Baxter 2025: Exactly Solved Models and Beyond-
Celebrating of Baxter’s Life and Achievements

On Baxter’s T-Q Relation

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I. History review

Baxter's T-Q relation

$$\mathbf{t}(u) = a(u)\frac{\mathbf{Q}(u-\eta)}{\mathbf{Q}(u)} + d(u)\frac{\mathbf{Q}(u+\eta)}{\mathbf{Q}(u)}$$

$$[\mathbf{t}(u), \mathbf{Q}(v)] = [\mathbf{Q}(u), \mathbf{Q}(v)] = 0$$

$$\mathbf{t}(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle$$

$$\mathbf{Q}(u)|\Psi\rangle = Q(u)|\Psi\rangle$$

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}$$

$$Q(u) = \prod_{j=1}^{M} f(u-\lambda_j), \qquad f(0) = 0$$

Regularity

$$a(\lambda_j)Q(\lambda_j - \eta) + d(\lambda_j)Q(\lambda_j + \eta) = 0$$

II. Some problems

For some models, there is no polynomial Q-solution in terms of homogeneous T-Q solution!

It violates either asymptotic behavior or periodicity

XYZ model with odd number of sites
Spin chains with non-diagonal boundaries
Toda chain
……

III. The Inhomogeneous T-Q relation

Two key invariants of the monodromy matrix (spin-1/2):

(1) Trace $\mathbf{t}(u)$ (2)Quantum determinant $\Delta_q(u)$

$[t(u),\ t(v)] = 0$

Intrinsic relationship between them!

$$\mathbf{t}(\theta_j)\mathbf{t}(\theta_j-\eta) = a(\theta_j)d(\theta_j-\eta) \times id \sim \Delta_q(\theta_j), \quad j=1,\cdots,N$$

$$a(\theta_j-\eta) = d(\theta_j) = 0$$

$$\mathbf{t}(u) = a(u)\frac{\mathbf{Q}(u-\eta)}{\mathbf{Q}(u)} + d(u)\frac{\mathbf{Q}(u+\eta)}{\mathbf{Q}(u)} + c(u)\frac{a(u)d(u)}{\mathbf{Q}(u)}$$

C(u) is nonsingular and matches asymptotic behavior or periodicity

Why?

$$\mathbf{t}(\theta_j) = a(\theta_j)\frac{\mathbf{Q}(\theta_j-\eta)}{\mathbf{Q}(\theta_j)} \qquad \mathbf{t}(\theta_j-\eta) = d(\theta_j-\eta)\frac{\mathbf{Q}(\theta_j)}{\mathbf{Q}(\theta_j-\eta)}$$

Regularity ⇒

$$a(\lambda_j)Q(\lambda_j-\eta) + d(\lambda_j)Q(\lambda_j+\eta) + c(\lambda_j)a(\lambda_j)d(\lambda_j) = 0$$

The operator product identities

Consider an R-matrix

$$R_{0,j}(u) = u + \eta P_{0,j} = u + \frac{1}{2}\eta(1 + \sigma_j \cdot \sigma_0)$$

Intrinsic properties:

$$\begin{aligned} &\text{Initial condition}: \ R_{1,2}(0) = P_{1,2}, \\ &\text{Unitary relation}: \ R_{1,2}(u)R_{2,1}(-u) = -\varphi(u) \times \text{id}, \\ &\qquad \varphi(u) = u^2 - 1, \\ &\text{Crossing relation}: \ R_{1,2}(u) = -\sigma_1^y R_{1,2}^{t_1}(-u-1)\sigma_1^y, \end{aligned}$$

Monodromy matrix:

$$\begin{aligned} T_0(u) &= R_{0,N}(u-\theta_N)\cdots R_{0,1}(u-\theta_1), \\ t(u) &= tr_0 T_0(u), \end{aligned}$$

YBE

$$[t(u),\ t(v)] = 0$$

$$\begin{aligned} t(\theta_j) &= tr_0\{R_{0,N}(\theta_j-\theta_N)\cdots R_{0,j+1}(\theta_j-\theta_{j+1}) \\ &\qquad \times P_{0,j} R_{0,j-1}(\theta_j-\theta_{j-1})\cdots R_{0,1}(\theta_j-\theta_1)\} \\ &= R_{j,j-1}(\theta_j-\theta_{j-1})\cdots R_{j,1}(\theta_j-\theta_1) \\ &\qquad \times tr_0\{R_{0,N}(\theta_j-\theta_N)\cdots R_{0,j+1}(\theta_j-\theta_{j+1})P_{0,j}\} \\ &= R_{j,j-1}(\theta_j-\theta_{j-1})\cdots R_{j,1}(\theta_j-\theta_1) \\ &\qquad \times R_{j,N}(\theta_j-\theta_N)\cdots R_{j,j+1}(\theta_j-\theta_{j+1}). \end{aligned}$$

