Are magnetic field-lines always identifiable with $1\frac{1}{2}$D Hamiltonian systems ?

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Why does Hamiltonian structure matter ?

Hamiltonian allows

- "nice" magnetic representation, used as starting point in many optimisation exercises

$$\boldsymbol{B} = \nabla\psi_t \times \nabla\theta + \nabla\varphi \times \nabla\psi_p(\psi_t, \theta, \varphi)$$

- Noether theorem to be invoked for integrability of field-lines

$$\text{(hidden) symmetry} \rightleftharpoons \text{flux-surfaces}$$

  in $\boldsymbol{B}$ above $\psi_p = \psi_p(\psi_t)$ so that $\boldsymbol{B} \cdot \nabla\psi_t = 0$

- methods of perturbation theory, KAM theorem to interpret "near"-integrability, magnetic islands, chaotic field-lines

$1\frac{1}{2}$D Hamiltonian systems $\hookrightarrow$ magnetic fields

on toroidal domain

Consider canonical coordinates $(q, p) \in \Sigma \subset \mathbb{R}^2$ and time-dependent Hamiltonian $H(t, q, p)$ with $t \in S^1$. Hamilton's equations read

$$\frac{dq}{dt}(t) = \frac{\partial H}{\partial p}(t, q(t), p(t))$$
$$\frac{dp}{dt}(t) = -\frac{\partial H}{\partial q}(t, q(t), p(t))$$

An *embedding* $F : S^1 \times \Sigma \hookrightarrow \mathbb{R}^3$, $(t, q, p) \mapsto (x, y, z)$ yields a magnetic field

$$\boldsymbol{B} = \nabla p \times \nabla q + \nabla t \times \nabla H$$

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$$\begin{aligned} \frac{dq}{dt}(t) &= \quad \frac{\partial H}{\partial p}(t,q(t),p(t)) = \frac{\boldsymbol{B}\cdot\nabla q}{\boldsymbol{B}\cdot\nabla t} \\ \frac{dp}{dt}(t) &= -\frac{\partial H}{\partial q}(t,q(t),p(t)) = \frac{\boldsymbol{B}\cdot\nabla p}{\boldsymbol{B}\cdot\nabla t} \end{aligned}$$

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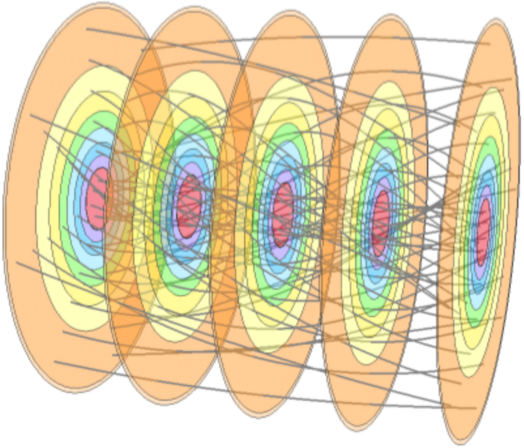
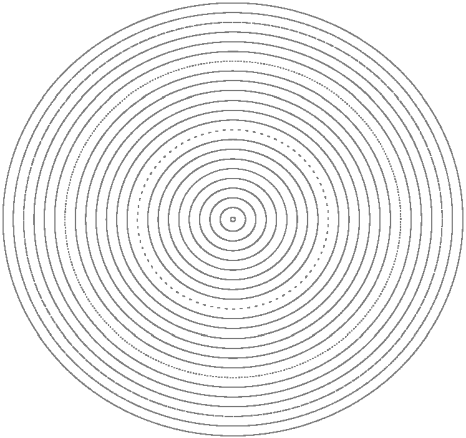
Example of Hamiltonian dynamics

$H(t, q, p) = p - \frac{1}{2}p^2 + p\epsilon_1 \cos(3q - 2t) + p\epsilon_2 \cos(4q - t)$.

