Boundary reflection matrices of perturbed minimal models

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+ Zoltan Bajnok & Paul Pearce [arXiv:2509.04286]

Baxter2025 Exactly Solved Models and Beyond:
Celebrating the Life and Achievements of Rodney James Baxter

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Jim McGuire

Rodney Baxter

Baxter's work has had — and will surely continue to have — a profound influence on every aspect of quantum integrability.

Journal of Statistical Physics, Vol. 35, Nos. 3/4, 1984

Eight-Vertex SOS Model and Generalized Rogers–Ramanujan-Type Identities

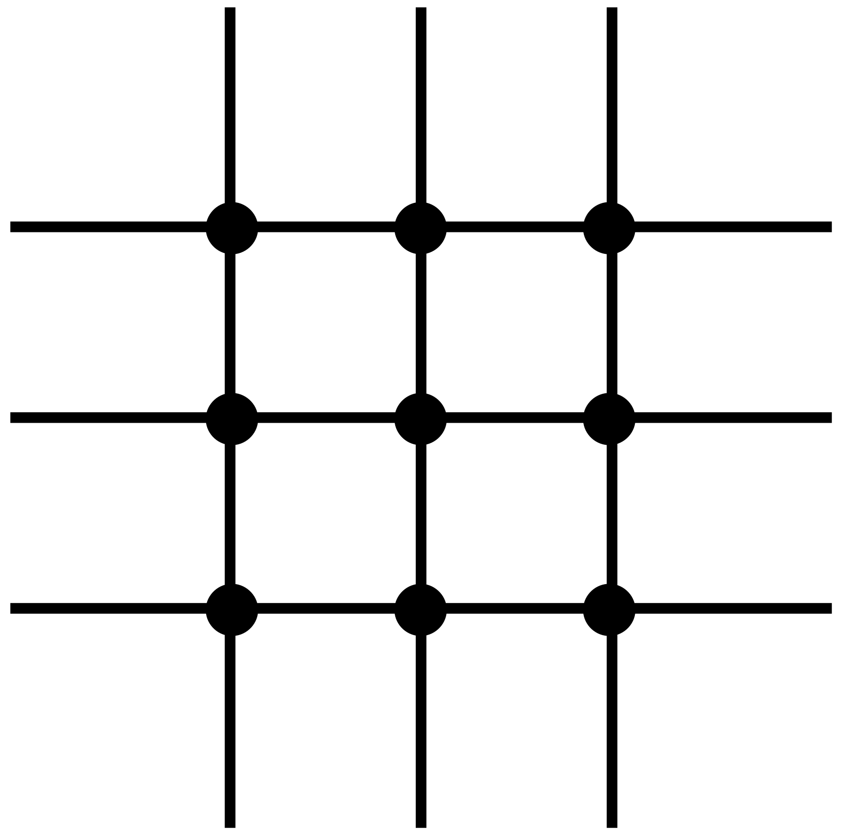
George E. Andrews,[1,2] R. J. Baxter,[3] and P. J. Forrester[3]

Today's talk is no exception, as it is partly based on the famous ABF paper.

Guiding example: 2-dim Ising model

2-dim Ising model

"spins" $\sigma_i = \pm 1$



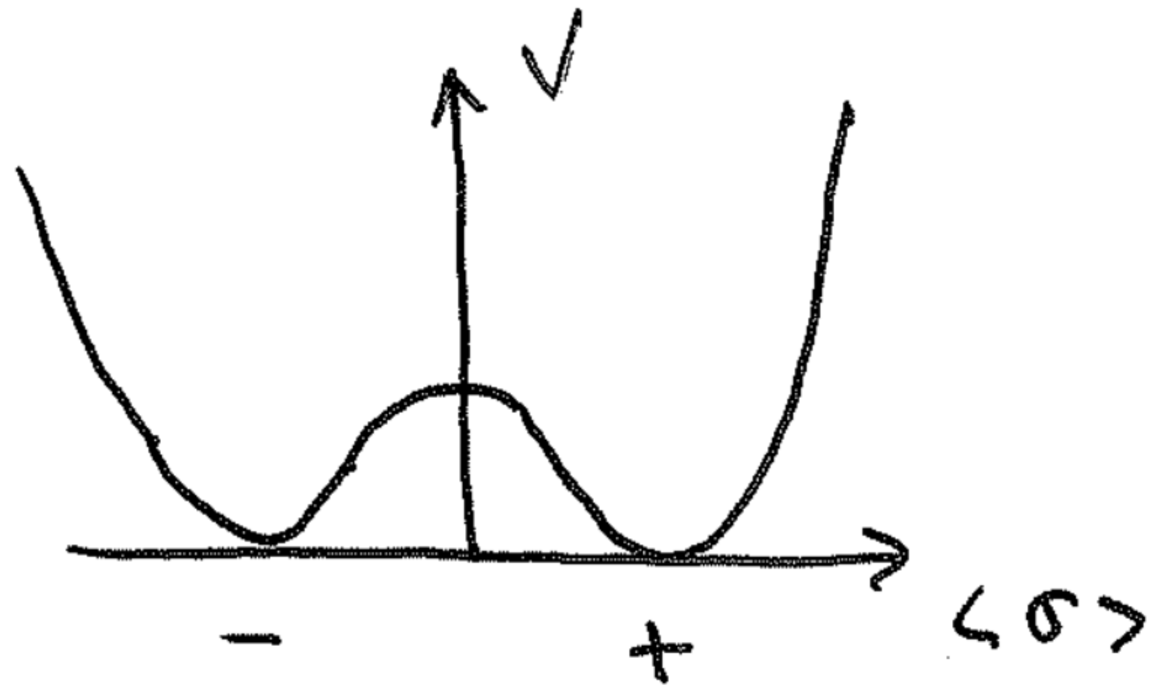
$$\mathcal{E} = - \sum_{\langle i,j \rangle} \sigma_i \, \sigma_j$$

nearest neighbors

critical temp T_C ~ CFT $c = \frac{1}{2}$

low-temp phase $T < T_C$: 2 degenerate equilibrium states +, -

Landau-Ginzburg effective potential



Equivalent to 1-dim quantum spin chain

$$H = \sum_n \left(-\sigma_n^x - \lambda\, \sigma_n^z\, \sigma_{n+1}^z \right) \,, \qquad \lambda = \frac{T_c}{T}$$

For $\lambda \to \infty \quad (T \to 0)$, 2 degenerate ground states:

$$|+\rangle = |\uparrow \cdots \uparrow\rangle$$
$$|-\rangle = |\downarrow \cdots \downarrow\rangle$$

Excitations are "kinks" $|\downarrow \cdots \downarrow\uparrow \cdots \uparrow\rangle$ mass $\propto |1-\lambda|$

kink operator $\mu_n^z = \sum_{m<n} \sigma^x$

$$\mu_n^z \,|+\rangle = |\downarrow \cdots \downarrow\uparrow \cdots \uparrow\rangle$$

Scaling region $T - T_c \to 0^-$

continuum QFT: free massive Majorana field

$$\mathcal{L}_{\text{bulk}} = \psi\partial_{\bar{z}}\psi - \bar{\psi}\partial_z\bar{\psi} + \boxed{M\bar{\psi}\psi} \qquad M \propto T_c - T > 0$$

bulk perturbation of dimension $(\frac{1}{2}, \frac{1}{2})$ of CFT $c = \dfrac{1}{2}$

spectrum:

