A Discontinuous Galerkin approach to MRxMHD

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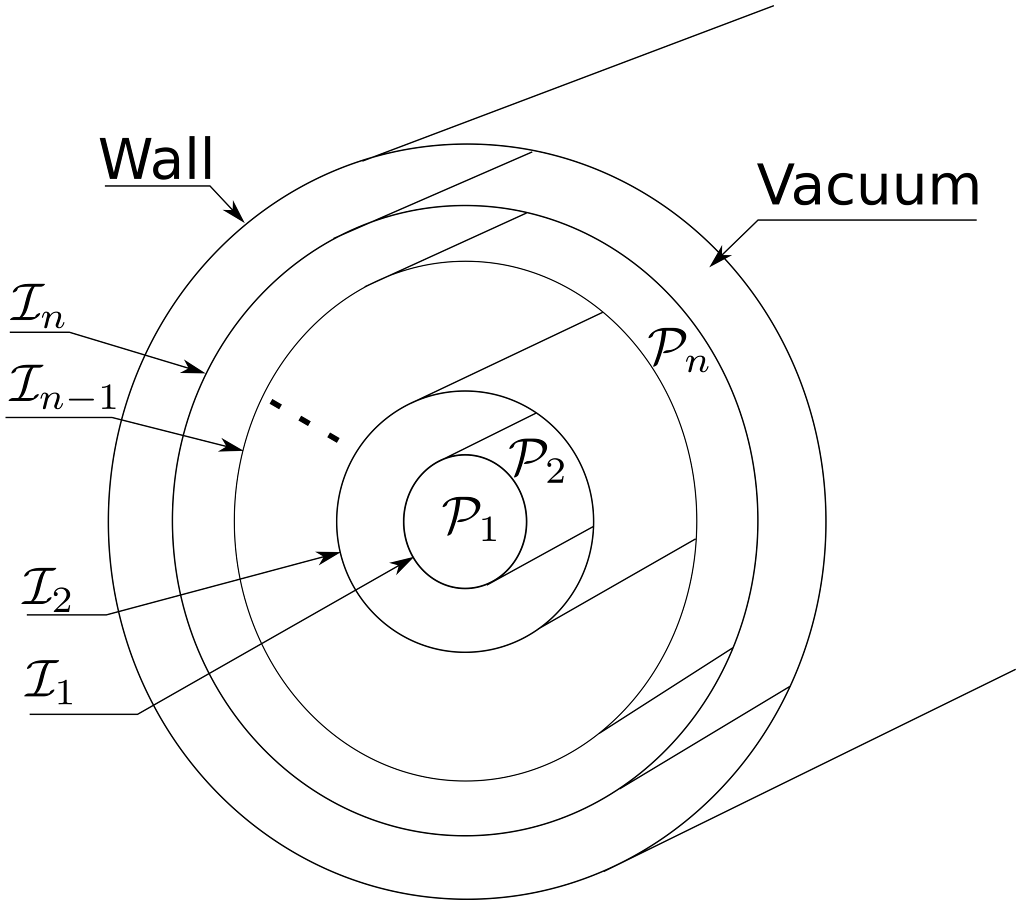
Multiple Region relaXed MHD

$$\left.\begin{aligned}\nabla\times\mathbf{B}&=\mu\mathbf{B}\\ \nabla\cdot\mathbf{B}&=0\end{aligned}\right\}\text{ in }\mathcal{P}_i$$

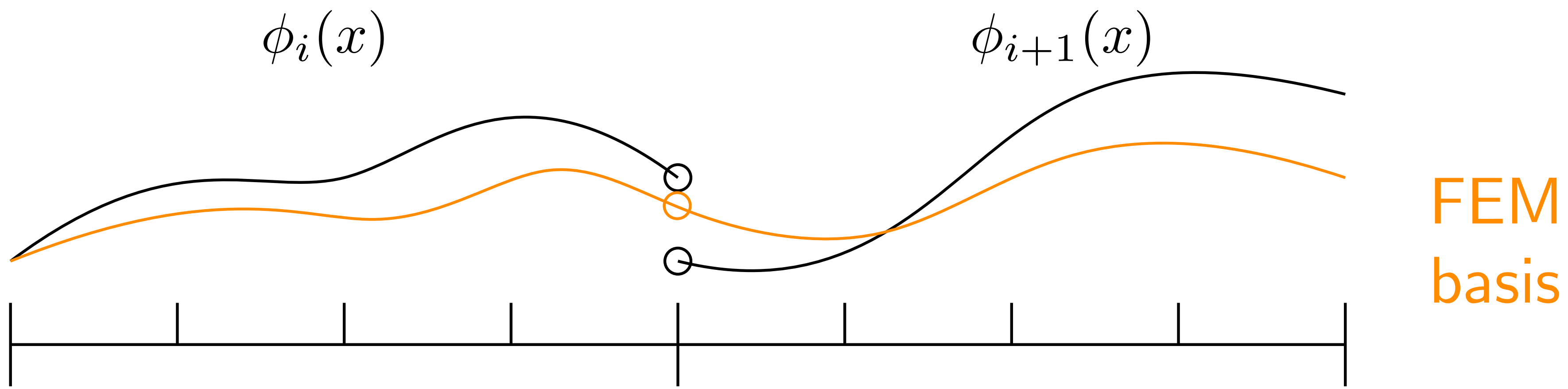
$$\left.\begin{aligned}\mathbf{B}\cdot\mathbf{n}&=0\\ \left[p+\frac{1}{2}\|\mathbf{B}\|^2\right]&=0\end{aligned}\right\}\text{ on }\mathcal{I}_i$$

