Baxter's work — a guiding star to complete integrability

Leon A. Takhtajan

Stony Brook University, Stony Brook NY, USA
Euler Mathematical Institute, Saint Petersburg, Russia

"Baxter 2025 Exactly Solved Models and Beyond: Celebrating the life and achievements of Rodney James Baxter"
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— a completely integrable Hamiltonian system on the real line, with the boundary condition

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- Теоретическая и Математическая Физика, **21**:2, 1974

СУЩЕСТВЕННО-НЕЛИНЕЙНАЯ ОДНОМЕРНАЯ МОДЕЛЬ КЛАССИЧЕСКОЙ ТЕОРИИ ПОЛЯ

Л. А. Тахтаджян, Л. Д. Фаддеев

Показано, что уравнение $u_{tt}-u_{xx}+\sin u=0$ с граничным условием $u(x,t)\to 0 (\mathrm{mod}\ 2\pi)$ при $|x|\to\infty$, описывающее классическое поле с существенно-нелинейным взаимодействием, является вполне интегрируемой гамильтоновой системой. Полученные результаты интерпретируются в терминах частиц, соответствующих полю $u(x,t)$.

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- "One relativistic field generates several particles".

- 1977 — continuous Heisenberg spin chain

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx}, \quad \vec{S} \in S^2$$

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- Volume 64A, number 2 PHYSICS LETTERS 12 December 1977

INTEGRATION OF THE CONTINUOUS HEISENBERG SPIN CHAIN THROUGH THE INVERSE SCATTERING METHOD

L.A. TAKHTAJAN

Leningrad Branch of Steclov Mathematical Institute of the USSR Academy of Sciences, 198152 Leningrad, USSR

Received 22 September 1977

The inverse scattering method is applied to the Heisenberg chain. We give the general scheme of the solution of the equations of motion. We describe the process of solitons scattering and show the existence of an infinite series of constants of motion.

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- ANNALS OF PHYSICS: **70**, 193–228 (1972)

Partition Function of the Eight-Vertex Lattice Model

RODNEY J. BAXTER

Research School of Physical Sciences, The Australian National University, Canberra, A.C.T. 2600, Australia

Received May 20, 1971

The partition function of the zero-field "Eight-Vertex" model on a square M by N lattice is calculated exactly in the limit of M, N large. This model includes the dimer, ice and zero-field Ising, F and KDP models as special cases. In general the free energy has a branch point singularity at a phase transition, with an irrational exponent.

- Commuting transfer matrices:

$$[\mathbf{T}\ \mathbf{T}']_{\boldsymbol{\alpha}|\boldsymbol{\beta}} = \mathrm{Tr}\left\{\prod_{J=1}^{N} \mathbf{S}(\alpha_J\,,\,\beta_J)\right\},$$

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the $\mathbf{S}(\alpha, \beta)$ are 4 by 4 matrices with elements

$$S^{\alpha,\beta}_{\lambda,\mu|\lambda',\mu'} = \sum_{\gamma} R(\alpha, \gamma \mid \lambda, \lambda')\; R'(\gamma, \beta \mid \mu, \mu')$$

$$= \sum_{j=1}^{4} \sum_{k=1}^{4} w_j w_k{}' (\boldsymbol{\sigma}^j \boldsymbol{\sigma}^k)_{\alpha,\beta}\; \sigma^j_{\lambda,\lambda'} \sigma^k_{\mu,\mu'}$$

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- Transfer matrices will commute, if we remove the trace (i.e., consider monodromy matrices) and assume that there exists a 4 by 4 nonsingular matrix $\mathbf{R}$ such that

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- This relation is a cornerstone of quantum integrability.

