

Baxter 2025: Exactly Solved Models and Beyond- Celebrating of Baxter's Life and Achievements

On Baxter's T-Q Relation



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I. History review

Baxter's T-Q relation

$$\mathbf{t}(u) = a(u) \frac{\mathbf{Q}(u - \eta)}{\mathbf{Q}(u)} + d(u) \frac{\mathbf{Q}(u + \eta)}{\mathbf{Q}(u)}$$

$$[\mathbf{t}(u), \mathbf{Q}(v)] = [\mathbf{Q}(u), \mathbf{Q}(v)] = 0$$

$$\mathbf{t}(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle$$

$$\mathbf{Q}(u)|\Psi\rangle = Q(u)|\Psi\rangle$$

$$\Lambda(u) = a(u) \frac{Q(u - \eta)}{Q(u)} + d(u) \frac{Q(u + \eta)}{Q(u)}$$

$$Q(u) = \prod_{j=1}^M f(u - \lambda_j), \quad f(0) = 0$$

Regularity

$$a(\lambda_j)Q(\lambda_j - \eta) + d(\lambda_j)Q(\lambda_j + \eta) = 0$$

II. Some problems

For some models, there is no polynomial Q-solution in terms of homogeneous T-Q solution!

It violates either asymptotic behavior or periodicity

XYZ model with odd number of sites

Spin chains with non-diagonal boundaries

Toda chain

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III. The Inhomogeneous T-Q relation

Two key invariants of the monodromy matrix (spin-1/2):

(1) **Trace** $\mathbf{t}(u)$ (2) **Quantum determinant** $\Delta_q(u)$

$$[t(u), t(v)] = 0$$

Intrinsic relationship between them!

$$\mathbf{t}(\theta_j)\mathbf{t}(\theta_j - \eta) = a(\theta_j)d(\theta_j - \eta) \times id \sim \Delta_q(\theta_j), \quad j = 1, \dots, N$$

$$a(\theta_j - \eta) = d(\theta_j) = 0$$

$$\mathbf{t}(u) = a(u) \frac{Q(u - \eta)}{Q(u)} + d(u) \frac{Q(u + \eta)}{Q(u)} + c(u) \frac{a(u)d(u)}{Q(u)}$$

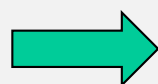
C(u) is nonsingular and matches asymptotic behavior or periodicity

Why?

$$\mathbf{t}(\theta_j) = a(\theta_j) \frac{Q(\theta_j - \eta)}{Q(\theta_j)}$$

$$\mathbf{t}(\theta_j - \eta) = d(\theta_j - \eta) \frac{Q(\theta_j)}{Q(\theta_j - \eta)}$$

Regularity



$$a(\lambda_j)Q(\lambda_j - \eta) + d(\lambda_j)Q(\lambda_j + \eta) + c(\lambda_j)a(\lambda_j)d(\lambda_j) = 0$$

The operator product identities

Consider an R-matrix

$$R_{0,j}(u) = u + \eta P_{0,j} = u + \frac{1}{2}\eta(1 + \sigma_j \cdot \sigma_0)$$

Intrinsic properties:

$$\text{Initial condition : } R_{1,2}(0) = P_{1,2},$$

$$\text{Unitary relation : } R_{1,2}(u)R_{2,1}(-u) = -\varphi(u) \times \text{id},$$

$$\varphi(u) = u^2 - 1,$$

$$\text{Crossing relation : } R_{1,2}(u) = -\sigma_1^y R_{1,2}^t(-u-1)\sigma_1^y,$$

