Baxter's *TQ*-relation and its applications

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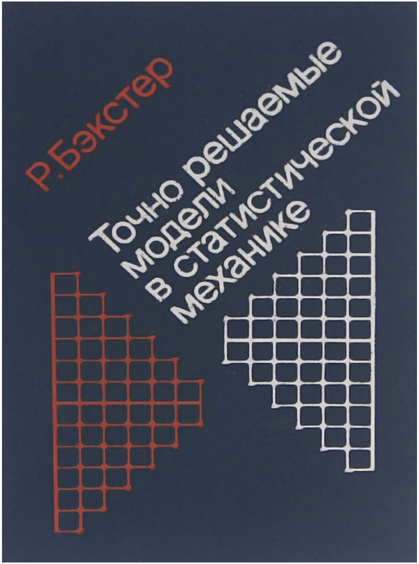
Baxter's book (in Russian)



The Baxter's *TQ*-relation

The famous Baxter's *TQ*-relation looks

$$T(\zeta)Q(\zeta) = a(\zeta)Q(q\zeta) + d(\zeta)Q(q^{-1}\zeta)$$

with the transfer matrix T and the Q-matrix which depend on spectral parameter ζ. a and d are some known functions.
Rodney described its derivation in Chapter 9 *"Alternative way of solving the ice-type model"* of his book. The derivation is based on *"pair propagation through a vertex"*:

$$R_{12}\, g_1\, g_2' \;=\; \tilde{g}_2'\, \tilde{g}_1$$

where $g, g', \tilde{g}, \tilde{g}'$ are some vectors sometimes called 'vacuum' vectors. The Q-matrix is related to the matrices Q_R (Q_L)

$$Q(\zeta) = Q_R(\zeta)Q_R^{-1}(\zeta_0) = Q_L^{-1}(\zeta_0)Q_L(\zeta)$$

whose columns (rows) are built from the 'vacuum' vectors.

Remark:

One immediate generalization of the 'vacuum' vectors to the 3D case looks:

$$R_{123}\, g_1\, g_2'\, g_3'' = \tilde{g}_3''\, \tilde{g}_2'\, \tilde{g}_1$$

It is interesting that this equation in case when R_{123} is Zamolodchikov's solution to the tetrahedron equation can be solved. Unfortunately, this solution did not lead to construction of the 3D analog of Q-matrices.

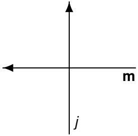
Applications of the *TQ*-relation

- Determination of the spectrum of the transfer matrix and the free energy via solving the Bethe ansatz equations (BAE)
- Calculation of the correlation functions via construction of fermionic operators and fermionic basis of local operators on the lattice called **'hidden Grassmann structure' (HGS)**
- Functions ρ and ω and Jimbo-Miwa-Smirnov theorem
- Applications to the QFT
  - CFT
  - sine-Gordon model
  - Bethe-Salpeter equation for the t'Hooft model

Fateev, Lukyanov, Zamolodchikov (09)

HGS of the Heisenberg XXZ spin chain

It is well known that the XXZ model is related to the six vertex model.
Integrable structure is generated by *R*-matrix or *L*-operator



$$\equiv \quad L_{j,\mathbf{m}}(\zeta) = q^{-\frac{1}{2}\sigma_j^z\sigma_{\mathbf{m}}^z} - \zeta^2 q^{\frac{1}{2}\sigma_j^z\sigma_{\mathbf{m}}^z} - \zeta(q - q^{-1})(\sigma_j^+\sigma_{\mathbf{m}}^- + \sigma_j^-\sigma_{\mathbf{m}}^+)$$

Consider the XXZ-model in infinite volume. Space of states is $\quad \mathfrak{H}_{\mathrm{S}} = \bigotimes_{j=-\infty}^{\infty} \mathbb{C}^2$

Also we introduce the Matsubara space: $\quad \mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbb{C}^2$

and monodromy matrix: $T_{\mathrm{S},\mathbf{M}} = \overset{\curvearrowright}{\prod_{j=-\infty}^{\infty}} T_{j,\mathbf{M}}\,, \ \ T_{j,\mathbf{M}} \equiv T_{j,\mathbf{M}}(1), \ \ T_{j,\mathbf{M}}(\zeta) = \overset{\curvearrowleft}{\prod_{\mathbf{m}=\mathbf{1}}^{\mathbf{n}}} L_{j,\mathbf{m}}(\zeta/\tau_{\mathbf{m}})$

In homogeneous case we take $\tau_{\mathbf{m}} = q^{1/2}$.