Fermions with mass M energy $e = M\cosh\theta$ momentum $p = M\sinh\theta$

Can decompose $\psi\,,\bar{\psi}$ in terms of creation $\boxed{A(\theta)^\dagger}$ and annihilation $A(\theta)$ operators

Regard as kink operator that interpolates between the two ground states

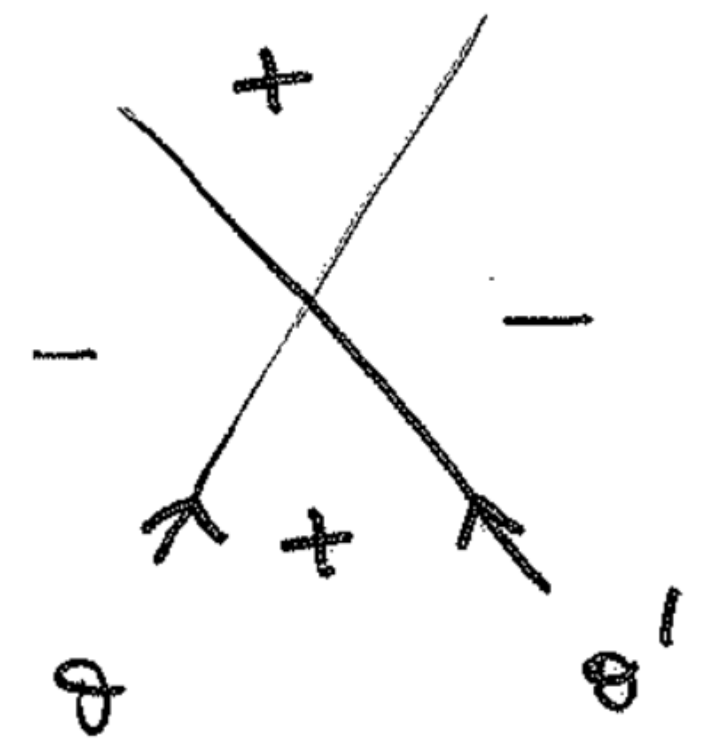
bulk scattering:

$$A(\theta)^\dagger\, A(\theta')^\dagger = -A(\theta')^\dagger\, A(\theta)^\dagger$$

$\Rightarrow$

$$S(\theta - \theta') = -1$$

free Fermion ✓



On half-line $(-\infty, 0]$ with boundary magnetic field h :

[Ghoshal, Zamolodchikov1994]

$$\mathcal{S} = \int_{-\infty}^{\infty} dy \left\{ \int_{-\infty}^{0} dx\, \mathcal{L}_{\text{bulk}} + \tfrac{1}{2} \left[\psi \bar{\psi} + a\, \dot{a} + h(\psi + \bar{\psi}) a \right] \Big|_{x=0} \right\}$$

On half-line $(-\infty, 0]$ with boundary magnetic field h : [Ghoshal, Zamolodchikov1994]

$$\mathcal{S} = \int_{-\infty}^{\infty} dy \left\{ \int_{-\infty}^{0} dx \, \mathcal{L}_{\text{bulk}} + \frac{1}{2} \left[\psi\bar{\psi} + a\dot{a} + h(\psi + \bar{\psi})a \right] \Big|_{x=0} \right\}$$

boundary degree of freedom

On half-line $(-\infty, 0]$ with boundary magnetic field h : [Ghoshal, Zamolodchikov1994]

$$\mathcal{S} = \int_{-\infty}^{\infty} dy \left\{ \int_{-\infty}^{0} dx\, \mathcal{L}_{\text{bulk}} + \tfrac{1}{2}\left[\psi\bar{\psi} + a\,\dot{a} + \boxed{h(\psi + \bar{\psi})a}\right]\Big|_{x=0} \right\}$$

boundary perturbation of dimension $\frac{1}{2}$ integrable

$M \to 0$

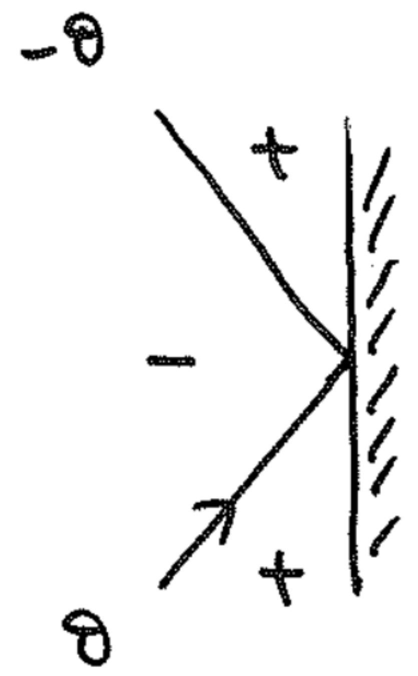
$h = 0$: "free" boundary conditions

ground states $|\pm\rangle$ still degenerate

$h \to \infty$: "fixed" boundary conditions $|+\rangle$

[Ghoshal, Zamolodchikov1994]

boundary scattering:



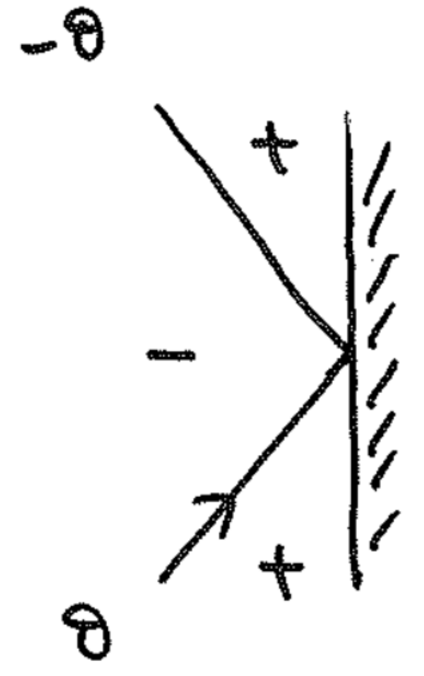
$$A(\theta)^\dagger = R(\theta) A(-\theta)^\dagger$$

$$R(\theta) = i \tanh\left(\frac{i\pi}{4} - \frac{\theta}{2}\right) \left(\frac{\sin\xi - i\sinh\theta}{\sin\xi + i\sinh\theta}\right)$$

"boundary reflection matrix"

$$\sin\xi = 1 - \frac{h^2}{2M}$$

[Ghoshal, Zamolodchikov1994]



boundary scattering:

$$A(\theta)^\dagger = \boxed{R(\theta)} A(-\theta)^\dagger$$

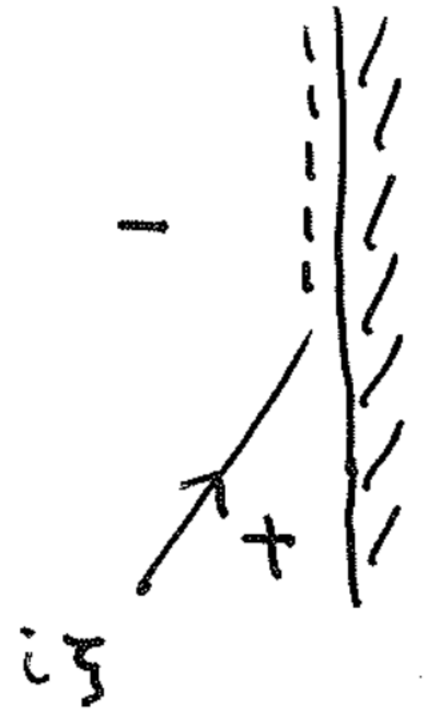
$$R(\theta) = i \tanh\left(\tfrac{i\pi}{4} - \tfrac{\theta}{2}\right) \left(\frac{\sin\xi - i\sinh\theta}{\sin\xi + i\sinh\theta} \right)$$