Monodromy matrix:

$$T_0(u) = R_{0,N}(u - \theta_N) \cdots R_{0,1}(u - \theta_1),$$

$$t(u) = \text{tr}_0 T_0(u),$$

YBE

$$[t(u), t(v)] = 0$$

$$\begin{aligned} t(\theta_j) &= \text{tr}_0 \{ R_{0,N}(\theta_j - \theta_N) \cdots R_{0,j+1}(\theta_j - \theta_{j+1}) \\ &\quad \times P_{0,j} R_{0,j-1}(\theta_j - \theta_{j-1}) \cdots R_{0,1}(\theta_j - \theta_1) \} \\ &= R_{j,j-1}(\theta_j - \theta_{j-1}) \cdots R_{j,1}(\theta_j - \theta_1) \\ &\quad \times \text{tr}_0 \{ R_{0,N}(\theta_j - \theta_N) \cdots R_{0,j+1}(\theta_j - \theta_{j+1}) P_{0,j} \} \\ &= R_{j,j-1}(\theta_j - \theta_{j-1}) \cdots R_{j,1}(\theta_j - \theta_1) \\ &\quad \times R_{j,N}(\theta_j - \theta_N) \cdots R_{j,j+1}(\theta_j - \theta_{j+1}). \end{aligned}$$

$$\begin{aligned} t(\theta_j - 1) &= \text{tr}_0 \{ R_{0,N}(\theta_j - \theta_N - 1) \cdots R_{0,1}(\theta_j - \theta_1 - 1) \} \\ &= (-1)^N \text{tr}_0 \{ \sigma_0^y R_{0,N}^{t_0}(-\theta_j + \theta_N) \cdots R_{0,1}^{t_0}(-\theta_j + \theta_1) \sigma_0^y \} \\ &= (-1)^N \text{tr}_0 \{ R_{0,1}(-\theta_j + \theta_1) \cdots R_{0,N}(-\theta_j + \theta_N) \} \\ &= (-1)^N R_{j,j+1}(-\theta_j + \theta_{j+1}) \cdots R_{j,N}(-\theta_j + \theta_N) \\ &\quad \times R_{j,1}(-\theta_j + \theta_1) \cdots R_{j,j-1}(-\theta_j + \theta_{j-1}). \end{aligned}$$

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$$\begin{aligned} t(\theta_j)t(\theta_j - 1) &= a(\theta_j)d(\theta_j - 1), \quad j = 1, \dots, N, \\ a(u) &= \prod_{j=1}^N (u - \theta_j + 1), \quad d(u) = \prod_{j=1}^N (u - \theta_j). \end{aligned}$$

**Homogeneous
limit**

$$\frac{\partial^l}{\partial u^l} \{ t(u)t(u-1) - a(u)d(u-1) \} |_{u=0, \{\theta_j=0\}} = 0, \quad l = 0, \dots, N-1.$$

IV. XXX spin chain with non-diagonal boundaries

Hamiltonian

$$H = \sum_{j=1}^{N-1} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + \frac{1}{p} \sigma_1^z + \frac{1}{q} (\sigma_N^z + \xi \sigma_N^x)$$

The K-matrices

$$K^-(u) = \begin{pmatrix} p+u & 0 \\ 0 & p-u \end{pmatrix}$$

$$K^+(u) = \begin{pmatrix} q+u+\eta & \xi(u+\eta) \\ \xi(u+\eta) & q-u-\eta \end{pmatrix}$$

$$\begin{aligned} & R_{1,2}(u-v) K_1^-(u) R_{2,1}(u+v) K_2^-(v) \\ &= K_2^-(v) R_{1,2}(u+v) K_1^-(u) R_{2,1}(u-v), \\ & R_{1,2}(-u+v) K_1^+(u) R_{2,1}(-u-v-2\eta) K_2^+(v) \\ &= K_2^+(v) R_{1,2}(-u-v-2\eta) K_1^+(u) R_{2,1}(-u+v), \end{aligned}$$