The local operators

Introduce a local operator $\mathcal{O}$ on $\mathfrak{H}_S$ which acts non-trivially only on a finite segment of $\mathfrak{H}_S$. We call **quasi-local operator** with tail α the following product

$$q^{2\alpha S(0)}\mathcal{O}$$

Here we defined

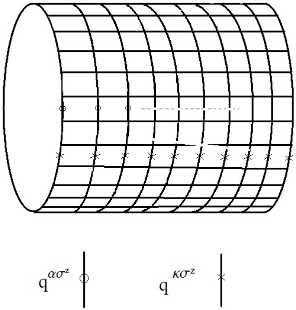
$$S(k)=\frac{1}{2}\sum_{j=-\infty}^{k}\sigma_j^z$$

So, $S(0)$ acts on the semi-infinite chain and $S=S(\infty)$ is the total spin.
We call

$$q^{2\alpha S(0)}$$

lattice 'primary field' and parameter α – **disorder field.**

Correlation functions as partition functions on cylinder



Cut corresponds to

insertion of local operator $\mathcal{O}$

κ – **“imaginary” magnetic field**

α – **disorder field**

Correlation functions =

Matsubara expectation values:

$$Z_{\mathbf{n}}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathrm{S},\mathbf{M}}\ \ q^{2\kappa S+2\alpha S(0)}\mathcal{O}\right)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathrm{S},\mathbf{M}}\ \ q^{2\kappa S+2\alpha S(0)}\right)}$$

Fermionic operators and fermionic basis

Describe the basis of quasi-local operators via certain creation operators $\mathbf{t}^*(\zeta)$, $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ and their expansion modes $\mathbf{t}^*_p$, $\mathbf{b}^*_p$, $\mathbf{c}^*_p$, $p \geq 1$:

Jimbo, Miwa, Smirnov, Takeyama, HB (07–09)

$$\mathbf{t}^*_{p_1} \cdots \mathbf{t}^*_{p_l} \mathbf{b}^*_{q_1} \cdots \mathbf{b}^*_{q_m} \mathbf{c}^*_{q'_{m'}} \cdots \mathbf{c}^*_{q'_1} \left(q^{2\alpha S(0)} \right)$$

Operators $\mathbf{t}^*_p$ are bosonic and correspond to commuting integrals of motion. They commute with all fermionic operators $\mathbf{b}^*_p, \mathbf{c}^*_p$.

Locality:

$$\text{length}(\mathbf{b}^*_p(X)) \leq \text{length}(X) + p, \quad \text{length}(\mathbf{c}^*_p(X)) \leq \text{length}(X) + p$$

$$\text{length}(\mathbf{t}^*_p(X)) \leq \text{length}(X) + p$$

Relation to correlators, JMS-theorem

- Correlation functions of quasi-local operators $\mathcal{O}$ are generated by two transcendental functions ρ and ω. ρ is related to one-point function, ω is related to nearest neighbor correlators

$$\omega(\zeta,\zeta') = Z_{\mathbf{n}}\{\mathbf{b}^*(\zeta)\mathbf{c}^*(\zeta')q^{2\alpha S(0)}\}$$

  Both functions depend on physical parameters like temperature, disorder and magnetic field, we call them **physical part**.

- In contrast to this, the basis is pure algebraic. It is built using representation theory of quantum group. We call it **algebraic part**.

The JMS-theorem allows to explicitly calculate   Jimbo, Miwa, Smirnov (09)

$$Z_{\mathbf{n}}\Big\{\mathbf{t}^*(\zeta_1^0)\cdots\mathbf{t}^*(\zeta_p^0)\mathbf{b}^*(\zeta_1^+)\cdots\mathbf{b}^*(\zeta_q^+)\mathbf{c}^*(\zeta_q^-)\cdots\mathbf{c}^*(\zeta_1^-)\big(q^{2\alpha S(0)}\big)\Big\} =$$

$$= \prod_{i=1}^{p} 2\rho(\zeta_i^0)\cdot\det\big(\omega(\zeta_i^+,\zeta_j^-)\big)|_{i,j=1,\cdots q}$$ generating function for series in ζ^2-1