II. Quantum integrability

- Quantum Inverse Problem Method was formulated, on the example of the Sine-Gordon model, in the paper in Теоретическая и Математическая Физика, **40**:2, 1979

КВАНТОВЫЙ МЕТОД ОБРАТНОЙ ЗАДАЧИ. I

Е. К. Склянин, Л. А. Тахтаджян, Л. Д. Фаддеев

Предлагается квантовомеханический вариант метода обратной задачи. Для квантовой модели синус-Гордон получены точное решение и квантовые аналоги переменных действие – угол. Обсуждается связь предлагаемого метода с подстановкой Бете, а также проблемы перенормировки.

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- From Baxter's 4×4 matrix R of Boltzmann weights to local quantum L-operator — 2×2 matrix $L_n(\lambda)$ in the 'auxiliary space' $\mathbb{C}^2$ with entires — operators in the quantum Hilbert space $\mathfrak{h}_n$ on the n-th site, and to 4×4 matrix $R(\lambda)$ such that

$$R(\lambda - \mu)\,(L_n(\lambda) \otimes L_n(\mu)) = (L_n(\mu) \otimes L_n(\lambda))\,R(\lambda - \mu).$$

We have taken this argument from the well-known paper of Baxter [17], who used it only to prove commutativity of the trace of the monodromy matrix for the spin model he was considering. For our discussions, the use of (1.16) fully plays a very important part. One can say that this relation in conjunction with (1.14) is the basis for the proof of the complete integrability of our model. Using it, we find not only commuting integrals ("variables of action type"), but also eigenvectors of the energy operator ("variables of angle type").

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- Explicitly,

$$L_n(\lambda) = \begin{pmatrix} e^{-\frac{i\Delta\beta\pi_n}{4}} & \frac{m\Delta}{4}\left(\lambda e^{-\frac{i\beta\varphi_n}{2}} - \frac{1}{\lambda}e^{\frac{i\beta\varphi_n}{2}}\right) \\ \frac{m\Delta}{4}\left(\frac{1}{\lambda}e^{-\frac{i\beta\varphi_n}{2}} - \lambda e^{\frac{i\beta\varphi_n}{2}}\right) & e^{\frac{i\Delta\beta\pi_n}{4}} \end{pmatrix}$$

where

$$\hat{R}(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda,\mu) & c(\lambda,\mu) & 0 \\ 0 & c(\lambda,\mu) & b(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with

$$b(\lambda,\mu) = \frac{i\sin\gamma}{\mathrm{sh}(\alpha-\beta+i\gamma)} \quad \text{and} \quad c(\lambda,\mu) = \frac{\mathrm{sh}(\alpha-\beta)}{\mathrm{sh}(\alpha-\beta+i\gamma)},$$

where $\alpha = \log\lambda,\ \beta = \log\mu.$

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  where $\alpha = \log\lambda,\ \beta = \log\mu.$

- Generating vector, algebraic Bethe Ansatz, physical vacuum as filled Dirac sea, joint spectrum of quantum integrals of motion, etc.

ANNALS OF PHYSICS 76, 1–24 (1973)

Eight-Vertex Model in Lattice Statistics and One-Dimensional Anisotropic Heisenberg Chain. I. Some Fundamental Eigenvectors*

RODNEY BAXTER†

Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790

Received March 29, 1972

We obtain some simple eigenvectors of the transfer matrix of the zero-field eight-vertex model. These are also eigenvectors of the Hamiltonian of the one-dimensional anisotropic Heisenberg chain. We also obtain new equations for the matrix $\mathbf{Q}(v)$ introduced in earlier papers.

ANNALS OF PHYSICS 76, 25–47 (1973)

Eight-Vertex Model in Lattice Statistics and One-Dimensional Anisotropic Heisenberg Chain. II. Equivalence to a Generalized Ice-type Lattice Model*

RODNEY BAXTER†

Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790

Received May 24, 1972

We establish an equivalence between the zero-field eight-vertex model and an Ising model (with four-spin interaction) in which each spin has L possible values, labeled $1,..., L$, and two adjacent spins must differ by one (to modulus L). Such an Ising model can also be thought of as a generalized ice-type model and we will later show that the eigenvectors of the transfer matrix can be obtained by a Bethe-type ansatz.