"boundary reflection matrix"

$$\sin\xi = 1 - \frac{h^2}{2M}$$

pole at $\theta = i\xi$ is physical for $0 < \xi < \frac{\pi}{2}$ (i.e. $0 < h^2 < 2M$)

boundary bound state



$$e_- - e_+ = M\cos\xi$$

check: For $\xi = \frac{\pi}{2}$ $(\text{i.e. } h = 0)\,,\quad e_- = e_+$ ✓

Generalization: critical Ising model $\longrightarrow$ unitary minimal models $\mathcal{A}_m$

$$c = \frac{1}{2} \qquad\qquad c = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4, \ldots$$

Ising

$\bar{\psi}\psi \longrightarrow \Phi_{1,3}$ bulk perturbation

$$\left(\frac{1}{2}, \frac{1}{2}\right) \qquad\qquad \left(\frac{m-1}{m+1}, \frac{m-1}{m+1}\right)$$

$(\psi + \bar{\psi})a\Big|_{x=0} \longrightarrow \phi_{1,3}$ boundary perturbation

$$\frac{1}{2} \qquad\qquad \frac{m-1}{m+1}$$

integrable [Ghoshal, Zamolodchikov 1994]

What are the boundary reflection matrices?

$m = 4$ [Chim 1996; Miwa, Weston 1997]

$m \geq 5?$ account for non-invertible symmetry [Copetti, Cordova, Komatsu 2024; Shimamori, Yamaguchi 2025]

Outline

1. Bulk scattering theory (review)
2. CBCs and boundary subsets (mostly review)
3. Boundary scattering theory (review)
4. Construction of boundary reflection matrices
5. Discussion

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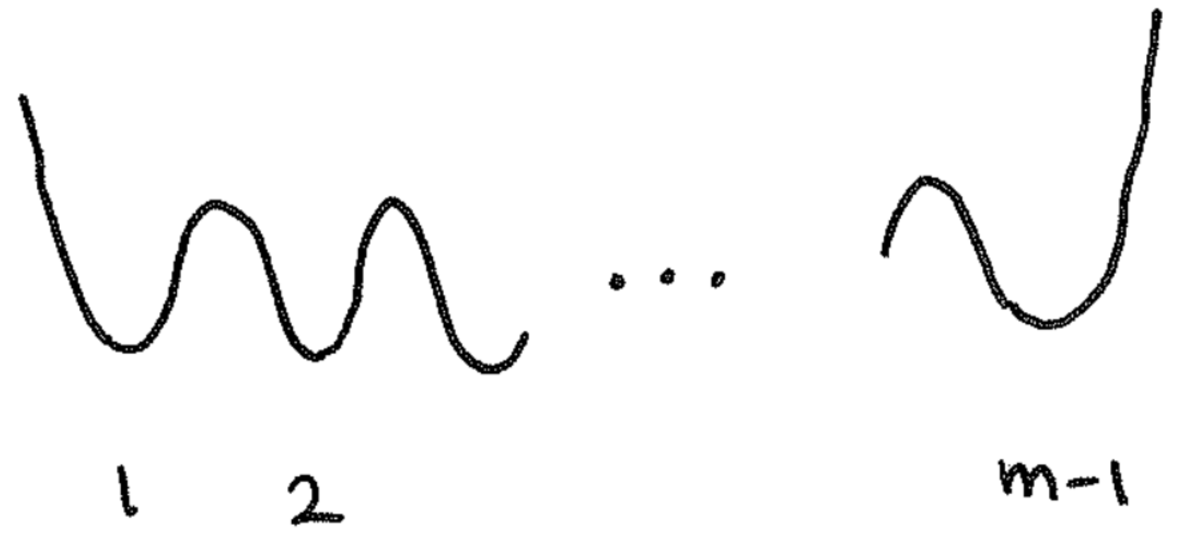
$$\mathcal{S} = \mathcal{S}_{\mathcal{A}_m} + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \ \Phi_{1,3}(x\,,y)\,, \qquad \lambda < 0$$

integrable [Zamolodchikov1987]

$(m-1)$ degenerate vacua



[Zamolodchikov 1989; Leclair 1989; Bernard, Leclair 1990; Reshetikhin, Smirnov 1990]

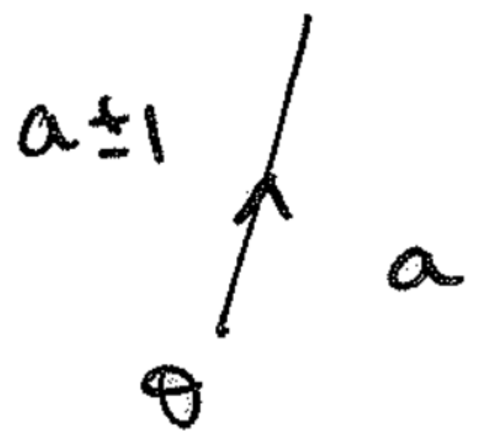
spectrum:

Kinks $K_{a,b}(\theta)$ with mass M energy $e = M\cosh\theta$ momentum $p = M\sinh\theta$

that interpolate between neighboring ground states

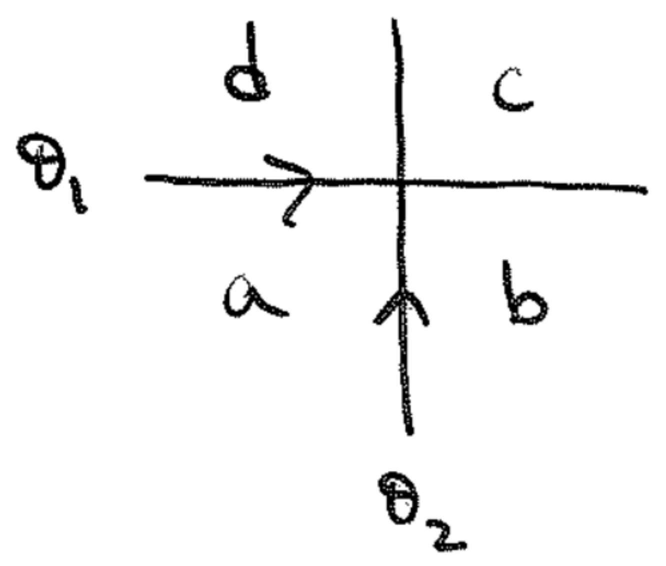
$a\,, b \in \{1\,, 2\,, \ldots\,, m-1\}$ "heights"

$|a-b| = 1$



S-matrix:

$$K_{d,a}(\theta_1)\ K_{a,b}(\theta_2) = \sum_d S^{d\ \ c}_{a\ \ b}(\theta_1 - \theta_2)\ K_{d,c}(\theta_2)\ K_{c,b}(\theta_1)$$



$$S^{d\ \ c}_{a\ \ b}(\theta) = U(\theta)\ \overline{S}^{d\ \ c}_{a\ \ b}(\theta)$$

"reduced"

