Parity of Fundamental Units

Number Theory Down Under, ANU, 4 September 2023

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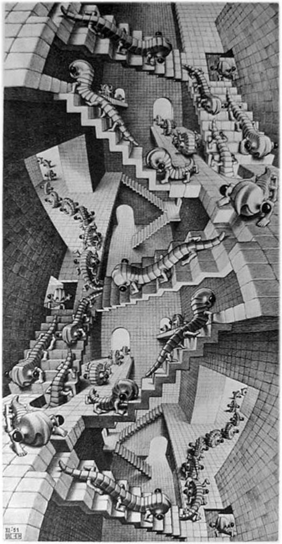
4 September 2023

Outline



1. Explicit Formulas
2. Fundamental Units
3. Link to cubic discriminants

Explicit formula

- $\chi : \mathbb{N} \to \mathbb{C}$ character
- $L(\chi, s) := \prod_p \left(\dfrac{1}{1 - \chi(p)p^{-s}} \right)$ associated L-function.
- $\dfrac{L'(\chi, s)}{L(\chi, s)} = -\sum_{n=1}^{\infty} \dfrac{\Lambda_\chi(n)}{n^s}$, where

$$\Lambda_\chi(n) = \begin{cases} \chi(p)^k \log p & \text{if } n = p^k \\ 0 & \text{otherwise.} \end{cases}$$

- Inverse Mellin transform:

$$\psi_\chi(x) := \sum_{n \le x} \Lambda_\chi(n)$$

$$=' -\sum_\rho \mathrm{ord}_{s=\rho}(L(\chi, s)) \frac{x^\rho}{\rho} - \frac{L'(\chi, 0)}{L(\chi, 0)}.$$

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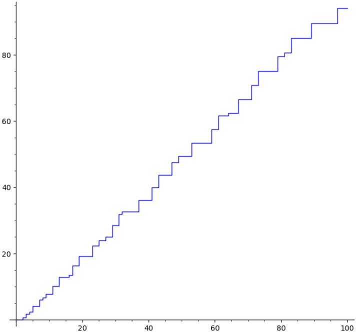
Example: $\chi = 1$, $L(\chi, s) = \zeta(s)$ Riemann zeta function

Chebychef ψ-function:

$$\psi(x) := \sum_{n \le x} \Lambda(n)$$

$$= \sum_{p^k \le x} \log p$$



$$\psi(x) =' \underbrace{\frac{x^1}{1}}_{\text{pole } s=1} - \underbrace{\sum_{n=1}^{\infty} \frac{x^{-2n}}{-2n}}_{\text{trivial zeros}} - \underbrace{\sum_{\rho} \frac{x^\rho}{\rho}}_{\text{critical zeros}} - \underbrace{\frac{\zeta'(0)}{\zeta(0)}}_{= \log 2\pi}$$

$$= x - \frac{1}{2}\log(1 - x^{-2}) - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi$$

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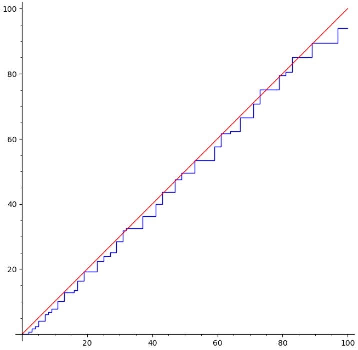

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Riemann Hypothesis: $\rho = \frac{1}{2} + it$

$$\frac{x^{\rho}}{\rho} + \frac{x^{\bar{\rho}}}{\bar{\rho}} = \sqrt{x}\frac{\cos(t\log x) + 2t\sin(t\log x)}{\frac{1}{4} + t^2} \approx 2\sqrt{x}\frac{\sin(t\log x)}{t}$$

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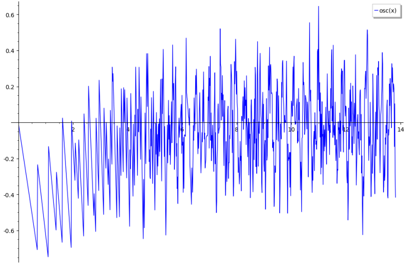