Monodromy matrix

$$\begin{aligned} T_0(u) &= R_{0,N}(u-\theta_N) \cdots R_{0,1}(u-\theta_1), \\ \hat{T}_0(u) &= R_{1,0}(u+\theta_1) \cdots R_{N,0}(u+\theta_N), \end{aligned}$$

$$\mathcal{U}_0(u) = T_0(u) K_0^-(u) \hat{T}_0(u)$$

The transfer matrix

$$t(u) = \text{tr}_0 \{ K_0^+(u) \mathcal{U}_0(u) \}$$

$$[t(u), t(v)] = 0$$

$$H = \left. \frac{\partial \ln t(u)}{\partial u} \right|_{u=0, \{\theta_j=0\}} - N$$

IV. XXX spin chain with non-diagonal boundaries

Operator identities

$$t(\theta_j)t(\theta_j - 1) = a(\theta_j)d(\theta_j - 1), \quad j = 1, \dots, N.$$

$$t(0) = 2pq \prod_{j=1}^N (1 - \theta_j)(1 + \theta_j),$$

$$t(u) \sim 2u^{2N+2} + \dots, \quad \text{for } u \rightarrow \pm\infty.$$

Crossing symmetry

$$t(u) = t(-u - 1)$$

$$a(\lambda_j)Q(\lambda_j - 1) + d(\lambda_j)Q(\lambda_j + 1) = -2[1 - (1 + \xi^2)^{\frac{1}{2}}]\lambda_j(\lambda_j + 1)$$

$$\times \prod_{l=1}^N (\lambda_j + \theta_l)(\lambda_j - \theta_l)(\lambda_j + \theta_l + 1)(\lambda_j - \theta_l + 1), \quad j = 1, \dots, N,$$

Regularity

$$a(u) = \frac{2u+2}{2u+1}(u+p)[(1+\xi^2)^{\frac{1}{2}}u+q] \prod_{j=1}^N (u+\theta_j+1)(u-\theta_j+1),$$

$$d(u) = \frac{2u}{2u+1}(u-p+1)[(1+\xi^2)^{\frac{1}{2}}(u+1)-q] \prod_{j=1}^N (u+\theta_j)(u-\theta_j)$$

Crossing symmetry: $\Lambda(-u-1) = \Lambda(u)$,

Initial condition: $\Lambda(0) = 2pq \prod_{j=1}^N (1 - \theta_j)(1 + \theta_j) = \Lambda(-1)$,

Asymptotic behavior: $\Lambda(u) \sim 2u^{2N+2} + \dots, \quad u \rightarrow \pm\infty$,

$$\Lambda(\theta_j)\Lambda(\theta_j - 1) = \frac{\Delta_q(\theta_j)}{(1 - 2\theta_j)(1 + 2\theta_j)}$$

$$= a(\theta_j)d(\theta_j - 1), \quad j = 1, \dots, N$$

$$\Lambda(u) = a(u)\frac{Q(u-1)}{Q(u)} + d(u)\frac{Q(u+1)}{Q(u)} + 2[1 - (1 + \xi^2)^{\frac{1}{2}}]u(u+1)$$

$$\times \frac{\prod_{j=1}^N (u + \theta_j)(u - \theta_j)(u + \theta_j + 1)(u - \theta_j + 1)}{Q(u)}.$$

$$Q(u) = \prod_{j=1}^N (u - \lambda_j)(u + \lambda_j + 1)$$

Completeness of the solutions

Proposition 1: Each solution of the functional relations of $\Lambda(u)$ can be parameterized in terms of the inhomogeneous $T - Q$ relation

$$Q(u)\Lambda(u) = a(u)Q(u-1) + d(u)Q(u+1) + F(u)$$

with a polynomial $Q(u)$

$$Q(u) = (u^2 + u)^N + \sum_{n=0}^{N-1} I_n (u^2 + u)^n$$

Proof: Given a $\Lambda(u)$, we seek for solutions of Q .