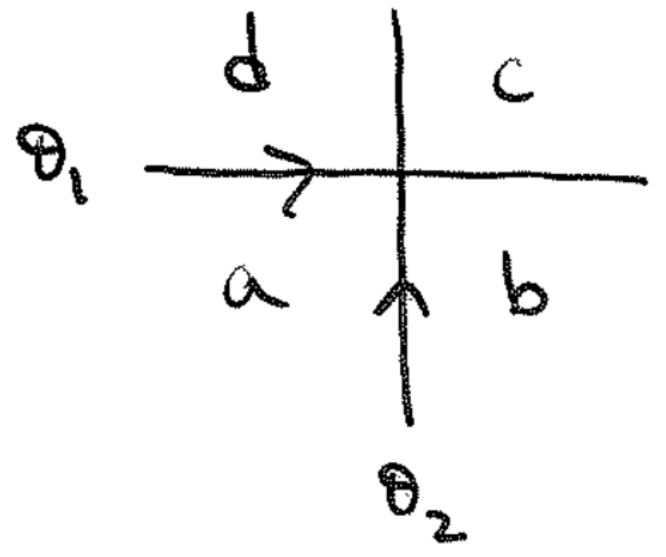
$$\overline{S}^{d\ \ c}_{a\ \ b}(\theta) = \sinh\left(\frac{i\pi - \theta}{m}\right)\delta_{ac} + \left(\frac{[a][c]}{[b][d]}\right)^{\frac{1}{2}} \sinh\left(\frac{\theta}{m}\right)\delta_{bd}$$

[Andrews, Baxter, Forrester 1984]

$$[a] = \frac{\sin(\frac{\pi a}{m})}{\sin(\frac{\pi}{m})}$$

S-matrix:

$$K_{d\,,a}(\theta_1)\ K_{a\,,b}(\theta_2) = \sum_d \boxed{S^{d\ \ c}_{a\ \ b}(\theta_1-\theta_2)}\ K_{d\,,c}(\theta_2)\ K_{c\,,b}(\theta_1)$$



$$S^{d\ \ c}_{a\ \ b}(\theta) = \boxed{U(\theta)}\,\overline{S}^{d\ \ c}_{a\ \ b}(\theta)$$

$$\overline{S}^{d\ \ c}_{a\ \ b}(\theta) = \sinh\left(\frac{i\pi-\theta}{m}\right)\delta_{ac} + \left(\frac{[a][c]}{[b][d]}\right)^{\frac{1}{2}} \sinh\left(\frac{\theta}{m}\right)\delta_{bd}$$

“reduced”

[Andrews, Baxter, Forrester 1984]

$$[a] = \frac{\sin(\frac{\pi a}{m})}{\sin(\frac{\pi}{m})}$$

$$U(\theta) = \frac{1}{\sinh(\frac{1}{m}(\theta-i\pi))} \exp\left(i\int_0^\infty \frac{dt}{t} \frac{\sin(\frac{\theta t}{\pi})\sinh(\frac{(m-1)t}{2})}{\sinh(\frac{mt}{2})\cosh(\frac{t}{2})}\right)$$

no bound states

Satisfies:

crossing

$$S_{a\ b}^{d\ c}(\theta) = \left(\frac{[a][c]}{[b][d]}\right)^{\frac{1}{2}} S_{d\ a}^{c\ b}(i\pi - \theta)$$

[Copetti, Cordova, Komatsu 2024]

[Smirnov 1991;
Colomo, Koubek, Mussardo 1992]

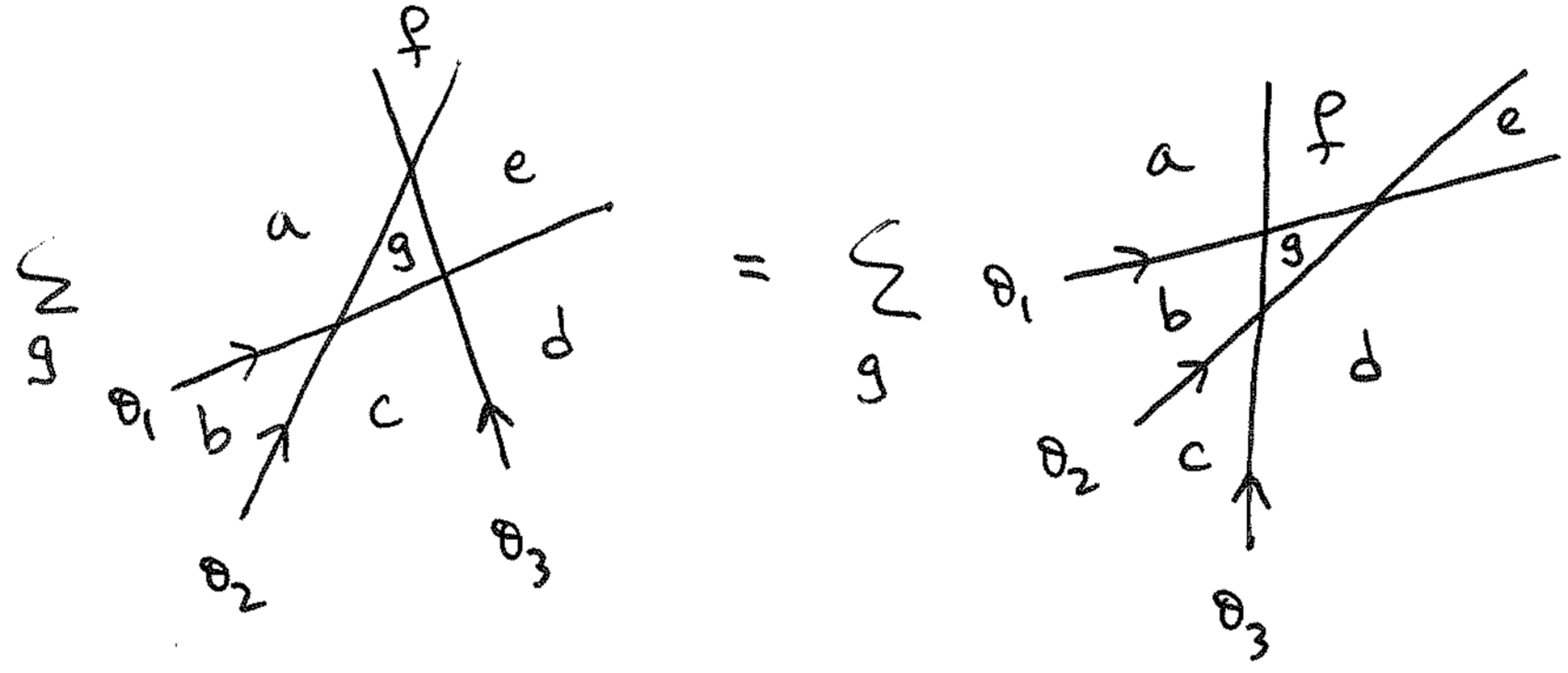
unitarity

$$\sum_{g} S_{a\ b}^{d\ g}(\theta)\ S_{g\ b}^{d\ c}(-\theta) = \delta_{a,c}\, G_{a,b}\, G_{a,d}$$

$$G_{a,b} = \delta_{a,b-1} + \delta_{a,b+1}$$

A_{m-1} adjacency matrix

Yang-Baxter



Outline

1. Bulk scattering theory (review)

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5. Discussion

A CFT can have various boundary conditions that are conformally invariant.