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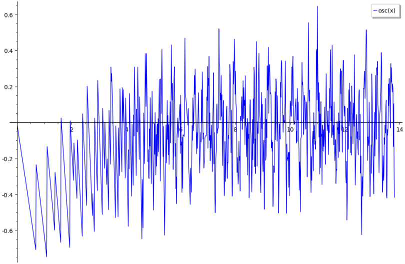
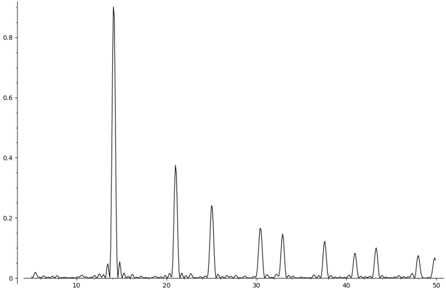


Fourier transform (power spectrum)



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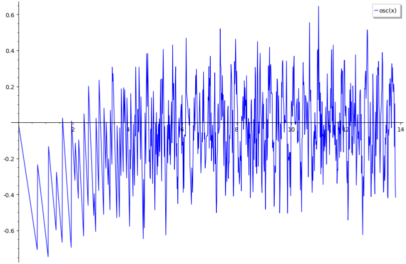

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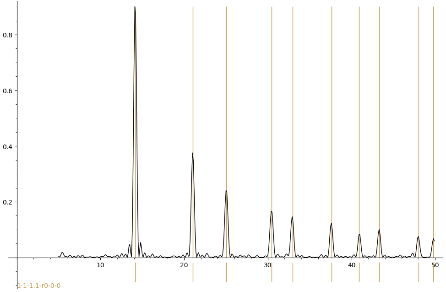

Example: χ Dirichlet character mod 4

$$\chi_4(p) = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ -1 & \text{if } p \equiv 3 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

$$\psi_{\chi_4}(x) := \sum_{n \leq x} \chi_4(n) \Lambda_{\chi_4}(n) = \sum_{p^k \leq x} \chi_4(p)^k \log p$$

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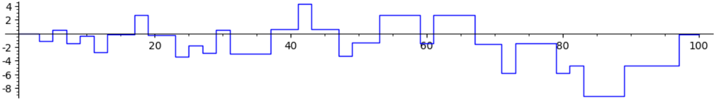

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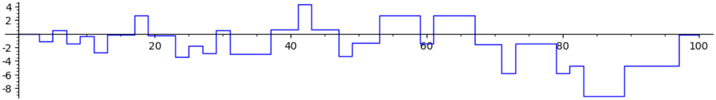

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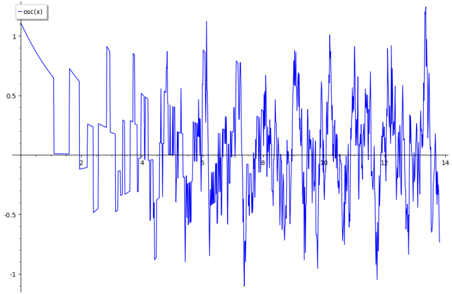

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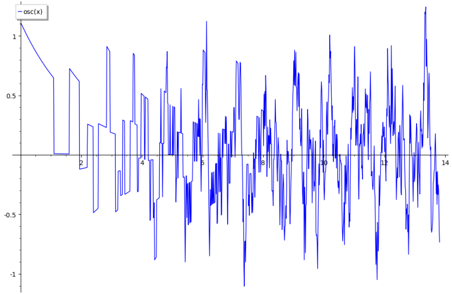

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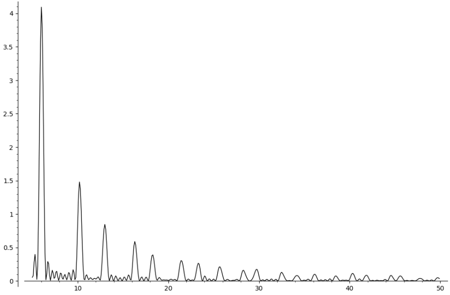

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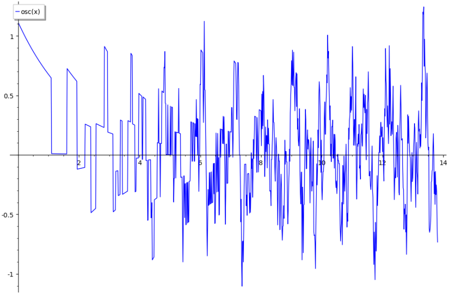