$$\begin{aligned} t(\theta_j - 1) &= tr_0\{R_{0,N}(\theta_j-\theta_N-1)\cdots R_{0,1}(\theta_j-\theta_1-1)\} \\ &= (-1)^N\, tr_0\{\sigma_0^y R_{0,N}^{t_0}(-\theta_j+\theta_N)\cdots R_{0,1}^{t_0}(-\theta_j+\theta_1)\sigma_0^y\} \\ &= (-1)^N\, tr_0\{R_{0,1}(-\theta_j+\theta_1)\cdots R_{0,N}(-\theta_j+\theta_N)\} \\ &= (-1)^N R_{j,j+1}(-\theta_j+\theta_{j+1})\cdots R_{j,N}(-\theta_j+\theta_N) \\ &\qquad \times R_{j,1}(-\theta_j+\theta_1)\cdots R_{j,j-1}(-\theta_j+\theta_{j-1}). \end{aligned}$$

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$$\begin{aligned} &t(\theta_j)t(\theta_j-1) = a(\theta_j)d(\theta_j-1), \quad j = 1,\ldots,N, \\ &a(u) = \prod_{j=1}^{N}(u-\theta_j+1), \quad d(u) = \prod_{j=1}^{N}(u-\theta_j). \end{aligned}$$

Homogeneous limit

$$\frac{\partial^l}{\partial u^l}\{t(u)t(u-1) - a(u)d(u-1)\}|_{u=0,\{\theta_j=0\}} = 0, \quad l = 0,\ldots,N-1.$$

IV. XXX spin chain with non-diagonal boundaries

Hamiltonian

$$H = \sum_{j=1}^{N-1} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + \frac{1}{p}\sigma_1^z + \frac{1}{q}(\sigma_N^z + \xi\sigma_N^x)$$

The K-matrices

$$K^-(u) = \begin{pmatrix} p+u & 0 \\ 0 & p-u \end{pmatrix}$$

$$K^+(u) = \begin{pmatrix} q+u+\eta & \xi(u+\eta) \\ \xi(u+\eta) & q-u-\eta \end{pmatrix}$$

$$\begin{aligned} &R_{1,2}(u-v)K_1^-(u)R_{2,1}(u+v)K_2^-(v) \\ &\quad = K_2^-(v)R_{1,2}(u+v)K_1^-(u)R_{2,1}(u-v), \\ &R_{1,2}(-u+v)K_1^+(u)R_{2,1}(-u-v-2\eta)K_2^+(v) \\ &\quad = K_2^+(v)R_{1,2}(-u-v-2\eta)K_1^+(u)R_{2,1}(-u+v), \end{aligned}$$

Monodromy matrix

$$\begin{aligned} T_0(u) &= R_{0,N}(u-\theta_N)\cdots R_{0,1}(u-\theta_1), \\ \hat{T}_0(u) &= R_{1,0}(u+\theta_1)\cdots R_{N,0}(u+\theta_N), \end{aligned}$$

$$\mathcal{U}_0(u) = T_0(u)K_0^-(u)\hat{T}_0(u)$$

The transfer matrix

$$t(u) = tr_0\{K_0^+(u)\mathcal{U}_0(u)\}$$

$$[t(u),\ t(v)] = 0$$

$$H = \left.\frac{\partial \ln t(u)}{\partial u}\right|_{u=0,\{\theta_j=0\}} - N$$

IV. XXX spin chain with non-diagonal boundaries

Operator identities

$$t(\theta_j)t(\theta_j-1)=a(\theta_j)d(\theta_j-1),\quad j=1,\cdots,N.$$

$$t(0)=2pq\prod_{j=1}^{N}(1-\theta_j)(1+\theta_j),$$

$$t(u)\sim 2u^{2N+2}+\cdots,\quad \text{for } u\rightarrow\pm\infty.$$

Crossing symmetry

$$t(u)=t(-u-1)$$

$$a(u)=\frac{2u+2}{2u+1}(u+p)[(1+\xi^2)^{\frac{1}{2}}u+q]\prod_{j=1}^{N}(u+\theta_j+1)(u-\theta_j+1),$$

$$d(u)=\frac{2u}{2u+1}(u-p+1)[(1+\xi^2)^{\frac{1}{2}}(u+1)-q]\prod_{j=1}^{N}(u+\theta_j)(u-\theta_j)$$

Crossing symmetry: $\Lambda(-u-1)=\Lambda(u)$,

Initial condition: $\Lambda(0)=2pq\prod_{j=1}^{N}(1-\theta_j)(1+\theta_j)=\Lambda(-1)$,