Example: free massless boson on half-line $(-\infty, 0]$ $\mathcal{L}_{\text{bulk}} = \frac{1}{2}\partial_{\bar{z}}\phi\,\partial_z\phi$ $c = 1$

$\phi\Big|_{x=0} = \phi_0$ Dirichlet $\partial_x\phi\Big|_{x=0} = 0$ Neumann

"conformal boundary conditions" (CBCs)

For $\mathcal{A}_m$ models: CBCs correspond to the primary fields $\phi_{r,s} = \phi_{r',s'}$ [Cardy 1989]

$$1 \leq r \leq m-1 \qquad r' = m - r$$

$$1 \leq s \leq m \qquad s' = m + 1 - s$$

Kac-table symmetry

Designate these CBCs by $(r,s) = (r',s')$

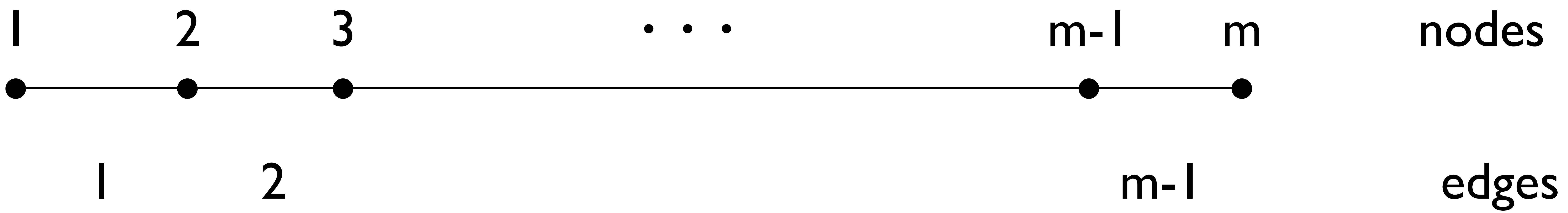
$m = 3$ **(Ising):** [Ghoshal, Zamolodchikov1994]

$$\mathcal{S} = \int_{-\infty}^{\infty} dy \left\{ \int_{-\infty}^{0} dx \, \mathcal{L}_{\text{bulk}} + \tfrac{1}{2} \left[\psi \bar{\psi} + a \, \dot{a} + h(\psi + \bar{\psi}) a \right] \Big|_{x=0} \right\}$$

$M \to 0$		Cardy's designation	CBC (r, s)
$h = 0:$	"free" boundary conditions ground states $\|\pm\rangle$ still degenerate	(f)	$(1, 2)$
$h \to \infty:$	"fixed" boundary conditions $\|+\rangle$	$(+)$	$(1, 1)$

Graphical representation of $\mathcal{A}_m$ CBCs

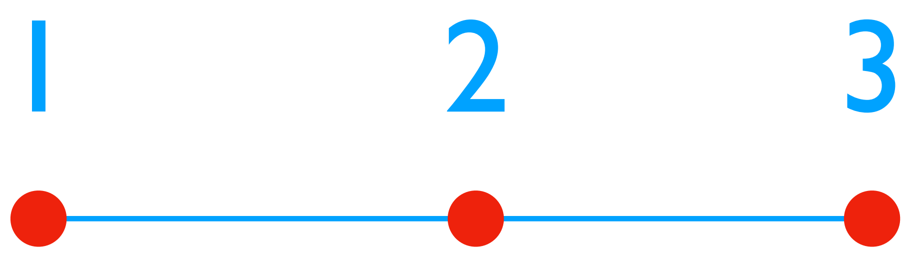
[Behrend, Pearce 2001; Graham 2002]

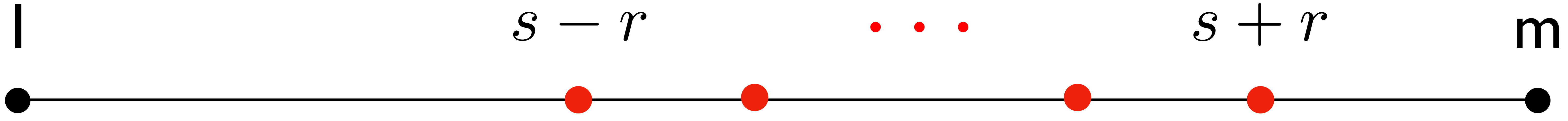


CBC (r, s)

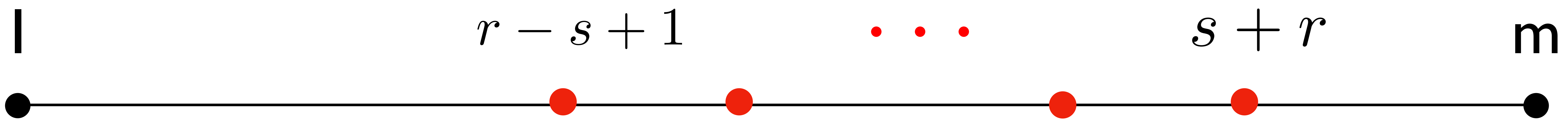
$r < s$: subgraph of connected nodes from $s - r$ to $s + r$

m=3 (1,2)





$r \geq s$: subgraph of connected nodes from $r - s + 1$ to $s + r$



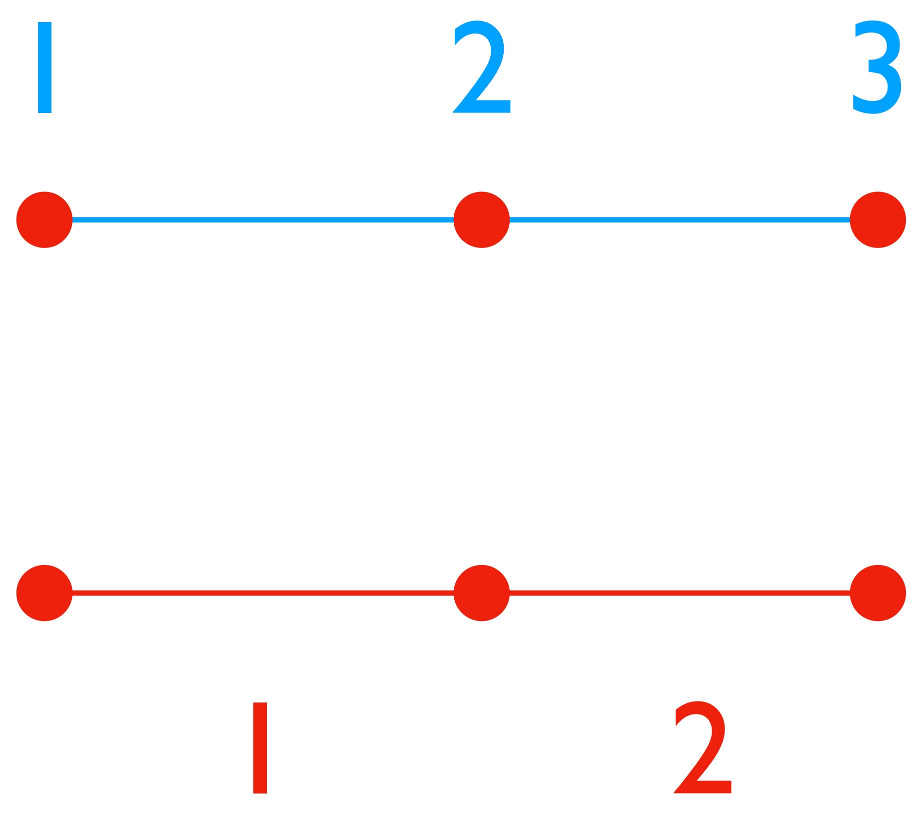
Important concept:

For a given CBC (r,s), only a subset $\mathcal{U}_{(r,s)} \subseteq \{1\,,2\,,\ldots\,,m-1\}$ of heights are allowed on the boundary