When $\epsilon_{1,2} = 0$, integrable system $\rightleftharpoons$ flux-surfaces with *rotational transform* $\iota(p) = 1 - p$



Dynamics on $S^1 \times D^2$, $\epsilon_1 = 0$, $\epsilon = 0$



Poincaré section $t = 2n\pi$, $\epsilon_1 = 0$, $\epsilon_2 = 0$

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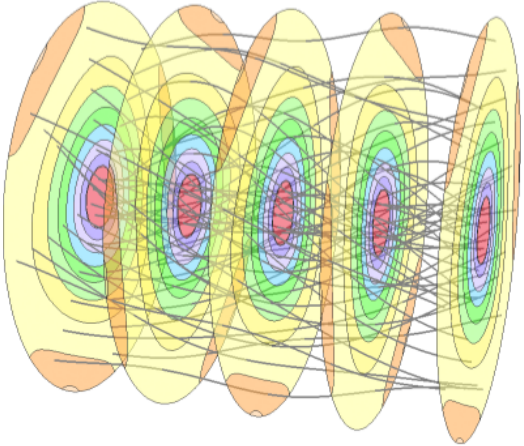
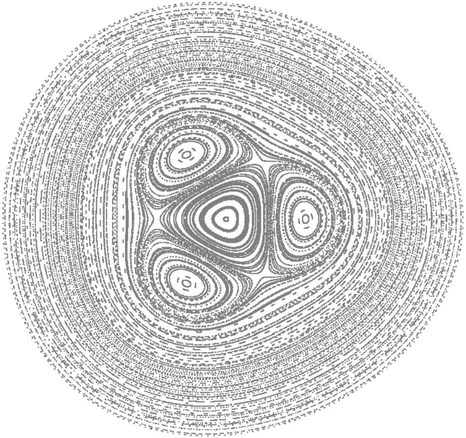


Dynamics on $S^1 \times D^2$, $\epsilon_1 = 0.02$, $\epsilon_2 = 0$



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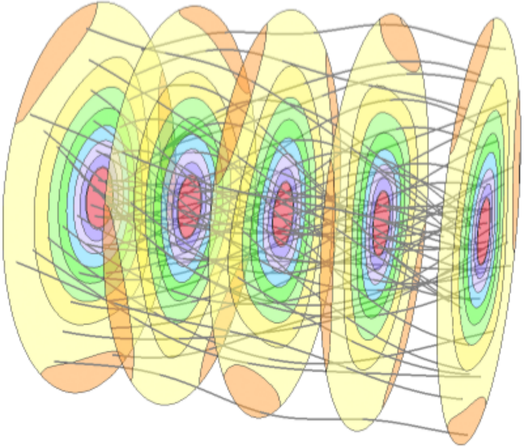
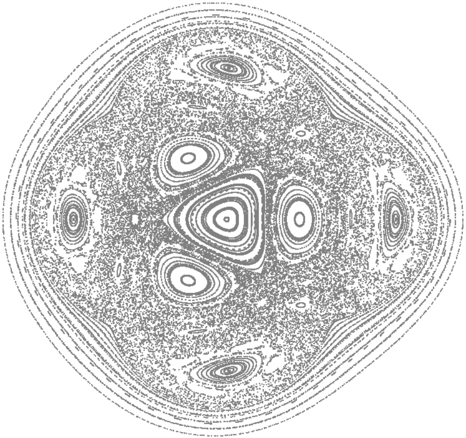


Dynamics on $S^1 \times D^2$, $\epsilon_1 = 0.02$, $\epsilon = 0.01$



Poincaré section $t = 2n\pi$, $\epsilon_1 = 0.02$, $\epsilon = 0.01$

Observations

- Field-lines in $\mathbb{R}^3$ are images of Hamiltonian dynamics on $S^1 \times \Sigma$ through F
- time is forward $\Rightarrow \boldsymbol{B} \cdot \nabla t > 0$
- $\boldsymbol{B} \cdot \nabla t = (\nabla p \times \nabla q) \cdot \nabla t = -1/J_F$ where J_F is Jacobian of embedding F

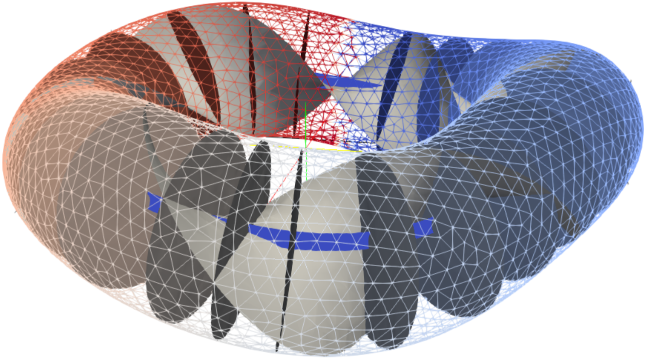
Necessary conditions for B being Hamiltonian

but not sufficient

If arbitrary magnetic field $\boldsymbol{B}$ on domain $\Omega \subset \mathbb{R}^3$ originated from a $1\frac{1}{2}D$ Hamiltonian system, then

