Decorated Newton polygons and cluster reductions

Andrei Marshakov
Dept. Math. HSE and Theory Dept. LPI

September, 2025

Baxter2025 Exactly Solved Models and Beyond:
Celebrating the Life and Achievements of Rodney James Baxter

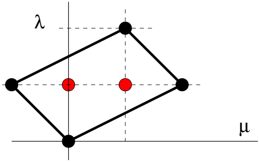
Based on

joint works with M. Bershtein, P. Gavrylenko and M. Semenyakin,
natural development of my talk *Cluster integrable systems and supersymmetric gauge theories* at *Baxter2020: Frontiers in Integrability*.

- Cluster Reductions, Mutations, and q-Painlevé Equations, with MB-PG-MS, arXiv:2411.00325
- preceeding
  - Cluster integrable systems, q-Painleve equations and their quantization, with MB-PG, JHEP 1802:077, 2018, arXiv:1711.02063;
  - Cluster Toda chains and Nekrasov functions, with MB-PG, L. Faddeev's volume in TMPh, arXiv:1804.10145;
  - Cluster integrable systems and spin chains, with MS, JHEP 2019, 100 (2019), arXiv:1905.09921

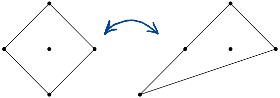
Newton polygons

- A convex hull of integer points in $N \subset \mathbb{Z}^2 \subset \mathbb{R}^2$
- defines a plane curve $\Sigma \subset \mathbb{C}^\times \times \mathbb{C}^\times$



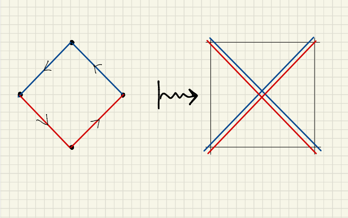
endowed with $\varpi = \frac{d\lambda}{\lambda} \wedge \frac{d\mu}{\mu}$

- Defines an integrable system modulo
  - $SA(2,\mathbb{Z}) = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ action;
  - Less obvious equivalence: (example of) polygon mutation

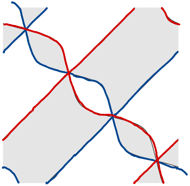


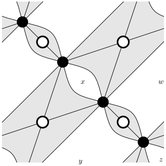
NP & GK system

- From NP



to Thurston diagram

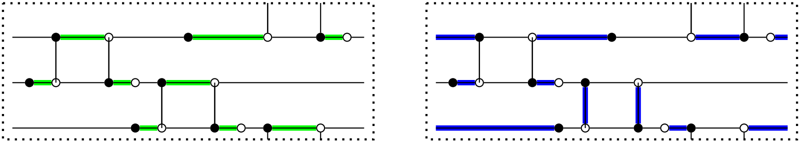




and bipartite graph $\Gamma \subset \mathbb{T}^2$.

Goncharov-Kenyon construction

- Γ is (consistent) bipartite (oriented edges!) graph on $\mathbb{T}^2$;
- *Dimer*: cover $D \subset E(\Gamma)$ and model $\mathrm{wt}\colon E(\Gamma) \to \mathbb{C}^*$;
- Partition function: $\mathcal{Z} = \sum_D \pm \mathrm{wt}(D)$



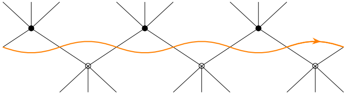
- Dimers $\mapsto$ loops: $\partial D = \sum \bullet - \sum \circ$

$$D - D_0 = \partial f + \gamma \quad (\in H_1(\mathbb{T}^2))$$

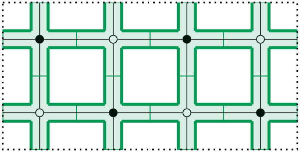
- Edge weights $\{\mathrm{wt}(e)\} \mapsto \{x_f = \mathrm{wt}(f) | q = \prod x_f = 1; (\lambda, \mu) \in H^1(\mathbb{T}^2)\}$
- Spectral curve equation: $\mathcal{Z}(\mathrm{x}|\lambda, \mu) = 0$, $\mathcal{C} \subset \mathbb{C}^\times \times \mathbb{C}^\times$

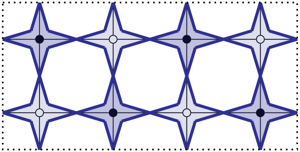
Duality

Zig-zag paths $\{\zeta | \sum_a \zeta_a = 0\}$, $\zeta \neq 0$ in $H_1(\mathbb{T}^2)$