"boundary subset" uniquely characterizes the CBC

edges of (r,s) subgraph

m=3 (1,2)



$$\mathcal{U}_{(1,2)} = \{1,2\}$$

$m = 3$ **(Ising):** heights $\{+\,,-\} \equiv \{1\,,2\}$

CBC (r,s)	Cardy's designation	boundary subset $\mathcal{U}_{(r,s)}$
$(1,1) = (2,3)$	$(+)$	$\{1\}$
$(1,2) = (2,2)$	(f)	$\{1\,,2\}$
$(1,3) = (2,1)$	$(-)$	$\{2\}$

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Both bulk and boundary perturbation: integrable [Ghoshal, Zamolodchikov1994]

$$\mathcal{S} = \mathcal{S}_{\mathcal{A}_m+(r,s)} + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx \; \Phi_{1,3}(x\,,y) + h \int_{-\infty}^{\infty} dy \; \phi_{1,3}(y)\,, \qquad \lambda < 0$$

boundary operator B_a height at the boundary $a \in \mathcal{U}_{(r,s)}$

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multi-kink states

$$K_{a_1\,,a_2}(\theta_1)\ K_{a_2\,,a_3}(\theta_2)\ldots K_{a_N\,,a}(\theta_N)\ B_a$$

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boundary operator B_a height at the boundary $a \in \mathcal{U}_{(r,s)}$

multi-kink states

$$K_{a_1\,,a_2}(\theta_1)\; K_{a_2\,,a_3}(\theta_2) \ldots K_{a_N\,,a}(\theta_N)\; B_a$$

boundary reflection matrix

Want to determine!

$$K_{a\,,b}(\theta)\; B_b = \sum_c R^{(r,s)}\, {}_a{}^{c}_{b}\, (\theta\,,\xi)\, K_{a\,,c}(-\theta)\; B_c$$

Both bulk and boundary perturbation: integrable [Ghoshal, Zamolodchikov1994]

$$\mathcal{S} = \mathcal{S}_{\mathcal{A}_m+(r,s)} + \boxed{\lambda}\int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx\ \Phi_{1,3}(x\,,y) + \boxed{h}\int_{-\infty}^{\infty} dy\ \phi_{1,3}(y)\,, \qquad \lambda < 0$$

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$$K_{a_1\,,a_2}(\theta_1)\ K_{a_2\,,a_3}(\theta_2)\dots K_{a_N\,,a}(\theta_N)\ B_a$$

boundary reflection matrix

$$K_{a\,,b}(\theta)\ B_b = \sum_c \boxed{R^{(r,s)}\ {}_{a}{}^{c}_{b}\ (\theta\,,\boxed{\xi})}\ K_{a\,,c}(-\theta)\ B_c$$

ξ is related to λ and h in some way $\qquad h = 0 \quad \leftrightarrow \quad \xi = \frac{\pi}{2}$

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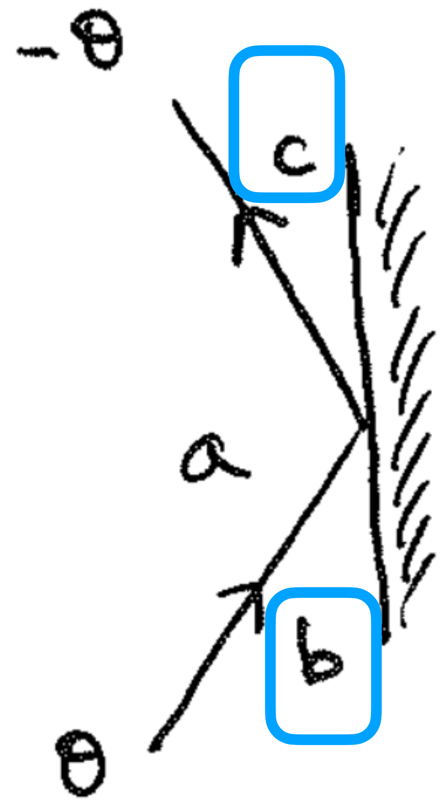
$$K_{a\,,b}(\theta)\; B_b = \sum_c R^{(r,s)}\,{}_{a}{}^{c}_{b}\;(\theta\,,\xi)\; K_{a\,,c}(-\theta)\; B_c$$



ξ is related to λ and h in some way $\qquad h = 0 \quad \leftrightarrow \quad \xi = \frac{\pi}{2}$

$R^{(r,s)}\,{}_{a}{}^{c}_{b} = 0$ unless both $b\,,c \in \mathcal{U}_{(r,s)}$

Must satisfy:

crossing

[Shimamori, Yamaguchi 2025]

$$R^{(r,s)}{}_{b}{}^{a}{}_{c}\left(\frac{i\pi}{2}-\theta\,,\xi\right)=\sum_{d}\left(\frac{[d]}{[b]}\right)^{\frac{1}{2}}S_{b}^{a}{}_{c}^{d}(2\theta)\;R^{(r,s)}{}_{d}{}^{a}{}_{c}\left(\frac{i\pi}{2}+\theta\,,\xi\right)$$

Must satisfy:

crossing

[Shimamori, Yamaguchi 2025]

$$R^{(r,s)}\ {}_{b}{}^{a}{}_{c}\ (\frac{i\pi}{2}-\theta\,,\xi)=\sum_{d}\left(\frac{[d]}{[b]}\right)^{\frac{1}{2}}\ S_{b\ c}^{a\ d}(2\theta)\ R^{(r,s)}\ {}_{d}{}^{a}{}_{c}\ (\frac{i\pi}{2}+\theta\,,\xi)$$