ANNALS OF PHYSICS 76, 48–71 (1973)

Eight-Vertex Model in Lattice Statistics and One-Dimensional Anisotropic Heisenberg Chain. III. Eigenvectors of the Transfer Matrix and Hamiltonian

RODNEY BAXTER*,†

Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790

Received September 5, 1972

We obtain the eigenvectors of the transfer matrix of the zero-field eight vertex model. These are also the eigenvectors of the Hamiltonian of the corresponding one-dimensional anistropic Heisenberg chain.

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- Faddeev and I gave algebraic formulation formulation of Baxter's method in the paper in Успехи Математических Наук, **34**:5, 1979

КВАНТОВЫЙ МЕТОД ОБРАТНОЙ ЗАДАЧИ И XYZ МОДЕЛЬ ГЕЙЗЕНБЕРГА

Л. А. Тахтаджян, Л. Д. Фаддеев

In 1972 Rodney Baxter in his remarkable papers [9]–[10] (the results were announced by him in 1971 in [11]–[12]) gave a solution for the *XYZ* model. He discovered a link between the quantum *XYZ* model and a problem of two-dimensional classical physics, the so-called eight-vertex model (its exact definition will be given in the main text), – in fact, it had been his principal aim in his paper to investigate this. Baxter made use of the ideas of Kramers–Wannier [13] and, in particular, of Onsager [14] on the transfer matrix, and of Lieb's solution [15]–[18] of the special case of the eight-vertex model, the so-called six-vertex model connected with the quantum *XXZ* model. In [9]–[10] he obtained a system of transcendental equations generalising the system derived from Bethe's method, and with its help he calculated the energy of the ground state of the *XYZ* model. In the subsequent series of papers [19]–[21] Baxter, by means of a very complicated and non-trivial generalization of Bethe's Ansatz, was able to construct the eigenvectors and to find the eigenvalues of the transfer matrix, and so to solve completely Heisenberg's *XYZ* model. In 1973 Johnson, Krinsky, and McCoy [22], using Baxter's results, calculated the energy of the excitations of the *XYZ* model.

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- Quantum L-operator is $L_n(\lambda) = \sum_{i=0}^{3} w_i(\lambda)\sigma_i \otimes \sigma_n^i$, and in the isotropic and semi-classical limit it becomes the Lax operator for the discretized Heisenberg spin chain.

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- Quantum R-matrix is the Baxter's matrix for the eight-vertex model (note $R_{41} = R_{14} \neq 0$), and the relation $RLL = LLR$ is

$$R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu).$$

In the notation (1.48)–(1.50) we can write (1.38) in the following form:

$$S^{iq}_{jp}(\lambda-\mu)\, S^{\alpha\gamma}_{pm}(\lambda)\, S^{\gamma\beta}_{qn}(\mu) = S^{\alpha\gamma}_{ip}(\mu) S^{\gamma\beta}_{jq}(\lambda)\, S^{pn}_{qm}(\lambda-\mu). \tag{1.51}$$

The relation (1.51) was first proposed by T. N. Yang in 1967 [49] in a discussion of many-particle factorizing S-matrices, therefore, it is natural to call (1.38) the Baxter–Yang relation. Later it was used in the papers by Karovskii and others [50], A. B. and Al. B. Zamolodchikov [51]–[52] in which conditions were investigated for the factorizibility of S-matrices in various models of the quantum field theory in two-dimensional space-time.

- Algebraic form of Baxter's generalized Bethe ansatz eigenvectors for the eight-vertex model:

$$\Psi(\lambda_1, \cdots, \lambda_n) = \sum e^{2\pi i l\theta} B_{l+1,l-1}(\lambda_1) \cdots B_{l+n,l-n}(\lambda_n) \Omega_N^{l-n}.$$

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- [45]. We should also like to express our respect and our gratitude to Rodney J. Baxter, whose papers we have read with so much pleasure.

  The authors dedicate this paper to Academician N. N. Bogolyubov on the occasion of his seventieth birthday.