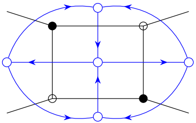
$\Gamma \subset \mathbb{T}^2$ as ribbon graph, *dual* ribbon graph $\Gamma^D \subset \Sigma_D \simeq \mathcal{C}$ (faces $\leftrightarrow$ zig-zags)





Dual cluster structures

- Intersection form $\langle \bullet, \bullet \rangle_{\mathcal{C}}$ on $H_1(\mathcal{C})$: Poisson quiver $\mathcal{Q}$



Poisson bracket in cluster seeds:

$$\{x_i, x_j\} := \{x_i, x_j\}_{\mathcal{Q}} = \epsilon_{ij} x_i x_j, \quad \{x_i\} \in \left(\mathbb{C}^\times\right)^{\dim \mathcal{X}} \tag{1}$$

Mutations of $\mathcal{Q}$: bi-rational maps

$$\mu_j : \quad x_j \to \frac{1}{x_j}, \qquad x_i \to x_i \left(1 + x_j^{\operatorname{sgn}(\epsilon_{ij})}\right)^{\epsilon_{ij}}, i \neq j$$

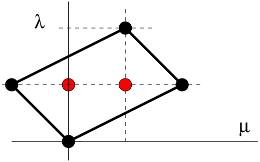
glueing seeds into cluster variety $\mathcal{X}$.

- Intersection form $\langle \bullet, \bullet \rangle_{\mathbb{T}^2}$ on $H_1(\mathbb{T}^2)$: *dual* zig-zag quiver $\mathcal{Q}_D$:
  - $\#$ of arrows $a \to b = \zeta_a \times \zeta_b$
  - $\operatorname{rank} \mathcal{Q}_D = 2$: Darboux variables $\varpi = \frac{d\lambda}{\lambda} \wedge \frac{d\mu}{\mu}$;
  - Mutations of $\mathcal{Q}_D$: rational transform of (λ, μ), preserving ϖ;
  - Strictly defined only for *decorated* polygons.

GK Integrable system

Poisson variety:

- Hamiltonians and Casimirs $\{\{H_k, C_a\} \in \mathrm{Fun}(\mathcal{X}) | \{H_i, H_k\} = 0, \{C_a, \bullet\} = 0\})$;
- Liouville-Arnold: $\dim\mathcal{X} = B - 2 + 2I$, $a = 1, \ldots, B - 2$, $k = 1, \ldots, I$;
- Proof: $V - E + F = 0$ for $\Gamma \subset \mathbb{T}^2$, and $V - E + B = 2 - 2I$ for $\Gamma \subset \mathcal{C}$, hence $F = E - V = B - 2 + 2I$.



Extra: cluster structure

$$\mathcal{X}_N \hookleftarrow \mathrm{F}(q=1) \longleftarrow \mathrm{Jac}(C)$$
$$\downarrow$$
$$\Big\{\text{Coeff. of } \mathcal{C}\Big\}$$

- $N \rightsquigarrow \mathcal{X}_N$ cluster variety.
- $\dim \mathcal{X} = 2\,\mathrm{Area}\, N$, $\quad \dim \mathrm{Jac}\, \mathcal{C} = I$, $\sharp\Big\{\text{Coeff. of } \mathcal{C}\Big\} = I + B - 3$
- $\mathcal{X}_N = \cup_s \mathcal{X}_s$ union of charts (tori)
- $s \rightsquigarrow$ quiver $\mathcal{Q}_s$, $[\mathcal{Q}_s]_\mu \rightsquigarrow \mathcal{X}_N$

$\mathcal{Q}$-mutations and spider moves

- Mutations in 4-valent $\mathcal{Q}$-vertices: spider moves of $\Gamma \subset \mathbb{T}^2$

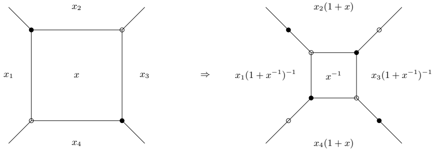


Figure: 4-gon face mutation (spider move)

- Mutations in $2l$-valent vertices with $l \geq 3$: out of the GK class

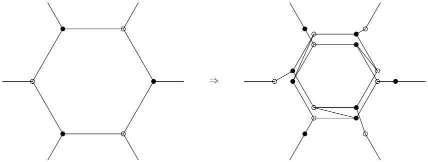
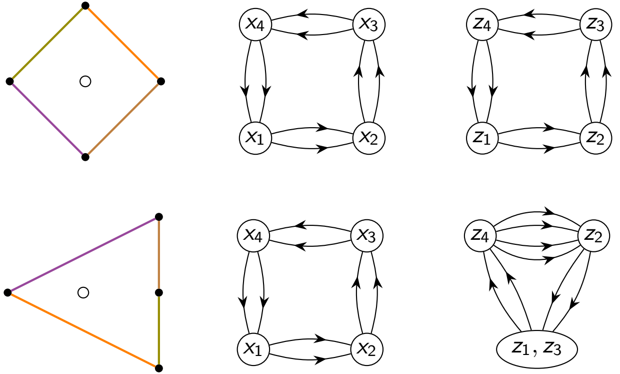


Figure: Mutation in 6-gon face

Dual polygon mutation



Newton polygon — “Face” quiver — “Zigzag” quiver

Mutations in $2l$-valent vertices with $l \geq 3$: out of the class of regular curves.

