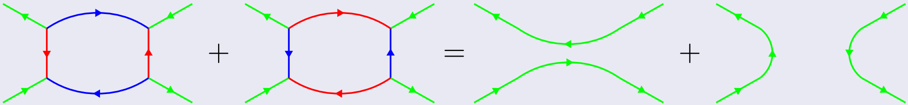



## Proof for the ‘square’ rule (3)



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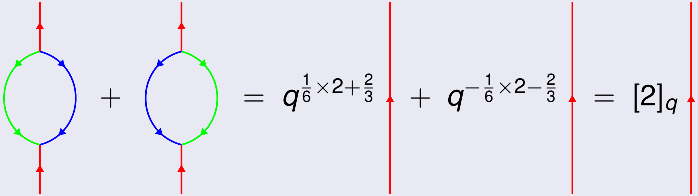


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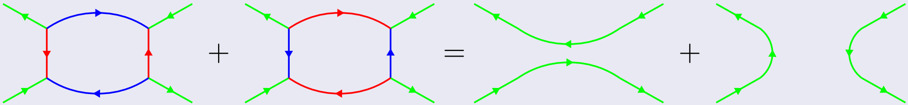
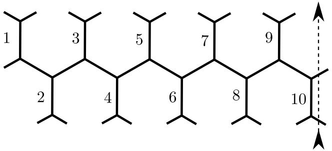



Other colours / arrangements of external legs work similarly.

# Defining the transfer matrix




Built of pieces $t_{(1)} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and $t_{(2)} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, so that

$$T = \left( \prod_{k=0}^{L-1} t_{2k+1} \right) \left( \prod_{k=1}^{L-1} t_{2k} \right)$$

with $t = t_{(2)} t_{(1)}$. Write $t_i$, with $i$ specifying the position.

Technically $T$ is an intertwiner of the quantum group action.

- Let $\{v_1, v_2, v_3\}$ be a basis of the first fundamental $V_1$ of $U_{-q}(\mathfrak{sl}_3)$.
- Let $\{w_1, w_2, w_3\}$ be a basis of the dual $V_1^*$, so that $w_i(v_j) = \delta_{ij}$.

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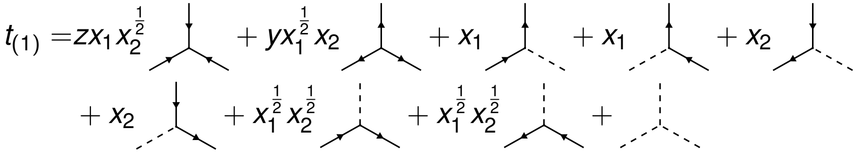

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  Amounts to drawing each link vertically and providing the corresponding powers of $q$.
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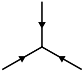
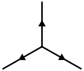
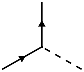
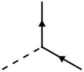
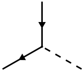
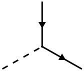
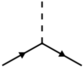
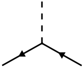
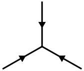
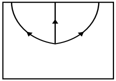
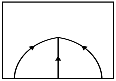
$t_{(1)} = z x_1 x_2^{\frac{1}{2}}$  $+ y x_1^{\frac{1}{2}} x_2$  $+ x_1$  $+ x_1$  $+ x_2$ 

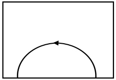
$+ x_2$  $+ x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$  $+ x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$  $+$ 

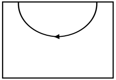
- Let us have a look at just the first term! 

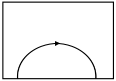
- Express each diagram in terms of the elementary blocks (maps)

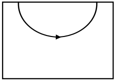
$v =$ 

$w =$ 

$\mathrm{ev} =$ 

$\mathrm{coev} =$ 

$\widetilde{\mathrm{ev}} =$ 

$\widetilde{\mathrm{coev}} =$ 

Their expressions follow from quantum group considerations.