The function ω

In this talk I mostly discuss the properties of the function ω.
In more general case:

$$\omega(\zeta,\zeta') = \omega(\zeta,\zeta'|\varkappa,\varkappa';\alpha), \quad \varkappa = (\kappa, S), \quad \varkappa' = (\kappa', S')$$

with different spin sectors S, S'

There are several equivalent definitions of ω

- via deformed Abelian integrals Jimbo, Miwa, Smirnov (09)
- via solution of linear and non-linear integral equations that come from thermodynamical description of six vertex model Göhmann, HB (09-12)
- via 'master' function Φ Göhmann, HB (12)

The function ω via integral equations

$$\frac{1}{4}\omega(\zeta,\xi|\varkappa,\varkappa';\alpha) = \frac{1}{4}\tilde{\omega}(\zeta,\xi|\varkappa,\varkappa';\alpha) + \omega_0(\zeta,\xi|\kappa,\kappa';\alpha),$$
$$\frac{1}{4}\tilde{\omega}(\zeta,\xi|\varkappa,\varkappa';\alpha) = -(f_L \star F_R)(\zeta,\xi), \quad \omega_0(\zeta,\xi) = \delta^-_\zeta \delta^-_\xi \Delta^{-1}_\zeta \psi(\zeta/\xi,\alpha)$$

$$F_R + K_\alpha \star F_R = f_R, \quad (F \star G)(\zeta,\xi) = \int_\gamma dm(\eta) F(\zeta,\eta)\, G(\eta,\xi)$$

$$dm(\eta) := \frac{d\eta^2}{\eta^2 \rho(\eta)(\mathfrak{a}(\eta)+1)}, \quad \mathfrak{a}(\eta) := \frac{a(\eta)}{d(\eta)} \frac{Q(q\,\eta)}{Q(q^{-1}\eta)}, \quad \rho(\eta) := \frac{T(\eta,\varkappa')}{T(\eta,\varkappa)}$$

BAE: $\mathfrak{a}(\lambda_i) = -1, \quad Q(\zeta) := Q(\zeta,\varkappa) = \zeta^{-\kappa+S} \prod_{i=1}^{\mathbf{n}/2-S} \left(1 - \frac{\zeta^2}{\lambda_i^2(\varkappa)}\right)$

integration contour γ goes around the points ζ^2, ξ^2 and all Bethe roots λ_i^2 in counterclockwise direction. Also we defined:

$$K_\alpha(\zeta,\xi) := \frac{1}{2\pi i}\Delta_\zeta \psi(\zeta/\xi,\alpha) \quad f_L(\zeta,\xi) := \frac{1}{2\pi i}\delta^-_\zeta \psi(\zeta/\xi,\alpha), \quad f_R(\zeta,\xi) := \delta^-_\xi \psi(\zeta/\xi,\alpha)$$
$$\Delta_\zeta f(\zeta) := f(q\zeta) - f(q^{-1}\zeta), \quad \delta^-_\zeta f(\zeta) := f(q\zeta) - \rho(\zeta) f(\zeta)$$

Reflection symmetries: $\alpha \to -\alpha$ and $\alpha \to 2-\alpha$

We can redefine all functions using ψ_+ instead of ψ where

$$\psi(\zeta,\alpha) := \frac{\zeta^\alpha}{2}\cdot\frac{\zeta^2+1}{\zeta^2-1} \quad\to\quad \psi_+(\zeta,\alpha) := \frac{\zeta^\alpha}{\zeta^2-1}$$

Since we have: $\psi(\zeta,\alpha) = -\psi(\zeta^{-1},-\alpha), \quad \psi_+(\zeta,\alpha) = -\psi_+(\zeta^{-1},2-\alpha),$

we can check the following reflection relations:

$$\omega(\zeta,\xi|\varkappa,\varkappa',\alpha) = \omega(\xi,\zeta|\varkappa,\varkappa',-\alpha),$$
$$\omega_+(\zeta,\xi|\varkappa,\varkappa',\alpha) = \omega_+(\xi,\zeta|\varkappa,\varkappa',2-\alpha)$$

