Exactly Solved Models in the Design of Quantum Architectures

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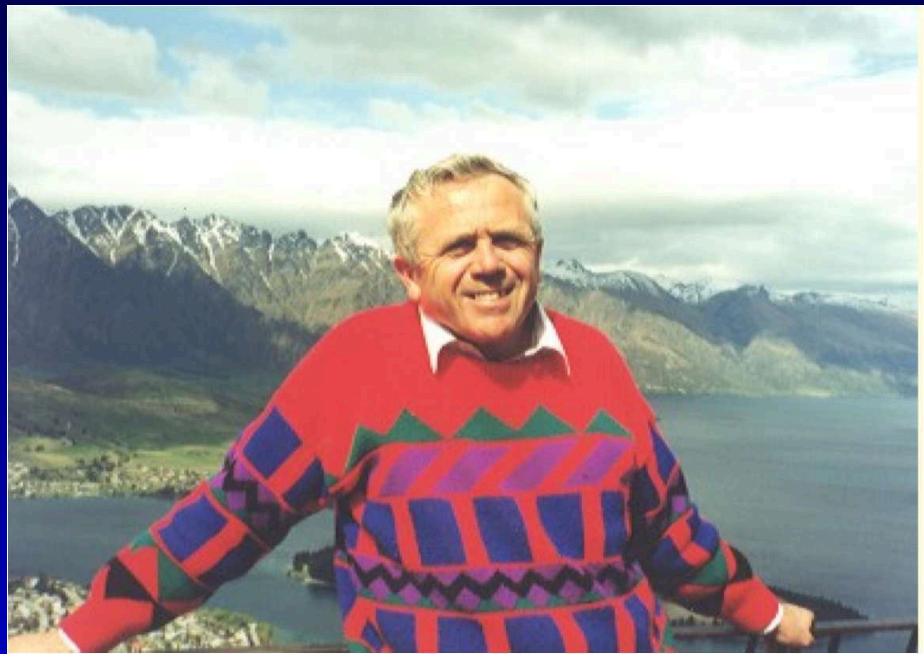
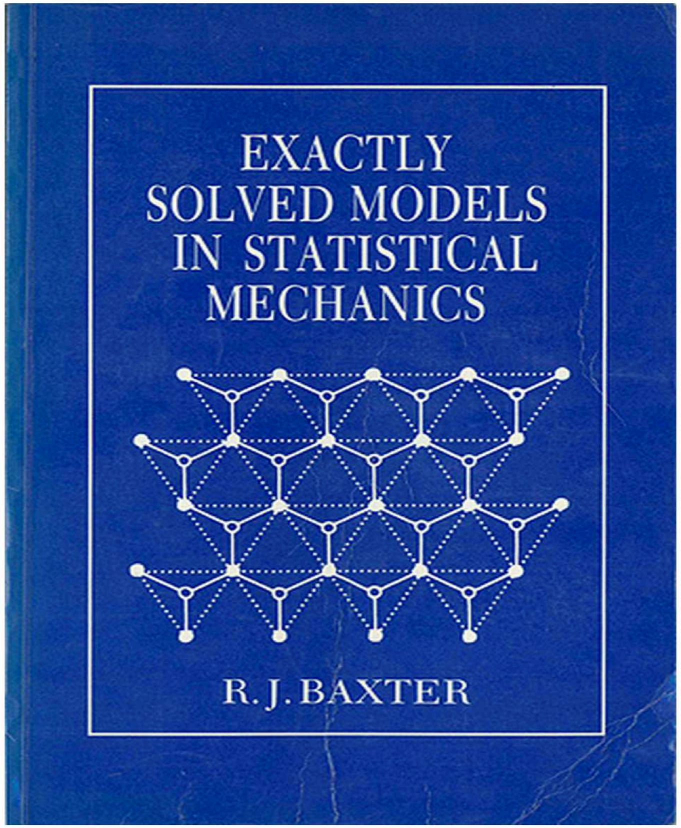
talk presented at: "Baxter 2025 Exactly Solved Models and Beyond: Celebrating the Life and Achievements of Rodney James Baxter"
September, Canberra, 2025

Few words about Rodney James Baxter:

He was a pioneer, a founder of the field of integrable models.



Example: pioneering work:



R. J. Baxter, "Exactly Solved Models in Statistical Mechanics" Academic Press, 1982
Copyright: Rodney J. Baxter 2004

His classic book has inspired all of us and continue to influence many generations of researchers.

It deals with 1D-Ising model, spherical model, ice-type models, 8-vertex model, Potts model,...