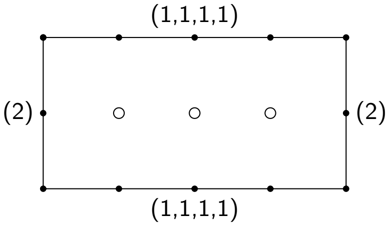
Decorated Newton polygons

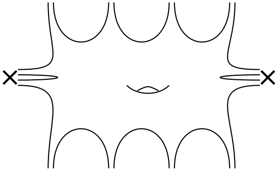
Definition

A *decorated Newton polygon* is a pair (N, H)

- Convex integral polygon N
- Set $\mathrm{H} = (H_E \mid E \in \text{sides of } N)$ of partitions $H_E = \{h_{E,i}\}$ of $|E|_{\mathbb{Z}}$.

Decorations prescribe singularities on $\overline{\mathcal{C}}$ of type $x^{h_{E,i}} = y^{h_{E,i}}$





$$genus(\overline{\mathcal{C}}) = I - \sum_{E,i} h_{E,i}(h_{E,i} - 1)/2 \qquad (2)$$

Polynomial mutations

Decorated Newton polygons $\Leftrightarrow$ Curves with reduction conditions

Multicross singularity with h branches on $\overline{\mathcal{C}}$ $\Leftrightarrow$ exists $\mathrm{SL}_2(\mathbb{Z})$ frame where

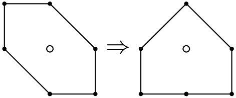
$$P(\lambda,\mu) = \sum_{k=-h'}^{h} \mu^k P_k(\lambda) \qquad \exists c: \quad (1 + c\lambda^{-1})^k \text{ divides } P_k(\lambda),\ \forall k > 0 \tag{3}$$

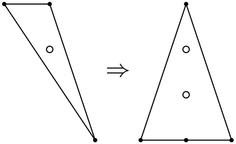
Definition

1. *Mutation of the polynomial P* is polynomial $\tilde{P}$ defined by

$$\tilde{P}(\lambda,\nu) = P(\lambda,\mu), \quad \text{where } \mu = \nu/(1 + c\lambda^{-1})$$

2. *Mutation of the polygon* is a corresponding transformation of N.
3. For decorated polygon (N, H), $k \in H_E$, decoration mutates.





Dual quivers and their mutations

For a (N, H) *dual quiver* $\mathcal{Q}_D$ is

- $\ell(H_E)$ vertices for every side E of N;
- Number of edges between vertices of E and of E' is $\dfrac{\det(E, E')}{|E|_{\mathbb{Z}}|E'|_{\mathbb{Z}}}$,

  (or $\epsilon^D_{aa'} = \zeta_a \times \zeta_{a'}$ between each pair of vertices).

Lemma

Mutations of (N, H) give rise to mutation of $\mathcal{Q}_D$.

Remark

Mutations of $\mathcal{Q}$ preserve N.

Conjecture

"Dual" mutations are isomorphisms of cluster varieties $\mathcal{X}_{N,\mathrm{H}}$ and $\mathcal{X}_{\widetilde{N},\widetilde{\mathrm{H}}}$.
They induce isomorphisms of reduced GK integrable systems.

Main proposal

Conjecture

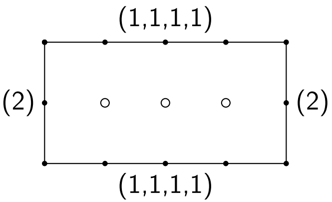
Under the certain conditions on (N, H) there exists integrable system such that

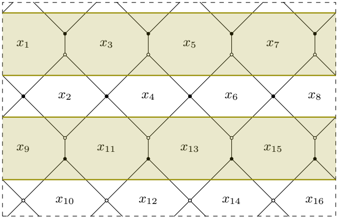
- It is a reduction of GK integrable system corresponding to N
- The phase space is given by $q = 1$ in the $\mathcal{X}$-cluster variety $\mathcal{X}_{N,\mathrm{H}}$ (remark: $q = \prod x = \prod w^{\#}$)
- $\dim \mathcal{X}_{N,\mathrm{H}} = 2\mathrm{Area}(N) - \sum_{E,i}(h_{E,i}^2 - 1)$
- $\mathrm{rk}\{\cdot,\cdot\}_{\mathcal{X}_{N,\mathrm{H}}} = 2I - \sum_{E,i} h_{E,i}(h_{E,i} - 1)$

Conjecture

There exists a seed in which reduction ideal if generated by binomials $m_y^{(I)} + 1$.