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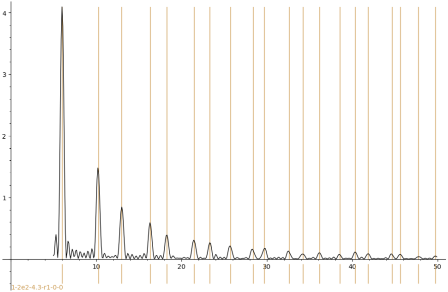

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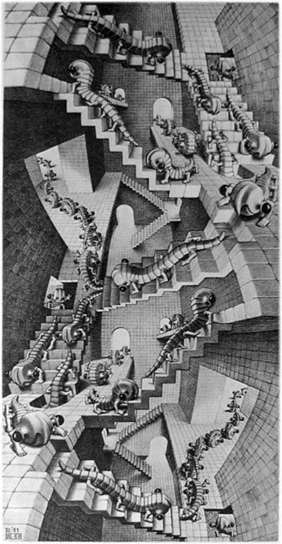

Fundamental units

- $p \equiv 5 \bmod 8$ prime
- $K = \mathbb{Q}(\sqrt{p})$ real quadratic field
- $\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{p}}{2}\right]$ ring of integers
- $\varepsilon_p = \frac{x_0+y_0\sqrt{p}}{2} \in \mathcal{O}_K^*$ fundamental unit
- (x_0, y_0) is a fundamental solution to $X^2 - pY^2 = -4$.
- $x_0 \equiv y_0 \bmod 2$.
- Is x_0 even or odd? I.e. what is $\varepsilon_p \bmod 2\mathcal{O}_K$?
- $2\mathcal{O}_K$ is prime, so three possibilities for

$$\varepsilon_p \bmod 2\mathcal{O}_K \in (\mathcal{O}_K/2\mathcal{O}_K)^* \cong \mathbb{F}_4^*.$$

Each should be equally likely.

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Associated L-function?

Suppose we can define a character $\chi : \mathbb{N} \to \mu_3 \cup \{0\}$ such that

$$\chi(p) = \begin{cases} 1 & \text{if } p \equiv 5 \bmod 8 \text{ and } \varepsilon_p \equiv 1 \bmod 2\mathcal{O}_K \\ \exp(\pm 2\pi i/3) \in \mu_3 \setminus \{1\} & \text{if } p \equiv 5 \bmod 8 \text{ and } \varepsilon_p \not\equiv 1 \bmod 2\mathcal{O}_K \\ 0 & \text{if } p \not\equiv 5 \bmod 8. \end{cases}$$

If the associated L-function

$$L(\chi, s) := \prod_p \left(\frac{1}{1 - \chi(p)p^{-s}} \right)$$

has nice analytic properties, we should have an explicit formula

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Problem

I don't know how to choose $\chi(p) \in \mu_3 \smallsetminus \{1\}$. But I do know

$$\mathrm{Re}\,\chi(p)^k = \begin{cases} 1 & \text{if } p \equiv 5 \bmod 8 \text{ and } \varepsilon_p \equiv 1 \bmod 2\mathcal{O}_K \text{ or } 3|k \\ -\frac{1}{2} & \text{if } p \equiv 5 \bmod 8 \text{ and } \varepsilon_p \not\equiv 1 \bmod 2\mathcal{O}_K \\ 0 & \text{if } p \not\equiv 5 \bmod 8 \end{cases}$$

So we can compute the real part of

$$\psi_\chi(x) = \sum_{p^k \leq x} \chi(p)^k \log p$$

Since the three possibilities for $\varepsilon_p \bmod 2\mathcal{O}_K \in \mathbb{F}_4^*$ should occur equally often, we expect

$$\psi_\chi(x) = o(x).$$

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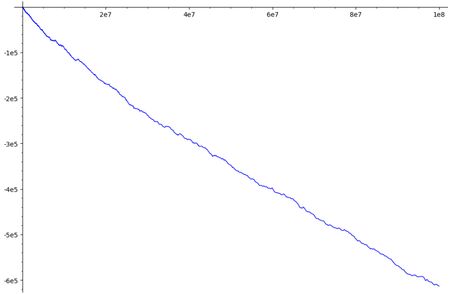
$\mathrm{Re}\psi_\chi(x)$ for $x \leq 10^8$



Main term: $\psi_\chi(x) \sim ax^b$ with $b = 0.8\ldots$.