How I Met Rodney Baxter:

- First encountered his work in 1990 during my PhD:
  My advisor asked me to read his book before any discussion.
- *Met him in person in 1997 at a conference in Australia, and after in many other conferences.*
- *Had the privilege of joining him for lunches at ANU in 2013 during my sabbatical, arranged by Murray Batchelor.*
- *Met him again at the conferences: Baxter2015 and Baxter2020.*

I - What I Learned from Rodney Baxter:

- His work shaped my approach to theoretical physics.
- Inspired me to value clarity, relevance, and elegance.
- And not only that... His quote:

“ You don’t necessarily make progress by thinking all the time when you should take a sabbatical.”
— R. J. Baxter

is a reminder that stepping back can also lead to insight.

II - What I Learned from Rodney Baxter:

"Basically, I suppose the justification for studying these models is very simple: they are relevant and they can be solved, so why not do so and see what they tell us?"
— R. J. Baxter

- This famous quote guided my research.
- Baxter left us more than models — he left a way of thinking.
- His influence continues to inspire me and many others.

From Baxter's Legacy to Modern Applications:

- Baxter's philosophy — to study models that are both relevant and exactly solvable — continues to guide our work.

- Here: inspired by Baxter's legacy, we discuss integrable multi-well quantum tunneling models in cold atoms and explore some possible applications.

Why are we studying integrable tunneling models?

They display high potential for applications in the development of emerging quantum technologies:

- Solvability and Control: *Integrable models offer analytical formulae and predictions, enabling precise control of quantum states for robust technology development.*
- Understanding Quantum Correlations: *Integrable models offer a platform for studying and manipulating quantum correlations in quantum computing and communication.*
- Developing Quantum Devices:*Integrable models guide the design and understanding of quantum devices, crucial for the implementation of quantum technologies.*

OUTLINE

1- Integrable bosonic quantum tunneling models

2- Application in a static frame: 4-sites in a ring: NOON states

3- Application in a rotating frame: 4-sites in a tetrahedral geometry: Quantum Turntable

4- Concluding remarks

1 - Integrable quantum tunneling models:

Two wells: Two-site Bose Hubbard Hamiltonian:

$$H = \frac{K}{8}(N_1 - N_2)^2 - \frac{\Delta\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}_J}{2}(a_1^\dagger a_2 + a_2^\dagger a_1)$$

- $N_i = a_i^\dagger a_i$: number of atoms in well $(i = 1, 2)$
- K: atom-atom interaction term
- $\Delta\mu$: external potential
- $\mathcal{E}_J$: tunneling strength

G. Milburn et al, Phys. Rev. A **55** *(1997) 4318; A. Leggett, Rev. Mod. Phys.* **73** *(2001) 307*

A. Tonel, J. Links, A. Foerster, JPA 38 (2005) 1235

The quantum dynamics of the model exhibits tunneling & self-trapping - experiment of Oberthaler et al - 2005

Two-wells: Integrability and BA-solution:

- *This model is integrable; it can be formulated by the Yang-Baxter equation and exactly solved via Bethe ansatz methods.*

Yang-Baxter equation: key ingredient in this construction

$$R_{12}(x-y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x-y)$$

- sufficient condition for integrability, proposed independently in different contexts by:
- C. N. Yang (China): Nobel Prize 1957
- R. Baxter (Australia): Boltzman Medal 1980, Lars Onsager Prize 2006, Royal Medal 2013, Henri Poincaré Prize 2021

Multiple-wells:

- We aim to extend this result to multiple wells or sites.
- Natural candidate: $\mathcal{L}$-site Bose-Hubbard model.
  It is not integrable in general: just 2 and ∞ sites.
- Until recently, there was a belief that constructing integrable multi-well systems was not possible.
- We solved this and found that integrability requires the presence of long-range interactions.
- The algebraic construction of these models is tricky, presenting challenges in obtaining all conserved quantities.

Integrable Multi-well tunneling models:

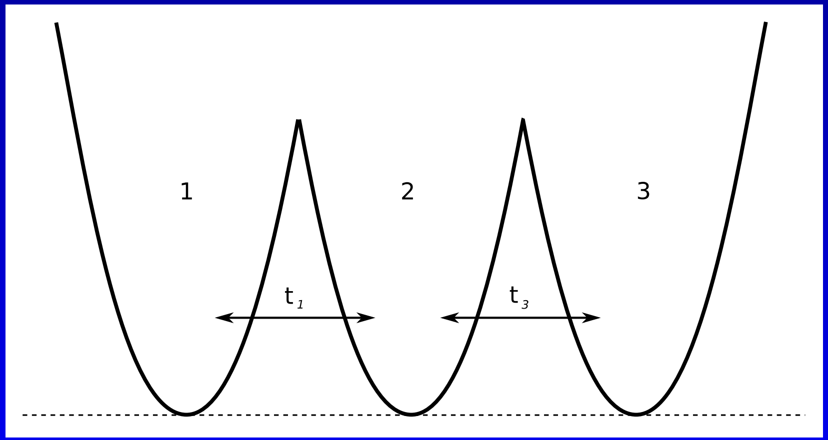
- 3 wells: Triple well Hamiltonian
- 4 wells: Four-well ring model
- ....
- Multi-well tunneling models