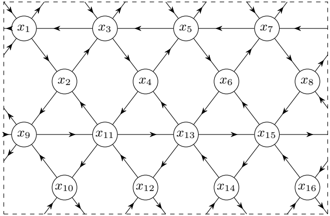
Example $E_7^{(1)}$

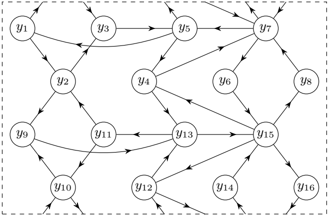




$$C_1 = x_1x_3x_5x_7 = 1, \qquad C_2 = x_9x_{11}x_{13}x_{15} = 1 \tag{4a}$$

$$H_1 = 1 + x_1(1 + x_3(1 + x_5)) = 0, \qquad H_2 = 1 + x_9(1 + x_{11}(1 + x_{13})) = 0 \tag{4b}$$





Example $E_7^{(1)}$

- Reduction conditions

$$y_1 = y_3 = y_9 = y_{11} = -1. \tag{5}$$

- Cluster variables after reduction

$$\begin{gathered} w_1 = y_7, \quad w_2 = y_{10}y_5y_{13}, \quad w_3 = y_2y_5y_{13}, \quad w_4 = y_{14}, \quad w_5 = y_6, \\ w_6 = y_{15}, \quad w_7 = y_{12}, \quad w_8 = y_4, \quad w_9 = y_{16}, \quad w_{10} = y_8. \end{gathered} \tag{6}$$

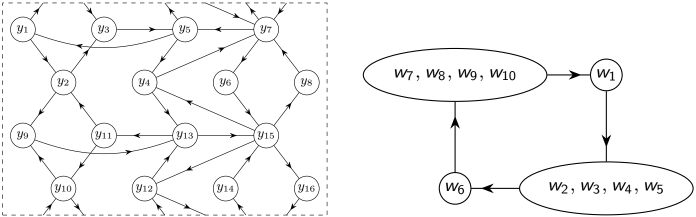


Figure: Quiver after mutation and quiver after reduction

q-Painlevé $E_7^{(1)}$

Results in:

- $$q = \prod_f x_f = w_1^2 w_2 w_3 w_4 w_5 w_6^2 w_7 x_7 w_9 w_{10}$$
- the Hamiltonian:
  $H = \frac{1}{\sqrt{w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9}}(1 + w_1^2 w_2 w_3 w_4 w_5 w_6^3 w_7 w_8 w_9((1 + w_7)(1 + w_9) + w_8(1 + w_7 + (1 + w_7 + w_1 w_7)w_9)) + w_6(1 + w_7 + w_9 + w_7 w_9 + w_1 w_7 w_9 + w_8(1 + w_7 + w_1 w_7 + (1 + w_1 + w_7 + w_1^2(1 + w_2)(1 + w_3)(1 + w_4)(1 + w_5)w_7 + w_1(2 + w_2 + w_3 + w_4 + w_5)w_7)w_9)) + w_1^2 w_6^2 w_7 w_8 w_9((w_2 + w_3)w_4 w_5 + w_2 w_3(w_4 + w_5 + w_4 w_5(2 + w_7 + w_8 + w_9))))$
- Invariant wrt $W(E_7^{(1)})$, generated by

$$\begin{aligned} s_1 &= (2,3), \quad s_2 = (3,4), \quad s_3 = (4,5), \quad s_4 = \mu_5(5,7)\mu_5, \\ s_5 &= (7,8), \quad s_6 = (8,9), \quad s_7 = (9,10), \quad s_0 = \mu_6(1,6)\mu_6. \end{aligned} \tag{7}$$

Outline

- A class of integrable systems on Poisson cluster varieties is defined by convex Newton polygons, they come from GK dimer construction;
- There exists an extension of the class of Goncharov-Kenyon integrable systems by their Hamiltonian reductions;
- Isomorphisms of such reductions are mutations in a “dual” cluster structure;

Remark

All q-Painlevé equations are deautonomizations of reduced GK integrable systems. They correspond to 5d supersymmetric gauge theories, and – when 4d limit exists – are solved by dual Nekrasov partition functions.