Proposition: $\omega(\zeta,\xi|\varkappa,\varkappa',\alpha) = \omega_+(\zeta,\xi|\varkappa,\varkappa',\alpha)$

if $\alpha = \kappa' - S' - \kappa + S, S' \geq S$ or $\alpha = -\kappa' + S' + \kappa - S, S' \leq S$

The function ω via the function Φ

Introduce function $\Phi(\zeta,\xi) := \Phi(\zeta,\xi|\varkappa,\varkappa';\alpha)$

$$\Phi(\zeta,\xi) = \tilde{\Phi}(\zeta,\xi) + \Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha),$$

where the following relation should be satisfied

$$\mathrm{res}_{\zeta^2=\lambda_i^2}\left(\frac{\Delta_\zeta\Phi(\zeta,\xi)}{\rho(\zeta)(1+\mathfrak{a}(\zeta))} + \tilde{\Phi}(\zeta,\xi)\right) = 0$$

Proposition 1: $\tilde{\Phi}(\zeta,\xi) = \sum_{i,j}\psi(\zeta/\lambda_i,\alpha)\,(U^{-1})_{i,j}\,\psi(\xi/\lambda_j,-\alpha)$

solves the above relation with the matrix U:

$$U_{i,j} = \frac{\delta_{i,j}}{m_i} + \psi(q\lambda_i/\lambda_j,\alpha) - \psi(q^{-1}\lambda_i/\lambda_j,\alpha), \quad m_i = \mathrm{res}_{\eta^2=\lambda_i^2}\; dm(\eta), \quad \bar{\mathfrak{a}}(\eta) := \frac{1}{\mathfrak{a}(\eta)}$$

Proposition 2: The function ω looks: $\frac{1}{4}\omega(\zeta,\xi) = H_\zeta\, H_\xi\;\Phi(\zeta,\xi)$

$$(H_\zeta f)(\zeta) := \frac{1}{1+\bar{\mathfrak{a}}(\zeta)}f(q\zeta) + \frac{1}{1+\mathfrak{a}(\zeta)}f(q^{-1}\zeta) - \rho(\zeta)f(\zeta)$$

Remark: The kernel $H_\zeta\left(\frac{Q(\zeta,\kappa')}{Q(\zeta,\kappa)}\right) = 0$. The TQ-relation is used for the proof.

Further properties

Using the above results, we can show **‘normalization condition’**:

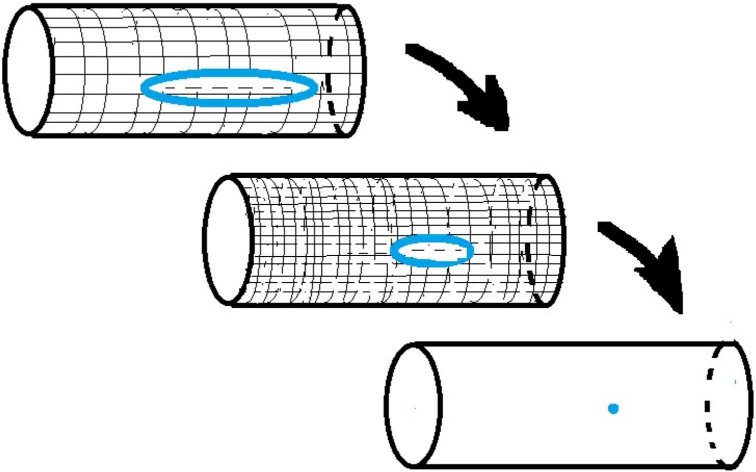
If the above relations of α, κ and κ' are satisfied and $0 < \alpha < 2$ then

$$\lim_{\zeta\to\infty} \omega(\zeta,\xi) = \lim_{\zeta\to 0} \omega(\zeta,\xi) = \lim_{\xi\to\infty} \omega(\zeta,\xi) = \lim_{\xi\to 0} \omega(\zeta,\xi) = 0$$

Proposition: $\varkappa \leftrightarrow \varkappa'$ symmetry Smirnov, HB (23)

$$\omega(\zeta,\xi|\varkappa,\varkappa';\alpha) = \rho(\zeta|\varkappa,\varkappa')\,\rho(\xi|\varkappa,\varkappa')\,\omega(\zeta,\xi|\varkappa',\varkappa;\alpha)$$

Application to QFT: continuum limit



Scaling limit and CFT

Aim is two-fold :