MRxMHD/SPEC diagram



Discontinuous Galerkin - Spectral Element Method

- FEM-like method but need to capture discontinuities
- Discontinuous Galerkin can capture jumps
- Challenge of nonlinear discontinuities on the interfaces
- Solution represented by peicewise polynomial basis functions
- Elements connected by fluxes
- Addition of penalties for provably stable schemes



$$F(x_{i+1}^{+}) - F(x_i^{-}) \neq 0$$

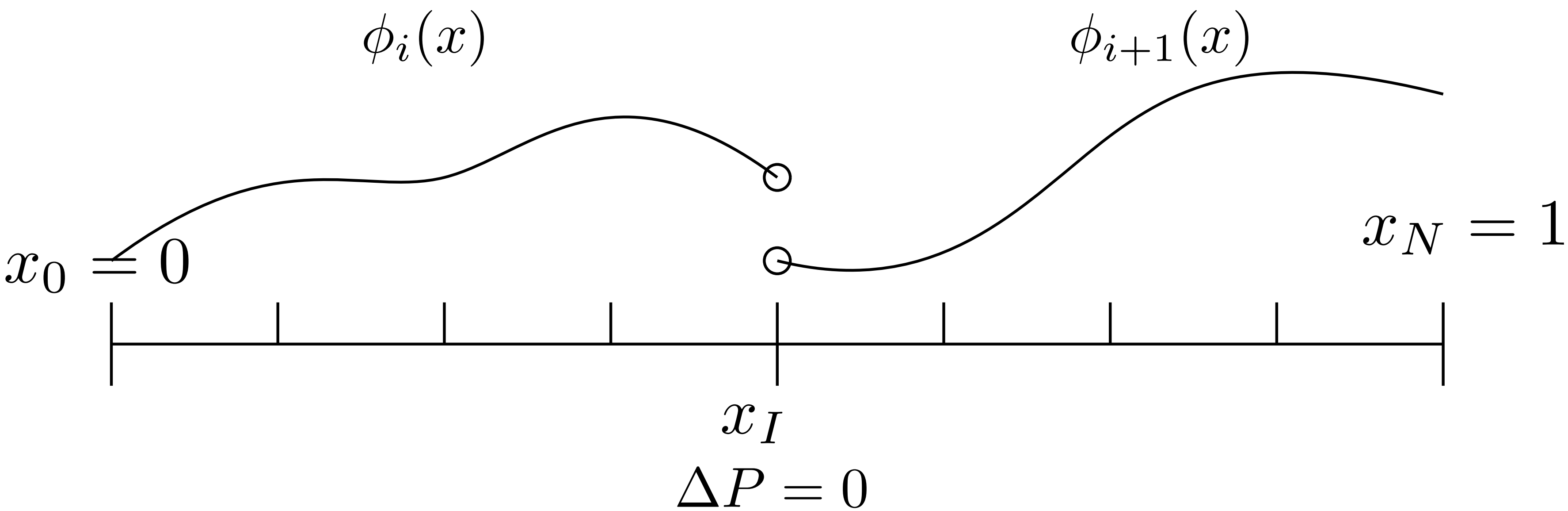
1D model problem

Helmholtz equation $$\Delta B(x) + \mu B(x) = f(x) \quad x \in (0,1)$$

Dirichlet boundaries $$B(x)\big|_{x_0} = g_0 \quad B(x)\big|_{x_N} = g_N$$

Pressure jump $$\left[p + \frac{1}{2}\|B\|^2\right] = 0 \quad \text{on} \quad x \in \mathcal{I} = \{x_I\}$$

Consider 1D domain with two elements with interface



1D model problem: Discretisation

Multiply Helmholtz by test function element-wise and integrate.

$$\sum_{i=1}^{2}\int_{\mathcal{P}_i} B_x(x)v_x(x)\mathsf{d}x - [B_x(x_I)v(x_I)] + B_x(1)v(1) - B_x(0)v(0) -$$
$$\sum_{i=1}^{2}\mu_i\int_{\mathcal{P}_i} B(x)v(x)\mathsf{d}x = \sum_{i=1}^{2}\int_{\mathcal{P}_i} f(x)v(x)\mathsf{d}x$$

$[\cdot]$ denotes difference: $[B(x_I)] = B(x_I^+) - B(x_I^-)$

1D model problem: Discretisation

Let $\{B(x_I)\} = \frac{B(x_I^+)-B(x_I^-)}{2}$, subscript x is derivative,

Add some terms

$$\sum_{i=1}^{2}\int_{\mathcal{P}_i} B_x(x)v_x(x)\mathrm{d}x - \{B_x(x_I)\}[v(x_I)] - \sum_{i=1}^{2}\mu_i\int_{\mathcal{P}_i} B(x)v(x)\mathrm{d}x$$

$$- \{v_x(x)\}[B(x)] + J_0 + J_1 = \sum_{i=1}^{2}\int_{\mathcal{P}_i} f(x)v(x) - \{v_x(x)\}[B(x)] + J_0$$