Triple well Hamiltonian:

$$\begin{aligned} H &= U(N_1 + N_3 - N_2)^2 + \Delta\mu(N_1 + N_3 - N_2) \\ &+ t_1(a_1^\dagger a_2 + a_1 a_2^\dagger) + t_3(a_2^\dagger a_3 + a_2 a_3^\dagger) \end{aligned} \tag{1}$$

- $N_i = a_i^\dagger a_i$: number of bosons in well i, $(i = 1, 2, 3)$,

  $N = N_1 + N_2 + N_3$ is constant, H is invariant by changing the indices 1 and 3
- U: controls on-site and inter-well interac. bet. bosons
- $\Delta\mu$: external potential, $t_i, i = 1, 3$: tunneling strength:



Application: atomtronic switching device

SciPost
SciPost Phys. Core 2, 003 (2020)

Entangled states of dipolar bosons generated in a triple-well potential

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Abstract

We study the generation of entangled states using a device constructed from dipolar bosons confined to a triple-well potential. Dipolar bosons possess controllable, long-range interactions. This property permits specific choices to be made for the coupling parameters, such that the system is integrable. Integrability assists in the analysis of the system via an effective Hamiltonian constructed through a conserved operator. Through computations of fidelity we establish that this approach, to study the time-evolution of the entanglement for a class of non-entangled initial states, yields accurate approximations given by analytic formulae.

COMMUNICATIONS PHYSICS

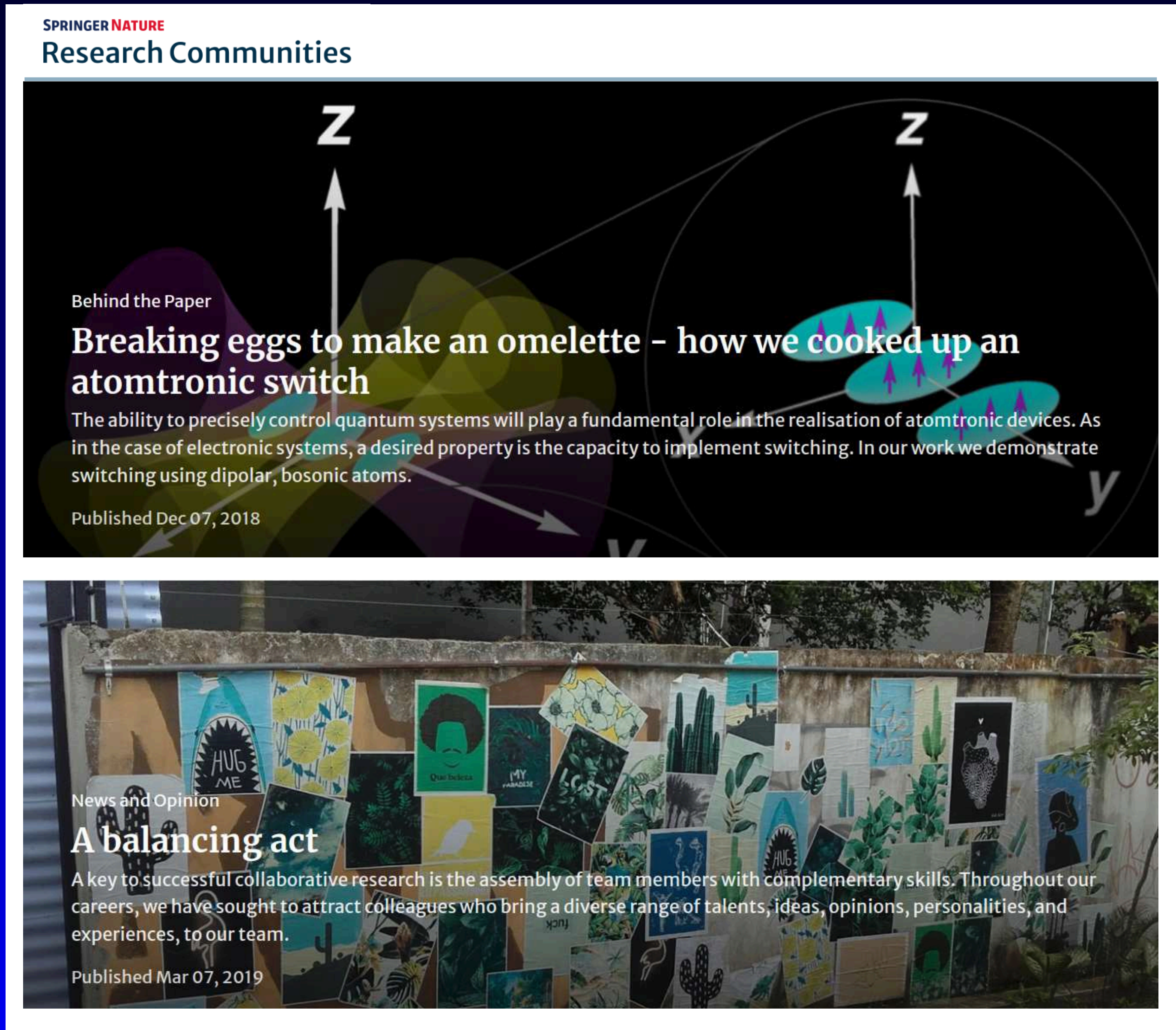
ARTICLE
DOI: 10.1038/s42005-018-0089-1 OPEN