- to obtain the CFT with non-trivial $c = 1 - 6\nu^2/(1-\nu)$, $q = e^{\pi i/\nu}$, $1/2 < \nu < 1$
- to consider asymptotic series for $\kappa \to \infty$

Scaling: Introduce lattice spacing a and take

$\tau_{\mathbf{m}} = q^{1/2}$, $\mathbf{n} \to \infty$, $a \to 0$, $\mathbf{n}a = 2\pi R$ with fixed radius of cylinder R

Lieb distribution gives: $\lambda_j \simeq \left(\pi j/\mathbf{n}\right)^{\nu}$. Spectral parameter must be re-scaled:

$$\zeta = \lambda \bar{a}^{\nu},\ \bar{a} = Ca,\ C = \frac{\Gamma\left(\frac{1-\nu}{2\nu}\right)}{2\sqrt{\pi}\Gamma\left(\frac{1}{2\nu}\right)}\Gamma(\nu)^{\frac{1}{\nu}}$$

$$\rho^{\mathrm{sc}}(\lambda) = \lim_{\mathrm{scaling}} \rho(\lambda\bar{a}^{\nu}), \quad \omega^{\mathrm{sc}}(\lambda,\mu) = \frac{1}{4}\lim_{\mathrm{scaling}} \omega(\lambda\bar{a}^{\nu},\mu\bar{a}^{\nu})$$

Conjectures on operators in scaling

- The creation operators are well-defined in the scaling limit for space direction when $ja = x$ is finite

$$\tau^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{t}^*(\lambda\bar{a}^\nu), \quad \beta^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{b}^*(\lambda\bar{a}^\nu), \quad \gamma^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{c}^*(\lambda\bar{a}^\nu)$$

Asymptotic expansions at $\lambda \to \infty$ look

$$\log\bigl(\tau^*(\lambda)\bigr) \simeq \sum_{j=1}^{\infty} \tau^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}$$

$$\frac{1}{\sqrt{\tau^*(\lambda)}}\beta^*(\lambda) \simeq \sum_{j=1}^{\infty} \beta^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}, \quad \frac{1}{\sqrt{\tau^*(\lambda)}}\gamma^*(\lambda) \simeq \sum_{j=1}^{\infty} \gamma^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}.$$

Asymptotic expansions

- Using the result by Bazhanov, Lukyanov, Zamolodchikov (BLZ) (96-99) , we get asymptotic expansion at $\lambda \to \infty$

$$\log \rho^{\rm sc}(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-\frac{2j-1}{\nu}} C_j \big(I^+_{2j-1} - I^-_{2j-1} \big) \to \tau^*_{2j-1} = C_j \mathbf{i}_{2j-1}$$

$$\omega^{\rm sc}(\lambda,\mu) \simeq \sqrt{\rho^{\rm sc}(\lambda)\rho^{\rm sc}(\mu)} \sum_{i,j=1}^{\infty} \lambda^{-\frac{2i-1}{\nu}} \mu^{-\frac{2j-1}{\nu}} \omega_{i,j}$$

Scaling limit of the determinant formula

$$Z_R^{\kappa,\kappa'} \Big\{ \tau^*(\lambda^0_1)\cdots\tau^*(\lambda^0_p)\beta^*(\lambda^+_1)\cdots\beta^*(\lambda^+_r)\gamma^*(\lambda^-_r)\cdots\gamma^*(\lambda^-_1)\big(\Phi_\alpha(0)\big)\Big\}$$
$$= \prod_{i=1}^{p} \rho^{\rm sc}(\lambda^0_i) \times \det\big(\omega^{\rm sc}(\lambda^+_i,\lambda^-_j)\big)_{i,j=1,\dots,r}.$$

Technical ρ-problem: We get coefficients $\omega_{i,j}$ by the Wiener-Hopf technique only for $\kappa = \kappa'$ when $\Delta_+ = \Delta_-$ and $\rho^{\rm sc}(\zeta) = 1$ i.e. modulo the integrals of motion. $\Delta_+ = \frac{\nu^2}{4(1-\nu)}(\kappa^2-1), \quad \Delta_- = \frac{\nu^2}{4(1-\nu)}(\kappa'^2-1)$

Correspondence to CFT 3-point correlator

- It is possible to state the correspondence

$$\frac{\langle\Delta_-|P_\alpha\big(\{\mathbf{I}_{-k}\}\big)\Phi_\alpha(0)|\Delta_+\rangle}{\langle\Delta_-|\Phi_\alpha(0)|\Delta_+\rangle} = \lim_{\mathbf{n}\to\infty, a\to 0, \mathbf{n}a=2\pi R} Z_{\mathbf{n}}\big\{q^{2\alpha S(0)}\mathcal{O}\big\}$$

between a polynomial $P_\alpha(\{\mathbf{I}_{-k}\})$ and some combinations of $\beta^*_{2j-1}, \gamma^*_{2j-1}$.