Asymptotic behavior: $\Lambda(u)\sim 2u^{2N+2}+\cdots,\quad u\rightarrow\pm\infty$,

$$\Lambda(\theta_j)\Lambda(\theta_j-1)=\frac{\Delta_q(\theta_j)}{(1-2\theta_j)(1+2\theta_j)}=a(\theta_j)d(\theta_j-1),\quad j=1,\ldots,N$$

$$\Lambda(u)=a(u)\frac{Q(u-1)}{Q(u)}+d(u)\frac{Q(u+1)}{Q(u)}+2[1-(1+\xi^2)^{\frac{1}{2}}]u(u+1)\times\frac{\prod_{j=1}^{N}(u+\theta_j)(u-\theta_j)(u+\theta_j+1)(u-\theta_j+1)}{Q(u)}.$$

$$Q(u)=\prod_{j=1}^{N}(u-\lambda_j)(u+\lambda_j+1)$$

Regularity

$$a(\lambda_j)Q(\lambda_j-1)+d(\lambda_j)Q(\lambda_j+1)=-2[1-(1+\xi^2)^{\frac{1}{2}}]\lambda_j(\lambda_j+1)\times\prod_{l=1}^{N}(\lambda_j+\theta_l)(\lambda_j-\theta_l)(\lambda_j+\theta_l+1)(\lambda_j-\theta_l+1),\quad j=1,\ldots,N,$$

Completeness of the solutions

Proposition 1: *Each solution of the functional relations of* $\Lambda(u)$ *can be parameterized in terms of the inhomogeneous* $T-Q$ *relation*

$$Q(u)\Lambda(u) = a(u)Q(u-1) + d(u)Q(u+1) + F(u)$$

with a polynomial $Q(u)$

$$Q(u) = (u^2+u)^N + \sum_{n=0}^{N-1} I_n (u^2+u)^n$$

Proof: Given a $\Lambda(u)$, we seek for solutions of Q.

$$Q(\theta_j)\Lambda(\theta_j) = a(\theta_j)Q(\theta_j - 1),$$
$$Q(\theta_j - 1)\Lambda(\theta_j - 1) = d(\theta_j - 1)Q(\theta_j)$$

⇓

N linear equations of $\{I_n | n = 0, \cdots, N-1\}$

Proposition 2: *For unparallel boundary fields* ($\xi \neq 0$), *the functional relations of* $\Lambda(u)$ *are the sufficient and necessary conditions to completely characterize the spectrum of the transfer matrix.*

$$\Lambda(u) = 2(u^2+u)^{N+1} + \sum_{n=1}^{N} c_n (u^2+u)^n + 2pq \prod_{j=1}^{N} (1-\theta_j^2)$$

Proof: The spectrum is simple.
All eigenvalues satisfy the functional relations ⇒ $\# \text{ of solutions} \geq 2^N$

$\Lambda(\theta_j)\Lambda(\theta_j - 1) = a(\theta_j)d(\theta_j - 1)$ ⇒ **N quadratic eqns.**

Bezout's theorem ⇒ $\# \text{ of solutions} \leq 2^N$

With each solution of the functional relations as an eigenvalue, in terms of the inhomogeneous T-Q relation, one can retrieve an eigenstate.

Bethe eigenvectors

Introduce a gauge matrix

$$G = \begin{pmatrix} \xi & (1+\xi^2)^{\frac{1}{2}} - 1 \\ \xi & -(1+\xi^2)^{\frac{1}{2}} - 1 \end{pmatrix}$$

$$\tilde{K}^+(u) = GK^+(u)G^{-1} = \begin{pmatrix} \tilde{K}^+_{11}(u) & 0 \\ 0 & \tilde{K}^+_{22}(u) \end{pmatrix}$$

$$\tilde{U}(u) = GT(u)K^-(u)\hat{T}(u)G^{-1} = \begin{pmatrix} \tilde{A}(u) & \tilde{B}(u) \\ \tilde{C}(u) & \tilde{D}(u) \end{pmatrix}$$

$$t(u) = \tilde{K}^+_{11}(u)\,\tilde{\mathcal{A}}(u) + \tilde{K}^+_{22}(u)\,\tilde{\mathcal{D}}(u)$$

$$_n\langle 1| = \xi\, _n\langle\uparrow| + ((1+\xi^2)^{\frac{1}{2}} - 1)\, _n\langle\downarrow|, \quad n = 1, \ldots, N,$$
$$_n\langle 2| = \xi\, _n\langle\uparrow| - ((1+\xi^2)^{\frac{1}{2}} + 1)\, _n\langle\downarrow|, \quad n = 1, \ldots, N.$$

$$|\Omega\rangle = \otimes_{j=1}^N |1\rangle_j, \quad \langle\bar{\Omega}| = \otimes_{j=1}^N\, _j\langle 2|.$$