Frank Calegari suggested maybe $b = \frac{5}{6}$. This gives a best fit

$$\psi_\chi(x) \sim (-0.134\ldots)x^{5/6}.$$

Then $L(\chi, s)$ has a real zero at $s = b = \frac{5}{6}$.
(Or maybe $b = \frac{4}{5}$...?)

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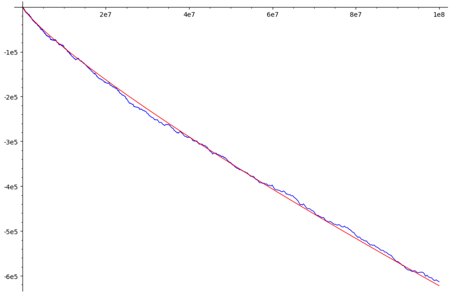

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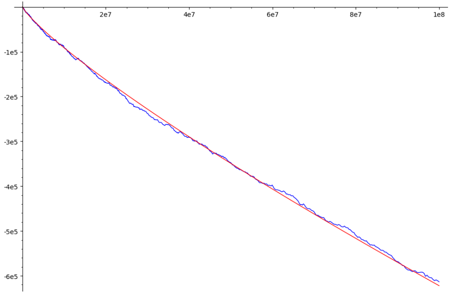

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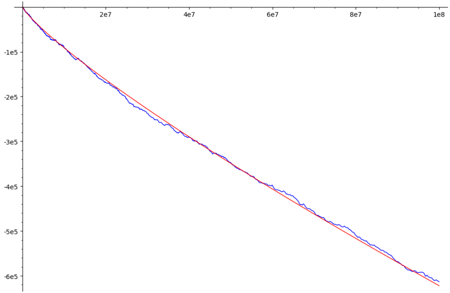

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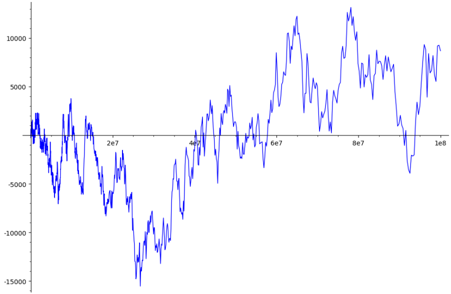
Then $L(\chi, s)$ has a real zero at $s = b = \frac{5}{6}$.
(Or maybe $b = \frac{4}{5}$...?)

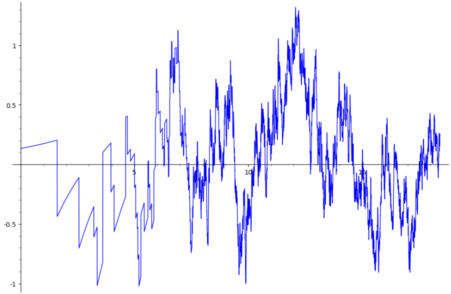
Oscillating term?

$\psi_\chi(x) + 0.134x^{5/6}$



Oscillating term?

$(\psi_\chi(x) + 0.134x^{5/6})/x^{5.7}$, ($\log x$ axis)



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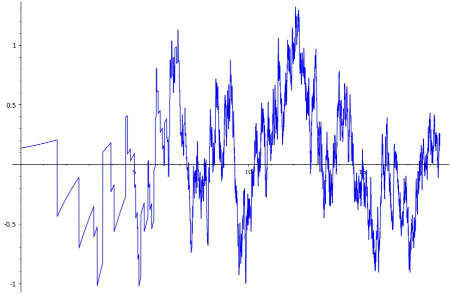
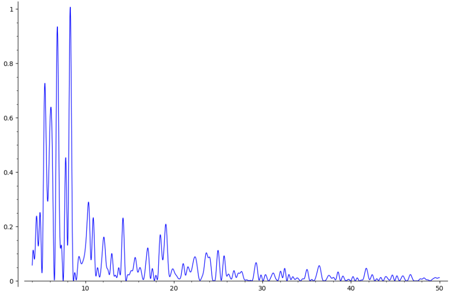


Fourier transform (power spectrum):