$$Q(\theta_j)\Lambda(\theta_j) = a(\theta_j)Q(\theta_j - 1),$$

$$Q(\theta_j - 1)\Lambda(\theta_j - 1) = d(\theta_j - 1)Q(\theta_j)$$



N linear equations of $\{I_n | n = 0, \dots, N-1\}$

Proposition 2: For unparallel boundary fields ($\xi \neq 0$), the functional relations of $\Lambda(u)$ are the sufficient and necessary conditions to completely characterize the spectrum of the transfer matrix.

$$\Lambda(u) = 2(u^2 + u)^{N+1} + \sum_{n=1}^N c_n (u^2 + u)^n + 2pq \prod_{j=1}^N (1 - \theta_j^2)$$

Proof: The spectrum is simple.
All eigenvalues satisfy the functional relations

of solutions $\geq 2^N$

$\Lambda(\theta_j)\Lambda(\theta_j - 1) = a(\theta_j)d(\theta_j - 1)$ N quadratic eqns.

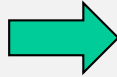
Bezout's theorem # of solutions $\leq 2^N$

With each solution of the functional relations as an eigenvalue, in terms of the inhomogeneous T - Q relation, one can retrieve an eigenstate.

Bethe eigenvectors

Introduce a gauge matrix

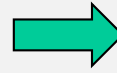
$$G = \begin{pmatrix} \xi & (1 + \xi^2)^{\frac{1}{2}} - 1 \\ \xi & - (1 + \xi^2)^{\frac{1}{2}} - 1 \end{pmatrix}$$



$$\tilde{K}^+(u) = GK^+(u)G^{-1} = \begin{pmatrix} \tilde{K}_{11}^+(u) & 0 \\ 0 & \tilde{K}_{22}^+(u) \end{pmatrix}$$



$$\tilde{U}(u) = GT(u)K^-(u)\hat{T}(u)G^{-1} = \begin{pmatrix} \tilde{A}(u) & \tilde{B}(u) \\ \tilde{C}(u) & \tilde{D}(u) \end{pmatrix}$$



$$t(u) = \tilde{K}_{11}^+(u) \tilde{A}(u) + \tilde{K}_{22}^+(u) \tilde{D}(u)$$

$$\begin{aligned} {}_n\langle 1| &= \xi {}_n\langle \uparrow| + ((1 + \xi^2)^{\frac{1}{2}} - 1) {}_n\langle \downarrow|, \quad n = 1, \dots, N, \\ {}_n\langle 2| &= \xi {}_n\langle \uparrow| - ((1 + \xi^2)^{\frac{1}{2}} + 1) {}_n\langle \downarrow|, \quad n = 1, \dots, N. \end{aligned}$$

$$|\Omega\rangle = \otimes_{j=1}^N |1\rangle_j, \quad \langle \bar{\Omega}| = \otimes_{j=1}^N {}_j\langle 2|.$$

$\tilde{\mathcal{C}}(u)$ eigenstates

$$[\tilde{\mathcal{C}}(u), \tilde{\mathcal{C}}(v)] = [\tilde{\mathcal{B}}(u), \tilde{\mathcal{B}}(v)] = 0.$$

SoV basis

$$\begin{aligned} \tilde{\mathcal{C}}(\theta_j)|\Omega\rangle &= \tilde{\mathcal{C}}(-\theta_j - 1)|\Omega\rangle = 0, \\ \langle \bar{\Omega}|\tilde{\mathcal{C}}(-\theta_j) &= \langle \bar{\Omega}|\tilde{\mathcal{C}}(\theta_j - 1) = 0, \end{aligned}$$

$$\begin{aligned} |\theta_{p_1}, \dots, \theta_{p_n}\rangle &= \tilde{\mathcal{A}}(\theta_{p_1}) \cdots \tilde{\mathcal{A}}(\theta_{p_n})|\Omega\rangle, \\ \langle \theta_{q_1}, \dots, \theta_{q_n}| &= \langle \bar{\Omega}|\tilde{\mathcal{D}}(-\theta_{q_1}) \cdots \tilde{\mathcal{D}}(-\theta_{q_n}), \end{aligned}$$