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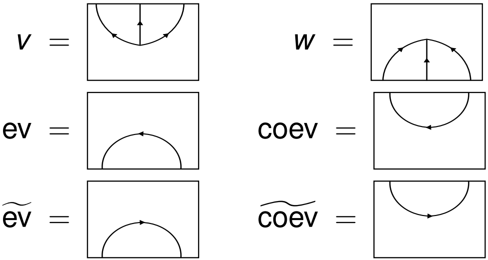
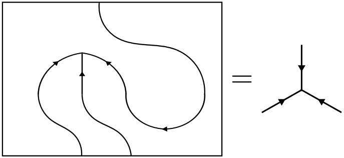



Their expressions follow from quantum group considerations.

- The first term is the composition of $\mathrm{coev}$ and $w$:



- In the bases $\{|\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\uparrow\uparrow\rangle\}$ of $V_1 \otimes V_1$ and $\{|\downarrow\rangle, |\downarrow\rangle, |\downarrow\rangle\}$ of $V_1^*$, we finally get

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{6}} & 0 & q^{-\frac{1}{6}} & 0 \\ 0 & 0 & q^{\frac{1}{6}} & 0 & 0 & 0 & q^{\frac{1}{6}} & 0 & 0 \\ 0 & q^{\frac{1}{6}} & 0 & q^{-\frac{1}{6}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Looks familiar?

- In the bases $\{|{\color{red}\uparrow}{\color{red}\uparrow}\rangle, |{\color{red}\uparrow}{\color{blue}\uparrow}\rangle, |{\color{red}\uparrow}{\color{green}\uparrow}\rangle, |{\color{blue}\uparrow}{\color{red}\uparrow}\rangle, |{\color{blue}\uparrow}{\color{blue}\uparrow}\rangle, |{\color{blue}\uparrow}{\color{green}\uparrow}\rangle, |{\color{green}\uparrow}{\color{red}\uparrow}\rangle, |{\color{green}\uparrow}{\color{blue}\uparrow}\rangle, |{\color{green}\uparrow}{\color{green}\uparrow}\rangle\}$ of $V_1 \otimes V_1$ and $\{|{\color{red}\downarrow}\rangle, |{\color{blue}\downarrow}\rangle, |{\color{green}\downarrow}\rangle\}$ of $V_1^*$, we finally get

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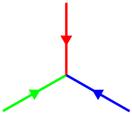
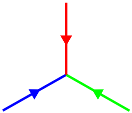
- Looks familiar?
- Hint:



$= zx^{\frac{3}{2}} q^{-\frac{1}{6}}$ ,



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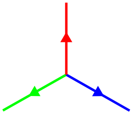

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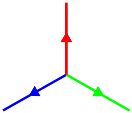


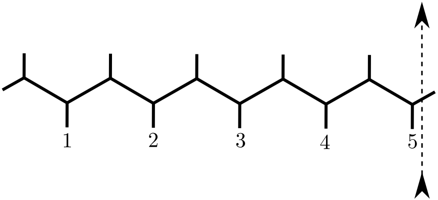
$= yx^{\frac{3}{2}} q^{-\frac{1}{6}}$

## Summary of this technical part

- The diagrams are intertwiners of $U_{-q}(\mathfrak{sl}_3)$.
- We can compute all elements of $T$ in this way.
- We are now ready to diagonalise $T$ numerically.

# Phase diagram of the web model

- More efficient to use the geometry

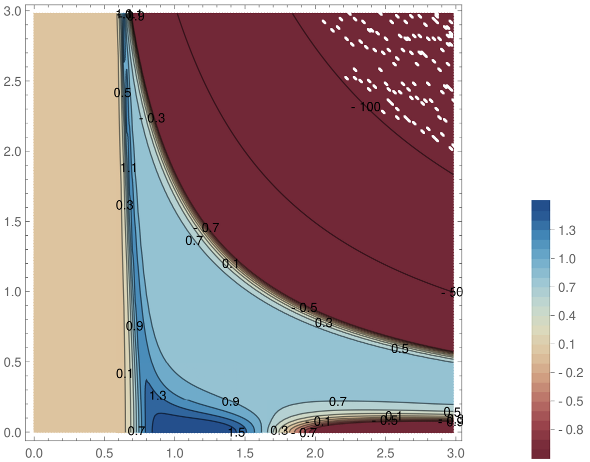



- Connection to the (effective) central charge of CFT:

