Zero-density estimates for L-functions associated to fixed-order Dirichlet characters

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Suppose that χ is a primitive Dirichlet character modulo q, fix $\sigma \in \left(\frac{1}{2}, 1\right)$ and $T \in (2, \infty)$, and consider the rectangle $R(\sigma, T) = [\sigma, 1] + i[-T, T]$.

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$$N(\sigma, T, \chi) = \#\{\varrho \in R(\sigma, T) : L(\varrho, \chi) = 0\},$$

where $L(s, \chi)$ is the L-function associated to the character χ.

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We will consider the families $\mathcal{O}_r$ of primitive Dirichlet characters of order $r \geqslant 2$.

Programme

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Adapting an approach used by Ingham (1937) to estimate the density of zeros of the ζ-function, Montgomery (1971) showed that

$$\sum_{Q<q\leqslant 2Q} \sum_{\chi \bmod q}^{*} N(\sigma, T, \chi) \ll_{\varepsilon} Q^{\frac{6-6\sigma}{2-\sigma}+\varepsilon} T^{\frac{3-3\sigma}{2-\sigma}+\varepsilon}.$$

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$$\sum_{\chi \in \mathcal{O}_2(Q)} N(\sigma, T, \chi) \ll_{\varepsilon} (QT)^{\frac{7-6\sigma}{6-4\sigma}+\varepsilon}.$$

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$$\sum_{\chi \in \mathcal{O}_r(Q)} N(\sigma, T, \chi) \ll_{\varepsilon} (QT)^{2(1-\sigma)+\varepsilon}.$$

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In the literature, generally $\mathfrak{S}_1(Q,T)$ and either $\mathfrak{L}_1(Q,T)$ or $\mathfrak{L}_2(Q,T)$ have been used to derive zero-density estimates.

We consider the polynomials $\Delta(Q, T, N)$ for which the bound

$$\mathfrak{S}_1(Q, T) \ll_{\varepsilon} (QN)^{\varepsilon} \Delta(Q, T, N) \sum_{n \leqslant N}{}^{'} |a_n|^2$$

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Lemma 1

Suppose that $X, Y \geqslant 2$ are such that $X \ll Y \ll (QT)^A$ for some absolute constant A. Then

$$\sum_{\chi \in \mathcal{F}(Q)} N(\sigma, T, \chi) \underset{\varepsilon}{\ll} (QT)^{\varepsilon} \Big(\mathfrak{L}_k(Q, T)^{\frac{1}{k+1}} \Delta(Q, T, X)^{\frac{k}{k+1}} Y^{\frac{k}{k+1}(1-2\sigma)} + \Delta(Q, T, X) X^{1-2\sigma} + \Delta(Q, T, Y) Y^{1-2\sigma} \Big)$$

for any $k \geqslant 1$, where the implied constant does not depend on k.

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Estimating $\mathfrak{S}_1(Q, T)$

Let $(A_h)_{h \leqslant H}$ and $(B_h)_{h \leqslant H}$ be sequences of non-negative reals, and define

$$D(Q, N) = \sum_{h \leqslant H} Q^{A_h} N^{B_h} \quad \text{and} \quad \Delta(Q, T, N) = \sum_{h \leqslant H} Q^{A_h} N^{B_h} T^{1-B_h}.$$

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If $D(Q, N)$ is such that

$$\sum_{\chi \in \mathcal{F}(Q)} \Big| \sum_{n \leqslant N}{}' a_n \chi(n) \Big|^2 \ll_{\varepsilon} (QN)^{\varepsilon} D(Q, N) \sum_{n \leqslant N}{}' |a_n|^2$$

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- (Balestrieri and Rome, 2023) For $\mathcal{F} = \mathcal{O}_r$ where $r \geqslant 2$, we have
$$D(Q, N) \ll \min\left(Q^{2} + N, Q^{\frac{4}{3}} + Q^{\frac{2}{3}} N\right).$$

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Lemma 2

Let $r \geqslant 3$, and suppose that $T^{2r-1} \gg Q^{2r-5}$. Then

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- (C., 2023) For $\mathcal{O}_r$ and $T \gg Q$ we have $\mathfrak{L}_2(Q, T) \ll_\varepsilon Q^{1+\varepsilon} T^{2+\varepsilon}$.

Main results

The following result improves on the aforementioned estimate of Jutila for all $Q, T \geqslant 2$, and the estimate of C. and Zhao whenever $T^{4-4\sigma} \gg Q^{2\sigma-1}$.

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The following result pertaining to $\mathcal{O}_r$ with $r \geqslant 3$ is valid only when $T^{2r-1} \gg Q^{2r-5}$.

Theorem 2

$$\sum_{\chi \in \mathcal{O}_r(Q)} N(\sigma, T, \chi) \ll_{\varepsilon} Q^{\min\left(\frac{6-4\sigma}{3}, \frac{6-6\sigma}{3-2\sigma}\right)+\varepsilon} T^{\frac{4-4\sigma}{3-2\sigma}+\varepsilon}.$$

Concluding Remarks

Lemma 1 is stronger for k than it is for $k-1$ if a sharp bound is known for $\mathfrak{L}_k(Q, T)$. However, for arbitrarily large k, there are no sharp bounds known on $\mathfrak{L}_k(Q, T)$.

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For $k \geqslant 2$, we can show using the same method as in Lemma 2 that

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However, we can get a better estimate simply appealing to the Weyl-bound $L\left(\frac{1}{2}+it,\chi\right) \ll_\varepsilon q^{\frac{1}{6}+\varepsilon}(|t|+1)^{\frac{1}{6}+\varepsilon}$ due to Petrow and Young (2023).

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We derive the following by taking k to be sufficiently large in Lemma 1.

Proposition 1

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Thank you for your attention