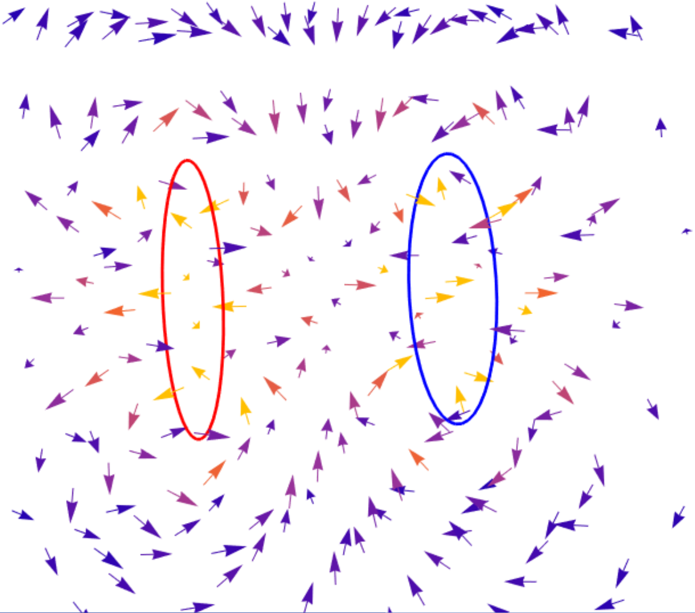
- necessary for $\boldsymbol{B}$ to have Poincaré sections, namely there exists

$$t : \Omega \to S^1, \text{ such that } \boldsymbol{B} \cdot \nabla t > 0$$



- necessary for $\boldsymbol{B} \neq 0$ (nowhere vanishing)

Counter-examples



- two loops with opposite (unit) currents
- $\boldsymbol{B}(\mathbf{0}) = \mathbf{0}$
- no (global) Hamiltonian formulation

Topology matters: more counter-examples

Theorem (Poincaré-Hopf*)

$\boldsymbol{B} \neq 0$ *on* $\Omega \Rightarrow$ *the Euler characteristic* $\chi(\Omega) = 0$.

- χ(surface) $=$ vertices - edges $+$ faces
- χ(ball) $= 1$
- χ(circle) $= 0$
- $\chi(M \times N) = \chi(M)\chi(N)$
- $\chi(M \cup N) \approx \chi(M) + \chi(N) - \chi(M \cap N)$

* $\boldsymbol{B} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$ but $\boldsymbol{B} \times \boldsymbol{n}|_{\partial\Omega} \neq 0$