Bethe eigenvectors

Scalar product

$$F_n(\theta_{p_1}, \dots, \theta_{p_n}) = \langle \Psi | \theta_{p_1}, \dots, \theta_{p_n} \rangle$$

$$\Lambda(\theta_{p_{n+1}}) F_n(\theta_{p_1}, \dots, \theta_{p_n}) = \langle \Psi | t(\theta_{p_{n+1}}) | \theta_{p_1}, \dots, \theta_{p_n} \rangle$$



$$\Lambda(\theta_{p_{n+1}}) F_n(\theta_{p_1}, \dots, \theta_{p_n}) = \frac{(2\theta_{p_{n+1}} + 1) \tilde{K}_{11}^+(\theta_{p_{n+1}}) + \tilde{K}_{22}^+(\theta_{p_{n+1}})}{2\theta_{p_{n+1}} + 1} \times F_{n+1}(\theta_{p_1}, \dots, \theta_{p_{n+1}}).$$

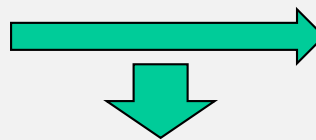


$$F_n(\theta_{p_1}, \dots, \theta_{p_n}) = \prod_{j=1}^n \frac{(2\theta_{p_j} + 1) \Lambda(\theta_{p_j})}{(2\theta_{p_j} + 1) \tilde{K}_{11}^+(\theta_{p_j}) + \tilde{K}_{22}^+(\theta_{p_j})} F_0$$

Bethe eigenvectors: With known eigenvalues and creation operator, we look for the reference state!

$$\langle \lambda_1, \dots, \lambda_N | = \langle 0 | \prod_{j=1}^N \tilde{\mathcal{C}}(\lambda_j)$$

requirement



$$\langle \lambda_1, \dots, \lambda_N | \theta_{p_1}, \dots, \theta_{p_n} \rangle \sim F_n(\theta_1, \dots, \theta_n)$$

$$\langle 0 | = \langle \uparrow | \otimes \dots \otimes \langle \uparrow |$$

V. Concluding remarks & perspective

Intrinsic properties of R-matrix satisfying YBE



$$\mathbf{t}(\theta_j)\mathbf{t}(\theta_j - \eta) = a(\theta_j)d(\theta_j - \eta) \times id \sim \Delta_q(\theta_j), \quad j = 1, \dots, N$$

+

Asymptotic behavior of the polynomial



$$\mathbf{t}(u) = a(u)\frac{Q(u - \eta)}{Q(u)} + d(u)\frac{Q(u + \eta)}{Q(u)} + c(u)\frac{a(u)d(u)}{Q(u)}$$

$$\begin{aligned} \mathbf{t}(u)|\Psi\rangle &= \Lambda(u)|\Psi\rangle \\ Q(u)|\Psi\rangle &= Q(u)|\Psi\rangle \end{aligned}$$



$$\Lambda(u) = a(u)\frac{Q(u - \eta)}{Q(u)} + d(u)\frac{Q(u + \eta)}{Q(u)} + c(u)\frac{a(u)d(u)}{Q(u)}$$



Retrieve
Eigenstates

Regularity



Bethe Ansatz equations

IV. Concluding remarks & perspective

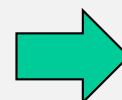
High rank systems: A_n , B_n , C_n , D_n

Fusion



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$$t(\theta_j)t_m(\theta_j - \eta) = t_{m+1}(\theta_j), \quad m = 1, \dots, n-1, \quad j = 1, \dots, N.$$



Nested T-Q

A large, vibrant bouquet of flowers is arranged in a white, woven basket. The bouquet features a variety of flowers including white daisies, yellow roses, purple chrysanthemums, and red carnations. Green foliage and small yellow flowers are interspersed throughout the arrangement. The basket is placed on a light-colored surface, and the background is a plain, light-colored wall.

Baxter forever!

Thanks!