$$f_L = -\frac{2}{\sqrt{3}L} \log(\Lambda_{\max}) \,,$$

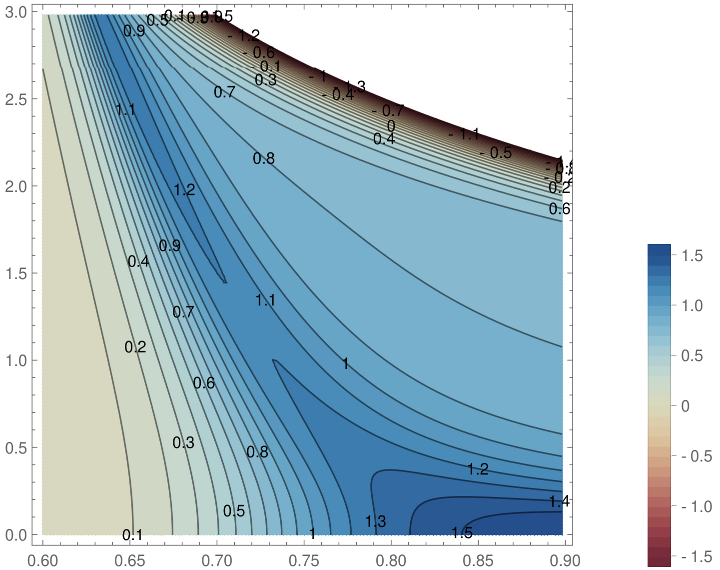
$$f_L = f_\infty - \frac{\pi c_{\text{eff}}}{6L^2} + o\left(\frac{1}{L^2}\right) \,.$$

# $c_{\text{eff}}$ for $q = e^{i\pi/5}$ in the $(\sqrt{x}, y)$ plane




- Based on sizes $L = 5$ and $L = 6$.
- Coulomb gas prediction: dilute $c = \frac{4}{5}$ and dense $c = \frac{6}{5}$ phases.

# Zoom of the interesting region



# Coulomb gas predictions

Set $q = \mathrm{e}^{i\gamma}$ with $\gamma \in [0, \pi]$.

## CG of two bosons compactified on the root lattice of $sl_3$

Coupling constant $g = 1 \pm \frac{\gamma}{\pi}$ in dilute $(+)$ or dense $(-)$ phase.

Central charge $c = 2 - 24\frac{(g-1)^2}{g}$.

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## Example I: $\gamma = \frac{\pi}{5}$ as in numerical figures

Coupling constant $g = \frac{6}{5}$ (dilute) or $g = \frac{4}{5}$ (dense).

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## Example II: $\gamma = \frac{\pi}{4}$ as at special point

Coupling constant $g = \frac{5}{4}$ (dilute) or $g = \frac{3}{4}$ (dense).
Central charge $c = \frac{4}{5}$ (dilute) or $c = 0$ (dense).
Corresponds to $Q = 3$ Potts model at $T = T_c$ or $T = \infty$.

# What about integrability?

- The $n = 2$ model (Nienhuis loops) is integrable in both the dilute and dense phases [Baxter 1986-87]

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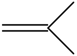
- The $n = 2$ model (Nienhuis loops) is integrable in both the dilute and dense phases [Baxter 1986-87]
- For webs, we study three different rank-2 models [Kuperberg]:
  - $A_2$ web model ($Z_{\mathbb{Z}_3} = 3Z_{A_2}$ at $q = e^{i\pi/4}$)

  - $G_2$ web model ($Z_{\mathbb{Z}_3} = 3Z_{G_2'}$ at $q = e^{i\pi/6}$, with only single lines)

  - $B_2$ web model ($Z_{\mathbb{Z}_4} = 4Z_{B_2}$ at $q = e^{i\pi/4}$)



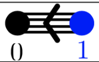
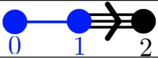
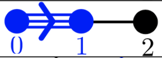
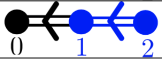
- They satisfy distinct spider relations and lead to models on $\mathbb{H}$.