Topology matters: more counter-examples

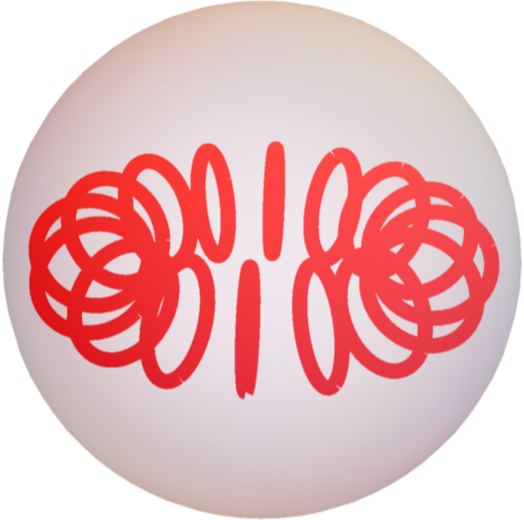
ball \ 1 coil

- removing loops (coils) from a ball still makes $\chi = 1$

$$\chi(B) = \chi(B \setminus S^1) + \cancel{\chi(S^1)} - \cancel{\chi(T^2)}$$

Topology matters: more counter-examples

ball $\setminus$ 16 TF coils



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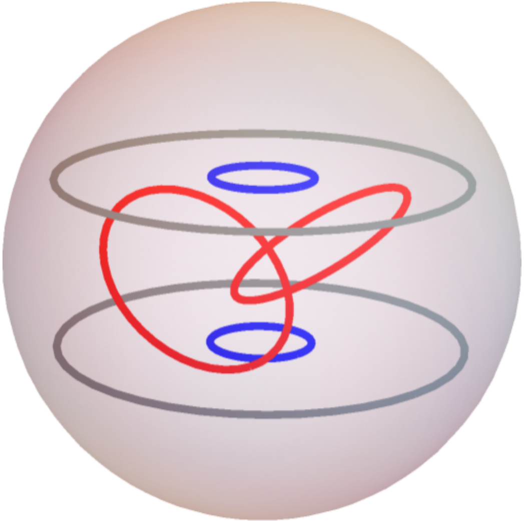
- technically, none of the fields in magnetic confinement are (globally) Hamiltonian
- toroidal domains have $\chi(S^1 \times \Sigma) = 0$

Topology matters: more counter-examples

ball $\setminus$ CNT coils [Pedersen and Boozer, 2002]



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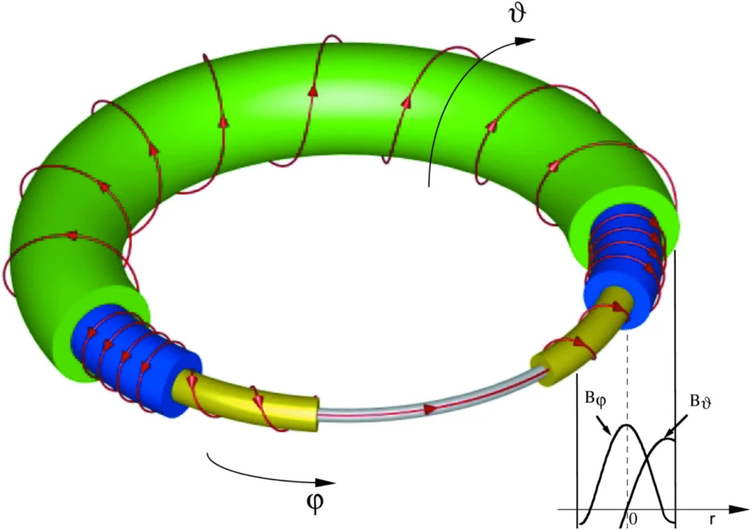
- technically, none of the fields in magnetic confinement are (globally) Hamiltonian
- toroidal domains have $\chi(S^1 \times \Sigma) = 0$
- not obvious that toroidal surface on which $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ exists

Integrable is not the same as Hamiltonian+symmetry



Reversed-Field Pinch (RFP)

- can display flux-surfaces (and straight-field line coordinates)
- toroidal field reverses

$$\boldsymbol{B}\cdot\nabla\varphi > 0, \quad r < r_c$$
$$\boldsymbol{B}\cdot\nabla\varphi < 0, \quad r > r_c$$

$\Rightarrow$ toroidal angle φ cannot be used as time

- Beltrami states $\nabla\times\boldsymbol{B} = \mu\boldsymbol{B}$ feature reversals

Toroidal domains and fields with Poincaré sections

In order to identify $\boldsymbol{B}$ with $1\frac{1}{2}$D Hamiltonian system, we work with conditions

1. toroidal domain (trivial bundle over the circle, planar section)

$$\Omega \cong S^1 \times \Sigma, \qquad \Sigma \subset \mathbb{R}^2, \qquad t : \Omega \to S^1$$

2. tangential fields on the boundary $\boldsymbol{B} \cdot \boldsymbol{n}\big|_{\partial\Omega} = 0$
3. fields with Poincaré section

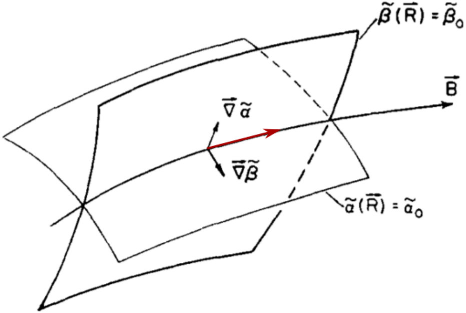
$$dt(\boldsymbol{B}) > 0$$

Local vs Global

Theorem (Darboux, variant)