"symmetriser" penalties

$$J_0 = \frac{\sigma_0}{h_{n,n}}[B(x_I)][v(x_I)] \qquad J_1 = \frac{\sigma_1}{h_{n,n}}[B'(x_I)][v'(x_I)]$$

σ_0, $\sigma_1 \in \mathbb{R}$ penalty parameters chosen for stability

1D model problem: Discretisation

Rewrite the pressure jump condition:

$$\left[p + \frac{1}{2}|B|^2 \right] = 0 \quad \text{as} \quad [p] + (B^+)^2 - (B^-)^2 = 0$$

Provided $[[p]] \neq 0$, then $\{B\} \neq 0$, want to enforce,

$$[B] + 2\frac{[p]}{\{B\}} = 0$$

Rewrite the penalty:

$$J_0 = \frac{\sigma_0}{h_{n,n}} \left([B(x)] + 2\frac{[P]}{\{B(x_I)\}} \right) [v(x)]$$

1D model problem

Stage 1: Solve uncoupled problem (continuous solution)

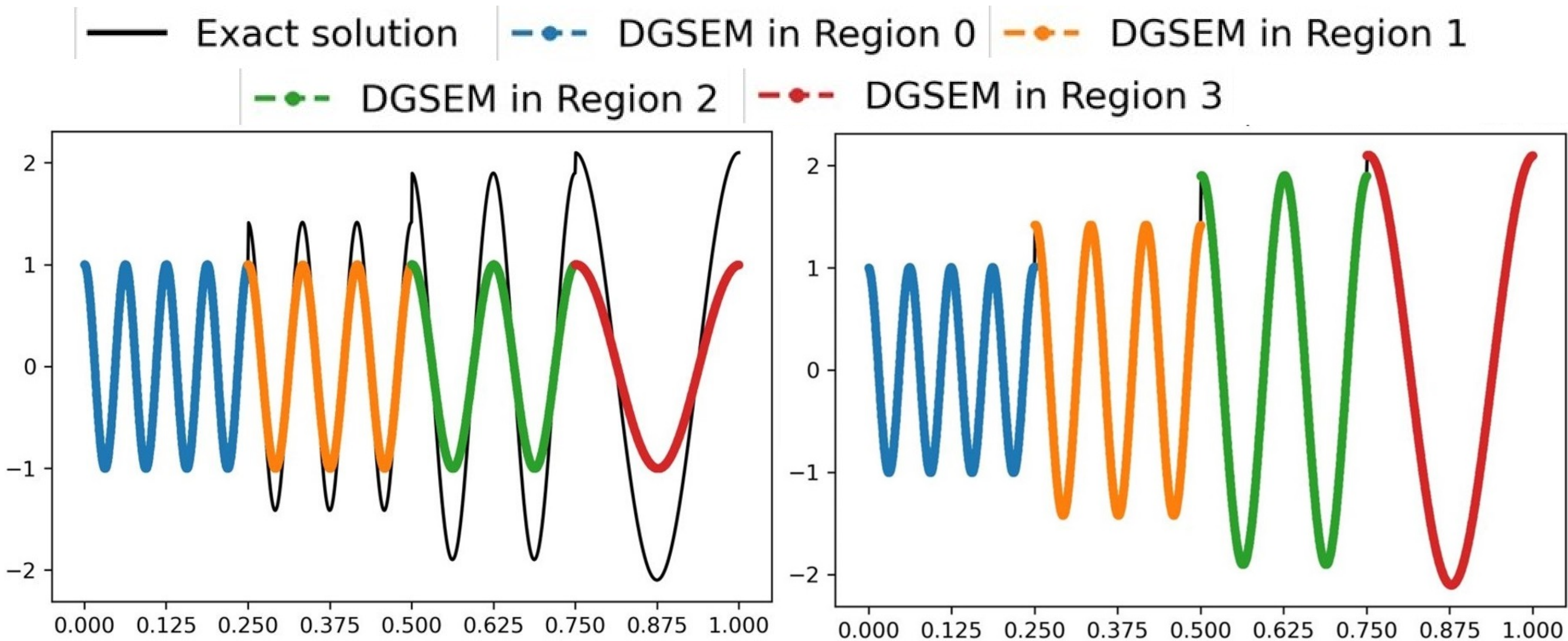
- $B^\star$ continuous solution (no pressure jumps)
- $\nabla \cdot B^\star \equiv \partial_x B^\star = 0$ on interfaces

Stage 2: Solve coupled problem

- Initial guess $B_0 = B_{uc}$
- Solution satisfies full PDE system w/ pressure jumps

1D model problem

manufactured solution $p = 5$ and 1024 elements.



w/o Δp correction | w/ Δp correction

Stage 1 | Stage 2