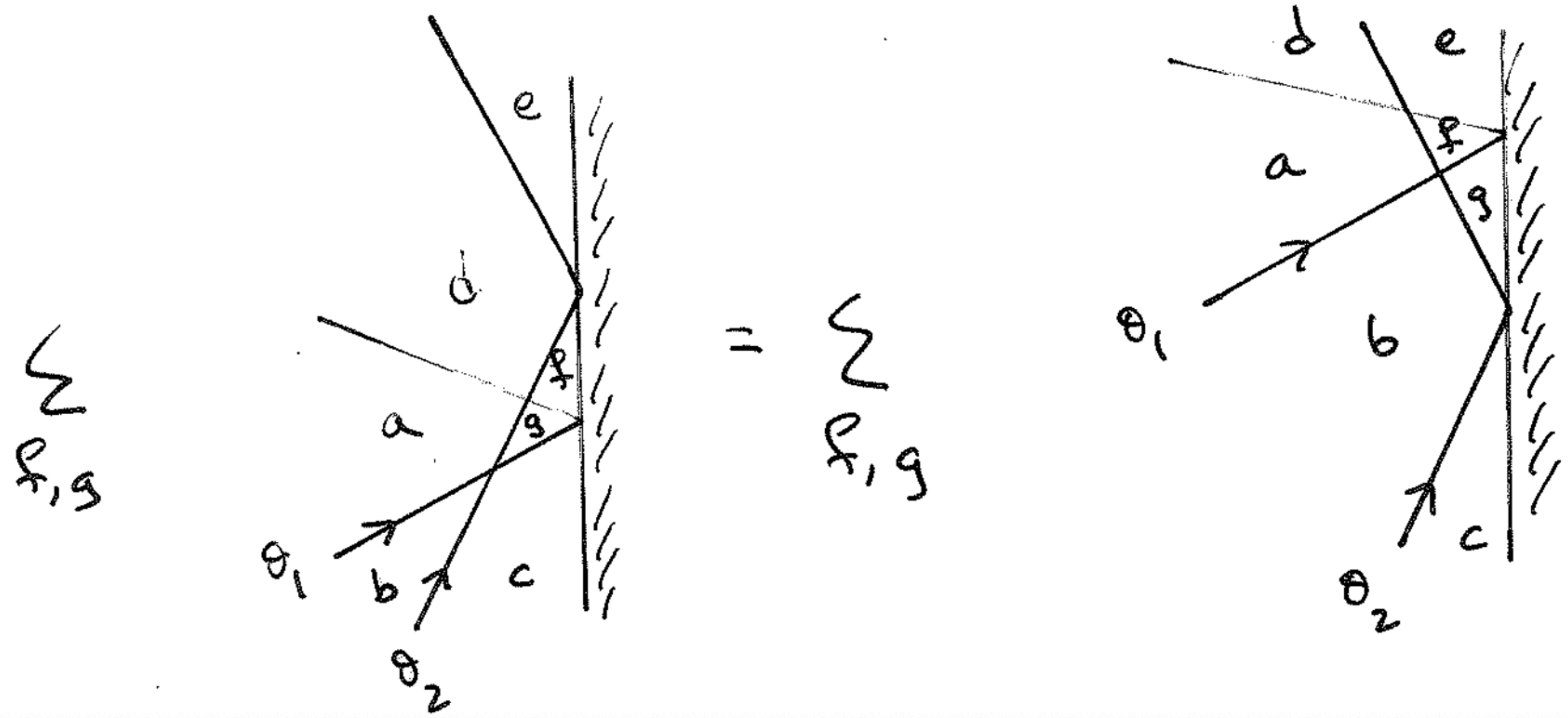
unitarity

$$\sum_{c}R^{(r,s)}\ {}_{a}{}^{c}{}_{b}\ (\theta\,,\xi)\ R^{(r,s)}\ {}_{a}{}^{d}{}_{c}\ (-\theta\,,\xi)=\delta_{b\,,d}\ G_{a\,,b}\ \mathcal{U}_d$$

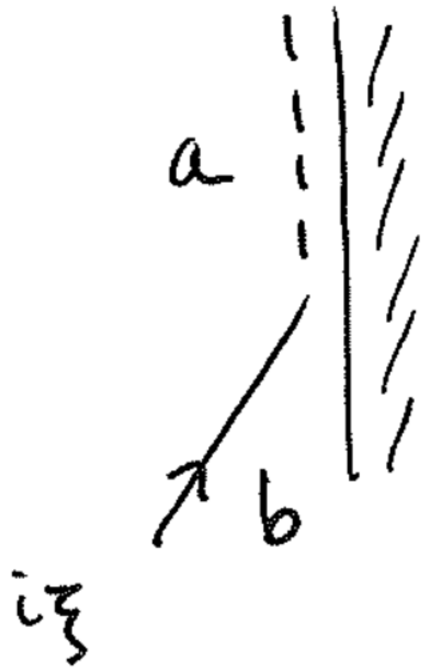
$$\mathcal{U}_d=\begin{cases}1 & \text{if } d\in\mathcal{U}_{(r,s)}\\ 0 & \text{otherwise}\end{cases}$$

boundary Yang-Baxter (BYB)

[Cherednik 1984]



boundary bound state

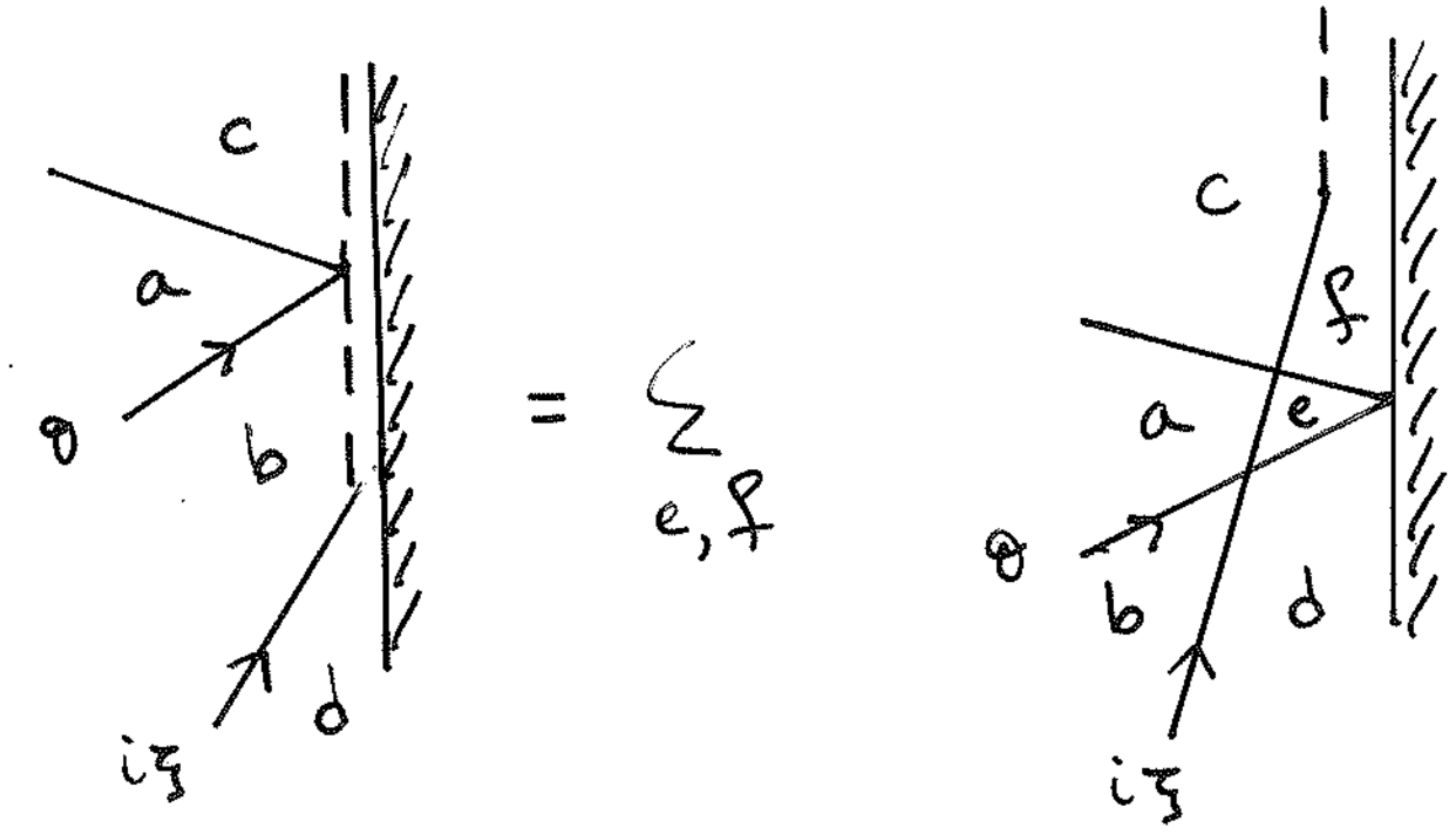


$$e_a = e_b + M\cos\xi \qquad 0 < \xi < \tfrac{\pi}{2}$$

$$B_a = \frac{1}{g_{b,a}(\xi)} K_{a\,,b}(i\xi)\; B_b$$

corresponds to pole of $R^{(r,s)}\;{}_{a}{}^{c}{}_{b}\;(\theta\,,\xi)$ at $\theta = i\xi$

boundary bound-state bootstrap



$$g_{d,b}(\xi)\; R^{(r,s)}\;{}_{a}{}^{c}{}_{b}\;(\theta\,,\xi) = \sum_{e,f} g_{f,c}(\xi)\; S_{b}^{a}{}_{d}^{e}(\theta - i\xi)\; R^{(r,s)}\;{}_{e}{}^{f}{}_{d}\;(\theta\,,\xi)\; S_{e}^{a}{}_{f}^{c}(\theta + i\xi)$$