On a neighbourhood U of any point $x \in \Omega$ where $\boldsymbol{B}(x) \neq 0$, there exists coordinates such that

$$\boldsymbol{B}|_U = \nabla\alpha \times \nabla\beta$$



[D'Haeseleer et al., 1991]

- topological obstructions to realising Clebsch representation globally
- flow box theorem: *every field-line can be locally straightened*
- usual procedure to identify $1\frac{1}{2}$D Hamiltonian system is local [Cary and Littlejohn, 1983; Boozer, 1983; Meiss, 1992; Helander, 2014]

Recalling (1) $\Omega \cong S^1 \times \Sigma$, $\Sigma \subset \mathbb{R}^2$; (2) $\boldsymbol{B} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$; (3) $dt(\boldsymbol{B}) > 0$.

Theorem (Duignan, Perrella and P.)

Assuming (1)-(3), there exists a (global) diffeomorphism $\Psi : \Omega \to S^1 \times \Sigma$ with $t = t \circ \Psi$ and a (global) Hamiltonian function $\mathcal{H}$ with $\mathcal{H}|_{\partial\Omega} =$ const such that

$$\boldsymbol{B} = \boldsymbol{B}_T + \nabla t \times \nabla \mathcal{H}$$

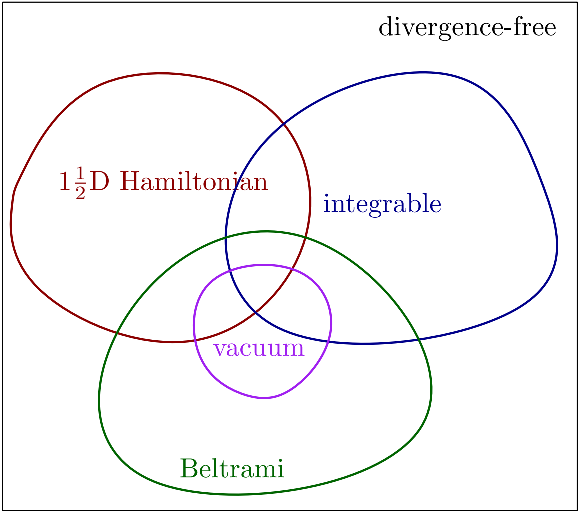
with $\nabla \cdot \boldsymbol{B}_T = 0$, $\boldsymbol{B}_T \propto \partial_t$.

In fact, on a solid torus ST $\cong S^1 \times D^2$ and on hollow torus HT $\cong S^1 \times S^1 \times [0,1]$, *toroidal coordinates* (ψ, θ, φ) allows to write

$$\boldsymbol{B} = \nabla\psi \times \nabla\theta + \nabla\varphi \times \nabla\mathcal{H}$$

On domains $\Omega \cong S^1 \times \Sigma$ where $\Sigma = D^2 \setminus \{p_1, \dots, p_n\}$ is a disk with $n \geq 2$ holes, there are no (global) *toroidal coordinates*, but $\boldsymbol{B}_T$ plays the role of a symplectic form on levels of t.

Summary



- not all magnetic fields are Hamiltonian
- which fields are best confining ?

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Example where flux coordinates cannot exist globally

despite presence of nested flux-surfaces (no islands)

In cylindrical coordinates (R, φ, Z), consider

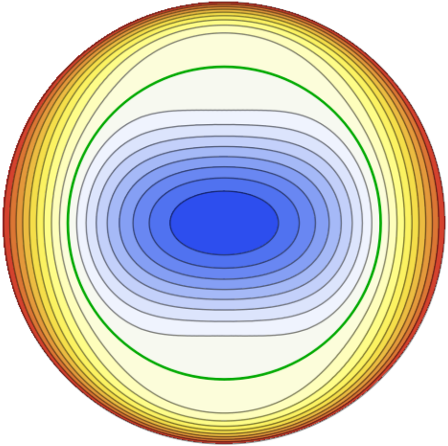
$$B = \nabla\psi \times \nabla\varphi + f\nabla\varphi$$

where

$$\psi = (\tilde{R}^2 + Z^2 - 1)(1 + Z^2(\tilde{R}^2 + Z^2 - 2))$$
$$f = Z + (\tilde{R}^2 + Z^2 - 1)$$