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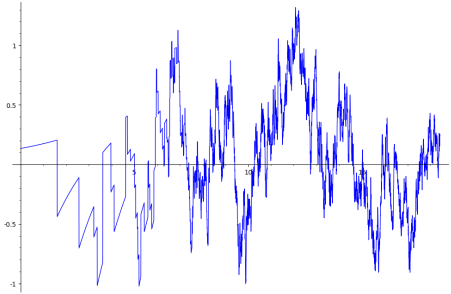

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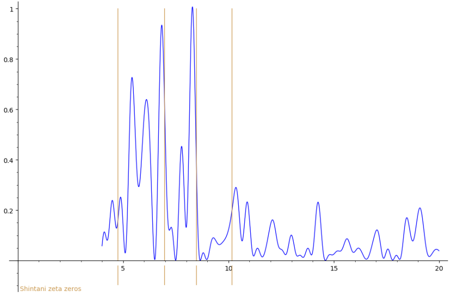
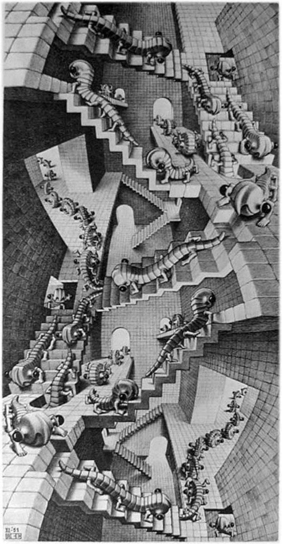


Outline



1. Explicit Formulas
2. Fundamental Units
3. Link to cubic discriminants

What is known

Let $d \in \mathcal{D} = \{d \in \mathbb{N} \mid d \equiv 5 \bmod 8$ and d is square-free$\}$

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$\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$ ring of integers
$\mathcal{O}_d = \mathbb{Z}[\sqrt{d}]$ order of conductor 2
$\varepsilon_d \in \mathcal{O}_K^*$ the fundamental unit.

$$0 \to (\mathcal{O}_d/2\mathcal{O}_d)^*/(\mathcal{O}_d^*/2\mathcal{O}_d)^* \to \mathrm{Cl}(\mathcal{O}_d) \to \mathrm{Cl}(\mathcal{O}_K) \to 0$$

The following are equivalent:

1. $\varepsilon_d \equiv 1 \bmod 2\mathcal{O}_K$
2. $\mathcal{O}_K^* = \mathcal{O}_d^*$
3. $\#\mathrm{Cl}(\mathcal{O}_d) = 3\#\mathrm{Cl}(\mathcal{O}_K)$

Stevenhagen (1996): Let $d \in \mathcal{D}$. The following are equivalent:

1. $\varepsilon_d \equiv 1 \bmod 2\mathcal{O}_K$ and the exact sequence splits;
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Counting cubic fields

Stevenhagen (1996):

- $\{d \in \mathcal{D},\ \varepsilon_d \equiv 1 \bmod 2\mathcal{O}_K\}$ has upper density $\leq \frac{1}{2}$ in $\mathcal{D}$.
- $\#\{d \in \mathcal{D},\ \varepsilon_d \equiv 1 \bmod 2\mathcal{O}_K, d \leq x\} \gg x^{1/2}$.

Uses **Davenport-Heilbronn (1971):**

$$\#\{\text{Isom class of cubic field with discriminant } \Delta_K \leq x\} = \frac{1}{12\zeta(3)}x + o(x)$$

Bhargava-Shankar-Tsimmerman (2013), Bhargava-Taniguchi-Thorne (2021):

$$\#\{\Delta_K \leq x\} = \frac{1}{12\zeta(3)}x + \underbrace{\frac{4\zeta(1/5)}{5\Gamma(2/3)^3\zeta(5/3)}}_{=-0.147685} x^{5/6} + O(x^{2/3}(\log x)^{2.09})$$

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Shintani zeta functions

$V(\mathbb{Z}) = \{au^3 + bu^2v + cuv^2 + dv^3 \mid a, b, c, d \in \mathbb{Z}\}$ binary cubic forms

$$\xi^{\pm}(s) := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}$$

Converges for $\mathrm{Re}(s) > 1$, analytic continuation to $\mathbb{C}$ with simple poles at $s = 1$ and $s = 5/6$. Satisfies a functional equation.

But...

F. Thorne (2014): $\xi^{\pm}(s)$ is not a finite sum of Euler products!

Any ideas?

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Thank you!