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Objective: Determine $R^{(r,s)}\,{}_{a}{}^{c}{}_{b}\,(\theta\,,\xi)$ for all CBC (r,s)

Observe: Certain "elementary" solutions of BYB are already known [Behrend, Pearce 2001]

But these solutions do *not* obey boundary bound-state bootstrap

Strategy: Construct "paired" solutions that obey bootstrap by forming "direct sums" of certain pairs of elementary solutions

Elementary solutions

[Behrend, Pearce 2001]

“reduced” solutions of BYB (with “reduced” bulk S-matrix $\overline{S}^{d\ c}_{a\ b}$)

$$\overline{B}^{(p,q)}{}_{c}{}^{c\pm1}_{c\pm1}(\theta,\xi,\mu) = \frac{1}{[p]\sqrt{[c][c\pm1]}}\Big\{[(p\mp c+q)/2][(c\pm q\mp p)/2]s(i\xi+\theta)\ s(i(p\pi+\xi)-\theta)$$

$$+[(p\pm c+q)/2][(c\mp q\pm p)/2]s(i\xi-\theta)\ s(i(p\pi+\xi)+\theta)\Big\}\boxed{F^{p}_{c\pm1,q}}$$

$$\overline{B}^{(p,q)}{}_{c}{}^{c\mp1}_{c\pm1}(\theta,\xi,\mu) = \frac{\sqrt{[(p-c+q)/2][(p+c-q)/2][(c+q-p)/2][(c+q+p)/2]}}{\sqrt[4]{[c-1][c+1]}\sqrt{[c]}}$$

$$\times\,\mu\, s(2\theta)\boxed{F^{p}_{c+1,q}F^{p}_{c-1,q}}$$

$$\mu=\pm1\,, \qquad s(x)=\frac{\sinh(\frac{x}{m})}{i\sin(\frac{\pi}{m})}\,, \qquad F^{p}_{a,b}=0\,,1$$

fused A_{m-1} adjacency matrices

Define <u>elementary</u> boundary subset

$$\mathcal{V}_{(p,q)} \subseteq \{1\,,2\,,\ldots\,,m-1\}$$

such that $a \in \mathcal{V}_{(p,q)}$ iff $F^{p}_{a,q} > 0$

$$\overline{B}^{(p,q)}\,{}_{a}{}^{c}{}_{b} = 0 \quad \text{unless both} \quad b\,,c \in \mathcal{V}_{(p,q)}$$

cf.

$$R^{(r,s)}\,{}_{a}{}^{c}{}_{b} = 0 \quad \text{unless both} \quad b\,,c \in \mathcal{U}_{(r,s)}$$

key fact: $\mathcal{U}_{(r,s)} = \mathcal{V}_{(s-1,r)} \cup \mathcal{V}_{(m-s,m-r)}$ disjoint union

“dressed” elementary solution

$$B^{(p,q)}\,{}_{a}{}^{c}{}_{b}\,(\theta\,,\xi,\mu) = \boxed{\frac{V^{(p,q)}(\theta\,,\xi,\mu)}{s(i\xi)\,s(i(p\pi+\xi))}\left(\frac{[b][c]}{[a]^2}\right)^{\frac{1}{4}}}\,\overline{B}^{(p,q)}\,{}_{a}{}^{c}{}_{b}\,(\theta\,,\xi,\mu)$$

“dressed” elementary solution

$$B^{(p,q)}\ {}_{a}{}^{c}{}_{b}\ (\theta\,,\xi,\mu) = \frac{V^{(p,q)}(\theta\,,\xi,\mu)}{s(i\xi)\,s(i(p\pi+\xi))}\left(\frac{[b][c]}{[a]^2}\right)^{\frac{1}{4}}\overline{B}^{(p,q)}\ {}_{a}{}^{c}{}_{b}\ (\theta\,,\xi,\mu)$$

Satisfies BYB (with complete bulk S-matrix $S_{a\ b}^{d\ c}$)

Also satisfies crossing & unitarity, with suitable $V^{(p,q)}(\theta,\xi,\mu)$

“dressed” elementary solution

$$B^{(p,q)}\,{}_a{}^c{}_b\,(\theta\,,\xi,\mu) = \frac{V^{(p,q)}(\theta\,,\xi,\mu)}{s(i\xi)\,s(i(p\pi+\xi))}\left(\frac{[b][c]}{[a]^2}\right)^{\frac{1}{4}}\overline{B}^{(p,q)}\,{}_a{}^c{}_b\,(\theta\,,\xi,\mu)$$

Satisfies BYB (with complete bulk S-matrix $S_{a\ b}^{d\ c}$)

Also satisfies crossing & unitarity, with suitable $V^{(p,q)}(\theta,\xi,\mu)$

But does *not* obey bootstrap:

$$g_{d,b}(\xi)B^{(m-p-1,m-q)}\,{}_a{}^c{}_b\,(\theta,\pi-\xi,\mu) = \sum_{e,f} g_{f,c}(\xi)S_{b\ d}^{a\ e}(\theta-i\xi)B^{(p,q)}\,{}_e{}^f{}_d\,(\theta,\xi,\mu)S_{e\ f}^{a\ c}(\theta+i\xi)$$

$$(m-p-1,m-q) \quad \leftrightarrow \quad (p,q)$$

Paired solutions

Main result

$$R^{(r,s)}\ {}_a{}^c{}_b\ (\theta,\xi,\mu) = B^{(s-1,r)}\ {}_a{}^c{}_b\ (\theta,\xi,\mu) \oplus B^{(m-s,m-r)}\ {}_a{}^c{}_b\ (\theta,\pi-\xi,\mu)$$

i.e.

$$R^{(r,s)}\ {}_a{}^c{}_b\ (\theta,\xi,\mu) = \begin{cases} B^{(s-1,r)}\ {}_a{}^c{}_b\ (\theta,\xi,\mu) & \text{if } b,c \in \mathcal{V}_{(s-1,r)} \\ B^{(m-s,m-r)}\ {}_a{}^c{}_b\ (\theta,\pi-\xi,\mu) & \text{if } b,c \in \mathcal{V}_{(m-s,m-r)} \end{cases}$$

(recall $\mathcal{U}_{(r,s)} = \mathcal{V}_{(s-1,r)} \cup \mathcal{V}_{(m-s,m-r)}$ disjoint union)

All constraints satisfied 🙂

Outline

1. Bulk scattering theory (review)

2. CBCs and boundary subsets (mostly review)

3. Boundary scattering theory (review)

4. Construction of boundary reflection matrices

5. Discussion

Boundary reflection matrices agree with those proposed for

$m = 3$ [Ghoshal, Zamolodchikov1994]

$m = 4$ [Chim 1996; Miwa, Weston 1997]

Open problems:

- Relation between parameters ξ in reflection matrix and λ , h in action?
- Reflection matrices for *superpositions* of Cardy CBCs?
- Boundary quantum group symmetry?
- Reduction of boundary sine-Gordon reflection matrix?

Thank you for your attention!