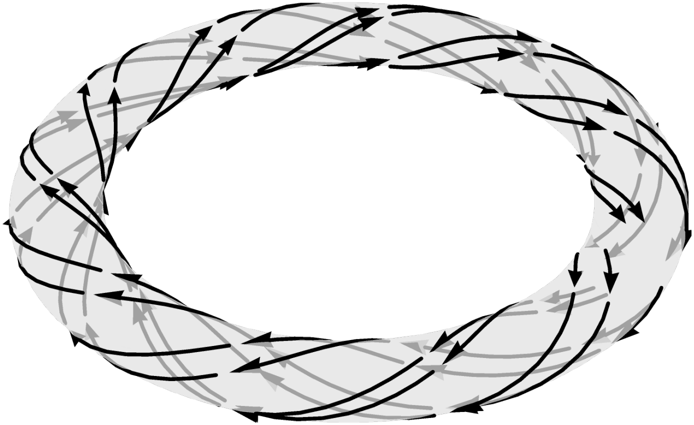
and $\tilde{R} = R - 2$.

1. $\psi =$ const are nested toroidal surfaces ✓
2. $|B| > 0$ ✓



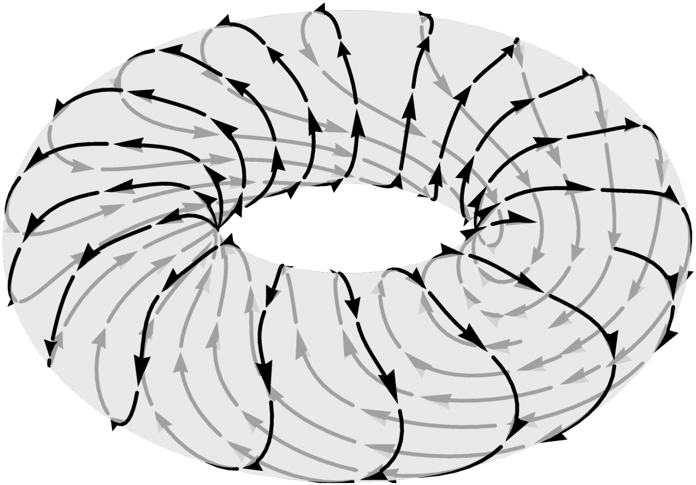
Poloidal cross-section of ψ levels.
Critical surface (green) at $\psi = 0$ where $\tilde{R}^2 + Z^2 = 1$.

Critical surface contains “Reeb component”



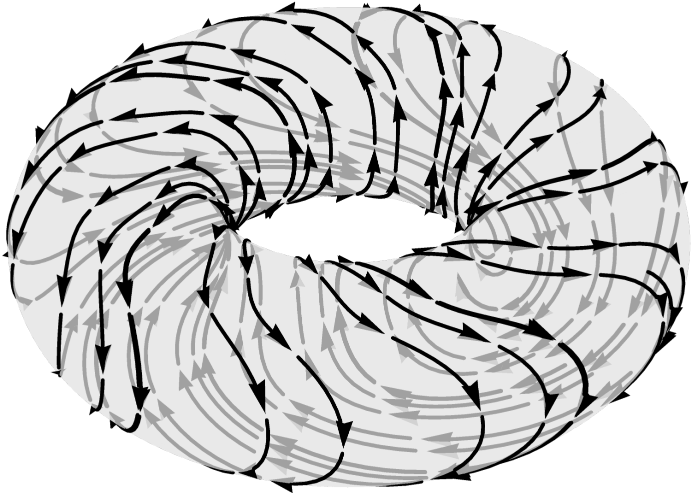
$\psi = -0.9$

Critical surface contains "Reeb component"



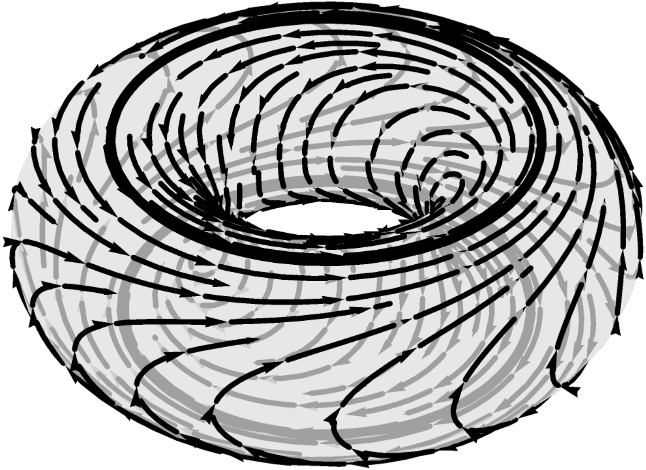
$\psi = -0.2$

Critical surface contains "Reeb component"