Introduce $\quad \beta^*_{2m-1} = D_{2m-1}(\alpha)\beta^{\mathrm{CFT}*}_{2m-1}, \quad \gamma^*_{2m-1} = D_{2m-1}(2-\alpha)\gamma^{\mathrm{CFT}*}_{2m-1}$

and take even and odd bilinear combinations

$$\phi^{\mathrm{even}}_{2m-1,2n-1} = (m+n-1)\frac{1}{2}\big(\beta^{\mathrm{CFT}*}_{2m-1}\gamma^{\mathrm{CFT}*}_{2n-1} + \beta^{\mathrm{CFT}*}_{2n-1}\gamma^{\mathrm{CFT}*}_{2m-1}\big),$$

$$\phi^{\mathrm{odd}}_{2m-1,2n-1} = d_\alpha^{-1}(m+n-1)\frac{1}{2}\big(\beta^{\mathrm{CFT}*}_{2n-1}\gamma^{\mathrm{CFT}*}_{2m-1} - \beta^{\mathrm{CFT}*}_{2m-1}\gamma^{\mathrm{CFT}*}_{2n-1}\big),$$

$$d_\alpha = \frac{\nu(\nu-2)}{\nu-1}(\alpha-1)$$

Identification with Virasoro Verma module

- If we accept an equivalence of the spaces spanned by

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}\mathbf{l}_{-2l_1}\cdots\mathbf{l}_{-2l_q}\big(\Phi_\alpha(0)\big) \qquad \text{and}$$

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}$$
$$\times\,\phi^{\mathrm{even}}_{2m_1-1,2n_1-1}\cdots\phi^{\mathrm{even}}_{2m_r-1,2n_r-1}\phi^{\mathrm{odd}}_{2\bar m_1-1,2\bar n_1-1}\phi^{\mathrm{odd}}_{2\bar m_{\bar r}-1,2\bar n_{\bar r}-1}\big(\Phi_\alpha(0)\big)$$

we can identify modulo integrals of motion ($\Delta\equiv\Delta_\alpha$)

Jimbo, Miwa, Smirnov, HB (10) (HGSIV), HB (11)

$$\phi^{\mathrm{even}}_{1,1}\cong\mathbf{l}_{-2},\qquad \phi^{\mathrm{even}}_{1,3}\cong\mathbf{l}^2_{-2}+\frac{2c-32}{9}\mathbf{l}_{-4},\qquad \phi^{\mathrm{odd}}_{1,3}\cong\frac{2}{3}\,\mathbf{l}_{-4}$$

$$\phi^{\mathrm{even}}_{1,5}\cong\mathbf{l}^3_{-2}+\frac{c+2-20\Delta+2c\Delta}{3(\Delta+2)}\mathbf{l}_{-4}\mathbf{l}_{-2}\;+\;\cdots\;\mathbf{l}_{-6}$$

$$\phi^{\mathrm{odd}}_{1,5}\cong\frac{2\Delta}{\Delta+2}\mathbf{l}_{-4}\mathbf{l}_{-2}+\frac{56-52\Delta-2c+4c\Delta}{5(\Delta+2)}\,\mathbf{l}_{-6}$$

$$\phi^{\mathrm{even}}_{3,3}\cong\mathbf{l}^3_{-2}+\frac{6+3c-76\Delta+4c\Delta}{6(\Delta+2)}\mathbf{l}_{-4}\mathbf{l}_{-2}\;+\;\cdots\;\mathbf{l}_{-6}$$