$\tilde{\mathscr{C}}(u)$ eigenstates $[\tilde{\mathscr{C}}(u), \tilde{\mathscr{C}}(v)] = [\tilde{\mathscr{B}}(u), \tilde{\mathscr{B}}(v)] = 0.$

$$\tilde{\mathscr{C}}(\theta_j)|\Omega\rangle = \tilde{\mathscr{C}}(-\theta_j - 1)|\Omega\rangle = 0,$$
$$\langle\bar{\Omega}|\tilde{\mathscr{C}}(-\theta_j) = \langle\bar{\Omega}|\tilde{\mathscr{C}}(\theta_j - 1) = 0,$$

SoV basis

$$|\theta_{p_1}, \ldots, \theta_{p_n}\rangle = \tilde{\mathscr{A}}(\theta_{p_1}) \cdots \tilde{\mathscr{A}}(\theta_{p_n})|\Omega\rangle,$$
$$\langle\theta_{q_1}, \ldots, \theta_{q_n}| = \langle\bar{\Omega}|\tilde{\mathscr{D}}(-\theta_{q_1}) \cdots \tilde{\mathscr{D}}(-\theta_{q_n}),$$

Bethe eigenvectors

Scalar product $$F_n(\theta_{p_1},\ldots,\theta_{p_n}) = \langle \Psi|\theta_{p_1},\ldots,\theta_{p_n}\rangle$$

$$\Lambda(\theta_{p_{n+1}})F_n(\theta_{p_1},\ldots,\theta_{p_n}) = \langle\Psi|t(\theta_{p_{n+1}})|\theta_{p_1},\ldots,\theta_{p_n}\rangle$$

$$\Lambda(\theta_{p_{n+1}})F_n(\theta_{p_1},\ldots,\theta_{p_n}) = \frac{(2\theta_{p_{n+1}}+1)\tilde{K}^+_{11}(\theta_{p_{n+1}}) + \tilde{K}^+_{22}(\theta_{p_{n+1}})}{2\theta_{p_{n+1}}+1} \times F_{n+1}(\theta_{p_1},\ldots,\theta_{p_{n+1}}).$$

$$F_n(\theta_{p_1},\ldots,\theta_{p_n}) = \prod_{j=1}^{n}\frac{(2\theta_{p_j}+1)\Lambda(\theta_{p_j})}{(2\theta_{p_j}+1)\tilde{K}^+_{11}(\theta_{p_j}) + \tilde{K}^+_{22}(\theta_{p_j})}F_0$$

Bethe eigenvectors: With known eigenvalues and creation operator, we look for the reference state!

$$\langle\lambda_1,\ldots,\lambda_N| = \langle 0|\prod_{j=1}^{N}\tilde{\mathscr{C}}(\lambda_j)$$

requirement

$$\langle\lambda_1,\cdots,\lambda_N|\theta_{p_1},\cdots,\theta_{p_n}\rangle \sim F_n(\theta_1,\cdots,\theta_n)$$

$$\langle 0| = \langle\uparrow|\otimes\cdots\otimes\langle\uparrow|$$

V. Concluding remarks & perspective

Intrinsic properties of R-matrix satisfying YBE

⬇

$$\mathbf{t}(\theta_j)\mathbf{t}(\theta_j-\eta)=a(\theta_j)d(\theta_j-\eta)\times id\sim\Delta_q(\theta_j),\quad j=1,\cdots,N$$

➕ Asymptotic behavior of the polynomial

⬇

$$\mathbf{t}(u)=a(u)\frac{\mathbf{Q}(u-\eta)}{\mathbf{Q}(u)}+d(u)\frac{\mathbf{Q}(u+\eta)}{\mathbf{Q}(u)}+c(u)\frac{a(u)d(u)}{\mathbf{Q}(u)}$$

$$\mathbf{t}(u)|\Psi\rangle=\Lambda(u)|\Psi\rangle$$
$$\mathbf{Q}(u)|\Psi\rangle=Q(u)|\Psi\rangle$$

➡

$$\Lambda(u)=a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(u)}+c(u)\frac{a(u)d(u)}{Q(u)}$$

➘ Retrive Eigenstates

Regularity ➡ Bethe Ansatz equations

IV. Concluding remarks & perspective

High rank systems: An, Bn, Cn, Dn

Fusion

JHEP 04 (2014) 143
JHEP 06 (2014) 128
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$$t(\theta_j)t_m(\theta_j - \eta) = t_{m+1}(\theta_j), \quad m = 1, \ldots, n-1, \quad j = 1, \ldots, N.$$

→ **Nested T-Q**


Baxter forever!
Thanks!