# Intertwiners

- Intertwining maps for cups, caps and trivalent vertices constructed as before, by invariance considerations.
- For $A_2$: 7-dim representation $V_1 \oplus V_2 \oplus \mathbb{C}$, where $V_1$ ($V_2$) are 3-dim fundamental representations of $U_q(A_2)$ of highest weight $w_1$ ($w_2$).
- For $G_2$: 8-dim representation $V \oplus \mathbb{C}$, where $V$ is 7-dim fundamental representation of $U_q(G_2)$ of highest weight $w_1$.
- For $B_2$: 10-dim representation $V_1 \oplus V_2 \oplus \mathbb{C}$, where $V_1$ ($V_2$) is 4-dim (5-dim) fundamental repr. of $U_q(B_2)$ of highest weight $w_1$ ($w_2$).

# Integrable $\check{R}(u,v)$: General strategy

- Test case $A_1$ (dilute loop model), then web cases $A_2$, $G_2$, $B_2$.
- Guess quantum affine algebra $U_t(\tilde{X}_n^{(k)})$ that contains as a Hopf subalgebra the non-affine quantum group $U_q(X_m)$ of the web.
- In practice, identify affine Dynkin diagram $\tilde{X}_n^{(k)}$ that reduces to the simple Dynkin diagram $X_m$ upon erasing one node.

| $U_t(\tilde{X}_n^{(k)})$ | $U_q(X_m)$ | $\tilde{X}_n^{(k)}$, $X_m$ |
|---|---|---|
| $U_t(A_2^{(2)})$ | $U_{t^4}(A_1)$ |   |
| $U_t(G_1^{(2)})$ | $U_{t^3}(A_2)$ |   |
| $U_t(D_4^{(3)})$ | $U_t(G_2)$ |   |
| $U_t(A_4^{(2)})$ | $U_{t^2}(B_2)$ |   |

- Find an irreducible evaluation representation $(\rho_u, V_u)$, $u \in \mathbb{C}$ of $U_t(\tilde{X}_n^{(k)})$ that decomposes under $U_q(X_m)$ as $V_u = V$, the $u$-independent local space of states of the web model.
- Then (following Jimbo) solve the equation for $\check{R}(u, v)$:

$$\check{R}(u, v)(\rho_u \otimes \rho_v)(a) = (\rho_v \otimes \rho_u)(a)\check{R}(u, v), \quad a \in U_t(\tilde{X}_n^{(k)})$$

- Since $\rho_u \otimes \rho_v$ is irreducible, this admits a unique solution, up to a multiplicative constant.
- Since $U_t(\tilde{X}_n^{(k)})$ has a universal $R$-matrix, $\check{R}(u, v)$ satisfies the spectral-parameter dependent YBE.
- Expanding $\check{R}(u, v)$ as a sum of intertwiners of $U_q(X_m)$ from $V \otimes V$ to itself, we get a linear system for the coefficients.
- Finally, identify values $(u^*, v^*)$ of $(u, v)$ so that only web diagrams that can appear in the transfer matrix on $\mathbb{H}$ have non-zero coefficients.

# Integrable $\check{R}(u, v)$: Results

- For $A_1$ we correctly recover Nienhuis' $A_2^{(2)}$ dilute model (9 intertwiners).
- For $A_2$ webs, solution with 33 intertwiners.
- For $G_2$ webs, solution with 15 intertwiners.
- For $B_2$ webs, solution with 43 intertwiners.

# Summary

- Web models generalise the $U_{-q}(\mathfrak{sl}_2)$ loop model to $U_{-q}(\mathfrak{sl}_n)$.
- Geometrical content with applications to $\mathbb{Z}_n$ spin interfaces.
- Dense and dilute critical points for $q = e^{i\gamma}$ and $\gamma \in [0, \pi]$.

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## More developments

- Coulomb gas description and fractal dimension of defects
- Statistical models for all rank-2 spiders: $A_2$, $G_2$ and $B_2$
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## Further possibilities

- SLE-like description of branching curves?