Functions Ψ and Θ

BLZ (97) introduced function Ψ that we also used in HGSIV-paper together with function Θ. Function Ψ is related to the integrals of motion in CFT found by Zamolodchkov in 1987:

$$I_{2n-1} = -i\,\Psi(\frac{i(2n-1)}{2\nu},\kappa)\; n(2n-1)(2\nu^2)^{n-1}(f\kappa)^{2n-1}R^{-2n+1}$$

$$\omega^{\rm sc}(\lambda,\mu) \simeq \sum_{r,s=1}^{\infty} \lambda^{-\frac{2r-1}{\nu}}\mu^{-\frac{2s-1}{\nu}} D_{2r-1}(\alpha)D_{2s-1}(2-\alpha)\,\frac{1}{\nu}\left(\frac{\sqrt{2}f\kappa\nu}{R}\right)^{2r+2s-2}$$
$$\times\Theta(\frac{i(2r-1)}{2\nu},\frac{i(2s-1)}{2\nu}|\kappa,\alpha) \quad \text{in case} \quad \kappa=\kappa', \quad (f^{-1}=2\sqrt{2(1-\nu)})$$

In HGSIV we applied the Wiener-Hopf factorization technique which is a bit complicated.

Alternative way of defining functions Ψ and Θ

We have tried to further simplify computations Adler, HB (23). We use renormalized Ψ (with $p = f\kappa$) and a functional $\mathcal{F}$:

$$\Psi_\nu(s,p) = \frac{1}{2\nu i}\Psi(\frac{s}{2\nu i},\frac{p}{2f\nu}), \quad S_\nu(s) = S(\frac{s}{2\nu i}), \quad S_\nu(s,\alpha) = S(\frac{s}{2\nu i},\alpha)$$

$$\mathcal{F}(s,p,\Phi_-) =$$

$$= \sum_{n\geq 0}\frac{p^{-n}}{n!}\,\mathrm{res}_{h_1}\cdots\mathrm{res}_{h_n}\,\Phi_-(h_1)\cdots\Phi_-(h_n)\Big(1+s-\sum_{j=1}^n h_j\Big)_n\,\zeta_0\big(n+s-\sum_{j=1}^n h_j,p\big)$$

with 'chiral' sources: $\Phi_+(h) := \sum_{n>0}\frac{a_n}{n}\,h^{n-1}, \quad \Phi_-(h) := \sum_{n<0} a_n\,h^n,$

bosonic modes: $[a_n,a_m] = n\,\delta_{n+m,0}$ and vacuum $|0>$: $a_n|0>=0$ for $n>0$.

$$\zeta_0(s,p) = \frac{p^s}{s}\zeta(s,p+1/2) = \frac{p}{s(s-1)} - \frac{1}{s}\sum_{m\geq 1} p^{-2m+1}(1-2^{-2m+1})\frac{B_{2m}}{(2m)!}(s)_{2m-1}$$

Proposition: The function Ψ fulfills the equation

$$\Psi_\nu(s,p) = \langle 0|\exp\Big\{\mathrm{res}_h\Big(\big(S_\nu(h)-1\big)\Psi_\nu(h,p)\Phi_+(h)\Big)\Big\}\mathcal{F}(s,p,\Phi_-)|0\rangle$$

Proposition: The function $\Theta_\nu(s,s';p,\alpha):=\frac{1}{2\nu}\Theta(\frac{s}{2\nu i},\frac{s'}{2\nu i}|\frac{p}{2f\nu},\alpha)$ fulfills equation

$$0=\mathrm{res}_h\,\mathrm{res}_{h'}\frac{S_\nu(h,2-\alpha)S_\nu(h',\alpha)}{s+h}\Theta_\nu(h',s';p,\alpha)\times$$
$$\langle 0|\exp\left\{\mathrm{res}_{h''}\Big((S_\nu(h'')-1)\Psi_\nu(h'',p)\Phi_+(h'')\Big)\right\}\mathcal{G}(-h-h',p,\Phi_-)|0\rangle$$

where $\mathcal{G}(s,p,\Phi_-)$ is a functional of fields Φ_-:

$$\mathcal{G}(s,p,\Phi_-)=$$
$$=\sum_{n\geq 0}\frac{p^{-n-1}}{n!}\mathrm{res}_{h_1}\dots\mathrm{res}_{h_n}\Phi_-(h_1)\dots\Phi_-(h_n)\big(1+s-\sum h_j\big)_{n+1}\;\zeta_0\big(n+1+s-\sum h_j,p\big)$$