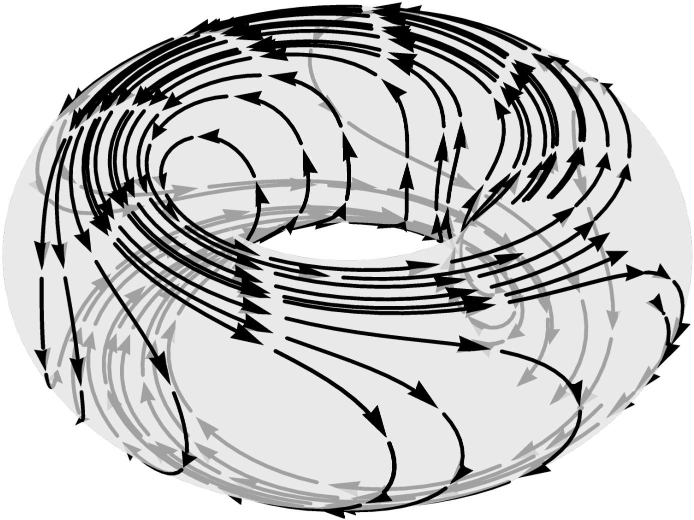
$\psi = -0.1$

Critical surface contains "Reeb component"



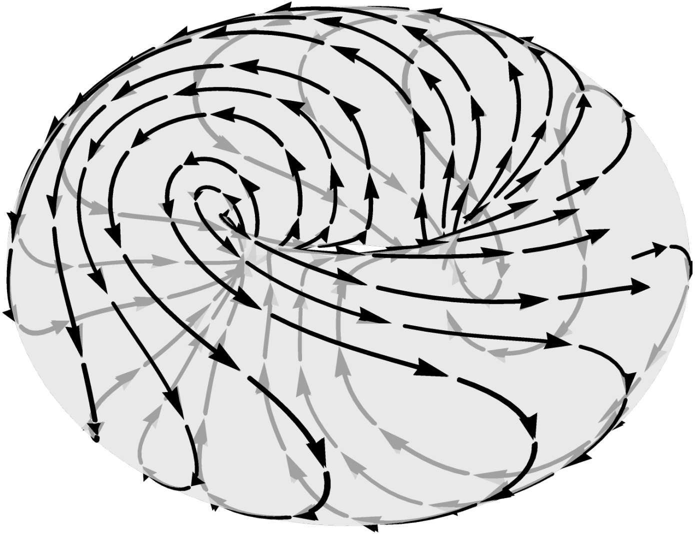
$\psi = 0$. The toroidal surface $\tilde{R}^2 + Z^2 = 1$ does not have straight field-line coordinates!

Critical surface contains “Reeb component”



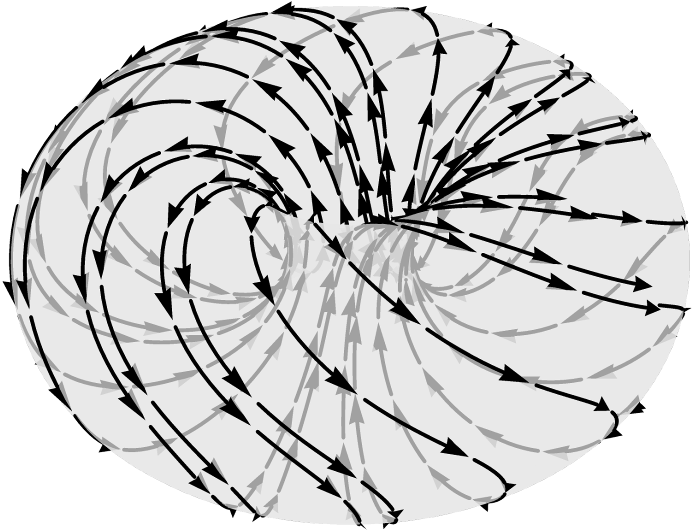
$\psi = 0.01$

Critical surface contains "Reeb component"



$\psi = 0.1$

Critical surface contains "Reeb component"



$\psi = 0.9$