Remark: To get formulae for excited states with L_+ particles and L_- holes defined by positions $I_r^{(\pm)}, r=1,\cdots,L_\pm$, we replace in the above formulas the function ζ_0 by

$$\zeta_0(s,p)\to\zeta_0(s,p)+E(s,p)$$

$$E(s,p)=\frac{1}{s}\sum_{r=1}^{L_+}\left(1-\frac{I_r^{(+)}}{2p}\right)^{-s}-\frac{1}{s}\sum_{r=1}^{L_-}\left(1+\frac{I_r^{(-)}}{2p}\right)^{-s}.$$

For a single particle-hole excitation $L_+=L_-=1$ we reproduce the result of HB(11).

Smirnov's result for free fermion case

In 2012 Fedya came up with explicit formula for $\tilde{\Phi}$ in free fermionic case
$\tilde{\Phi}(\zeta,\xi)\to\tilde{\Phi}(\lambda,\mu)$, $\zeta=\lambda\bar{a}^{\nu}$, $\xi=\mu\bar{a}^{\nu}$: Smirnov, unpubl. (12)

$$\tilde{\Phi}(\lambda,\mu;\kappa,\kappa',\alpha)=\frac{\lambda^{\alpha}\mu^{2-\alpha}}{(2\pi)^3}\Gamma(-\pi\lambda^2+\frac{1+\kappa}{2})\Gamma(-\pi\mu^2+\frac{1+\kappa}{2})$$

$$\times\int_0^{\infty}dk\,\frac{k\sinh\pi k\cosh\pi k}{\cosh\pi k+\sin\frac{\pi\alpha}{2}}\,\frac{F(\lambda,k;\kappa,\kappa')F(\mu,k;\kappa,\kappa')}{\Gamma(\frac{1+\kappa-\kappa'}{2}+ik)\Gamma(\frac{1+\kappa-\kappa'}{2}-ik)\Gamma(\frac{1+\kappa+\kappa'}{2}+ik)\Gamma(\frac{1+\kappa+\kappa'}{2}-ik)}$$

where

$$F(\lambda,k;\kappa,\kappa')=\int_{-i\infty-0}^{i\infty-0}ds\,\frac{\Gamma(-s)\Gamma(-s+\frac{\kappa+\kappa'}{2})\Gamma(-s+\frac{\kappa-\kappa'}{2})\Gamma(s+\frac{1}{2}+ik)\Gamma(s+\frac{1}{2}-ik)}{\Gamma(-s-\pi\lambda^2+\frac{1+\kappa}{2})}$$

It is explicitly symmetric: $\tilde{\Phi}(\lambda,\mu;\kappa,\kappa',\alpha)=\tilde{\Phi}(\mu,\lambda;\kappa,\kappa',2-\alpha)$

Questions:

- how about the above $\kappa\leftrightarrow\kappa'$ symmetry?
- how to generalize this result to the case $\nu\neq 1/2$

Conclusions

- There are still many technical questions that have to be answered like, for instance, the **ρ-problem**. Is there a way to escape the condition $\kappa = \kappa'$ in order to involve the contribution of the integrals of motion. We saw that the function ω had many symmetries, in particular, the $\kappa \leftrightarrow \kappa'$ symmetry which was satisfied inspite of the fact that these two parameters entered in rather different manner. Probably, there exists an explicitly symmetric description.
- Also the function ω is invariant under two reflections:

$$\sigma_1 : \alpha \to 2 - \alpha, \quad \text{and} \quad \sigma_2 : \alpha \to -\alpha$$

  where the first one is related to the natural symmetry of the CFT since $\Delta_\alpha = \Delta_{2-\alpha}$ while the second reflection originates from the sG model. The idea to use both these symmetries was promoted by Negro and Smirnov (13). Also it helped us together with Smirnov (18) to incorporate the integrals of motion for few particular cases of Virasoro levels. Would it be possible to extend it to further levels and relate it with the previous approach?

In this talk I discussed the point that many Rodney's results or methods can be extended to the other areas like quantum groups, knot theory, integrable QFT etc. even though Rodney's papers maybe were not directly related to them. Main example of such results here was the Baxter's *TQ*-relation. I believe there are many other such examples. I think many interesting new results may be obtained as generalizations or extensions of Rodney's results.

THANK YOU FOR YOUR ATTENTION!