Control of tunneling in an atomtronic switching device

Karin Wittmann Wilsmann[1], Leandro H. Ymai[2], Arlei Prestes Tonel[2], Jon Links[3] & Angela Foerster[1]

The precise control of quantum systems will play a major role in the realization of atomtronic devices. As in the case of electronic systems, a desirable property is the ability to implement switching. Here we show how to implement switching in a model of dipolar bosons confined to three coupled wells. The model describes interactions between bosons, tunneling of bosons between adjacent wells, and the effect of an external field. We conduct a study of the quantum dynamics of the system to probe the conditions under which switching behavior can occur. The analysis considers both integrable and non-integrable regimes within the model. Through variation of the external field, we demonstrate how the system can be controlled between various “switched-on” and “switched-off” configurations.

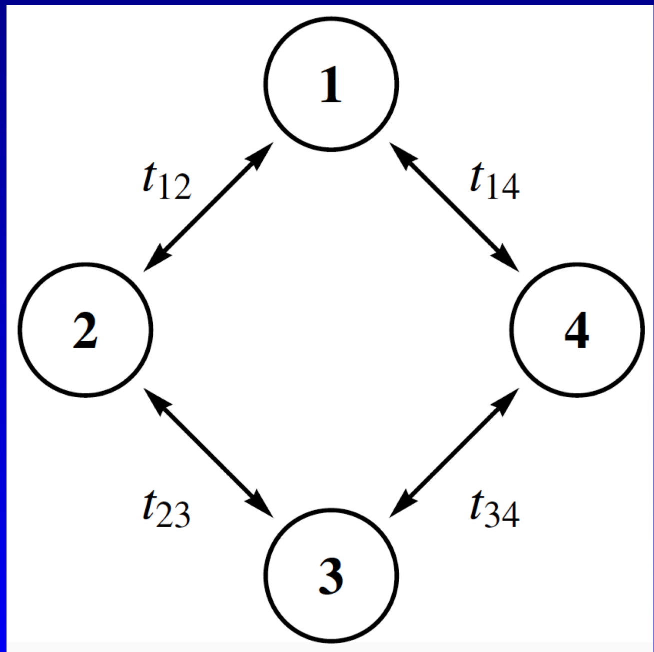
Contribution to the blogs of Nature Research Communities:



Four-well ring:

$$
\begin{aligned}
H &= U(N_1+N_3-N_2-N_4)^2+\Delta\mu(N_1+N_3-N_2-N_4) \\
&+ t_{12}(a_1a_2^\dagger+a_1^\dagger a_2)+t_{14}(a_1a_4^\dagger+a_1^\dagger a_4) \\
&+ t_{23}(a_3a_2^\dagger+a_3^\dagger a_2)+t_{34}(a_3a_4^\dagger+a_3^\dagger a_4)
\end{aligned} \quad (2)
$$

U: controls on-site and inter-well interactions; t_{ij} are not independent: $t_{12}t_{34}=t_{23}t_{14}$ but still admits sufficient freedom to investigate a range of anisotropic tunneling regimes



Application: NOON states

PHYSICAL REVIEW LETTERS **129,** 020401 (2022)

Integrable Atomtronic Interferometry

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High sensitivity quantum interferometry requires more than just access to entangled states. It is achieved through the deep understanding of quantum correlations in a system. Integrable models offer the framework to develop this understanding. We communicate the design of interferometric protocols for an integrable model that describes the interaction of bosons in a four-site configuration. Analytic formulas for the quantum dynamics of certain observables are computed. These expose the system's functionality as both an interferometric identifier, and producer, of NOON states. Being equivalent to a controlled-phase gate acting on 2 hybrid qudits, this system also highlights an equivalence between Heisenberg-limited interferometry and quantum information. These results are expected to open new avenues for integrability-enhanced atomtronic technologies.

DOI: 10.1103/PhysRevLett.129.020401

communications
physics

ARTICLE

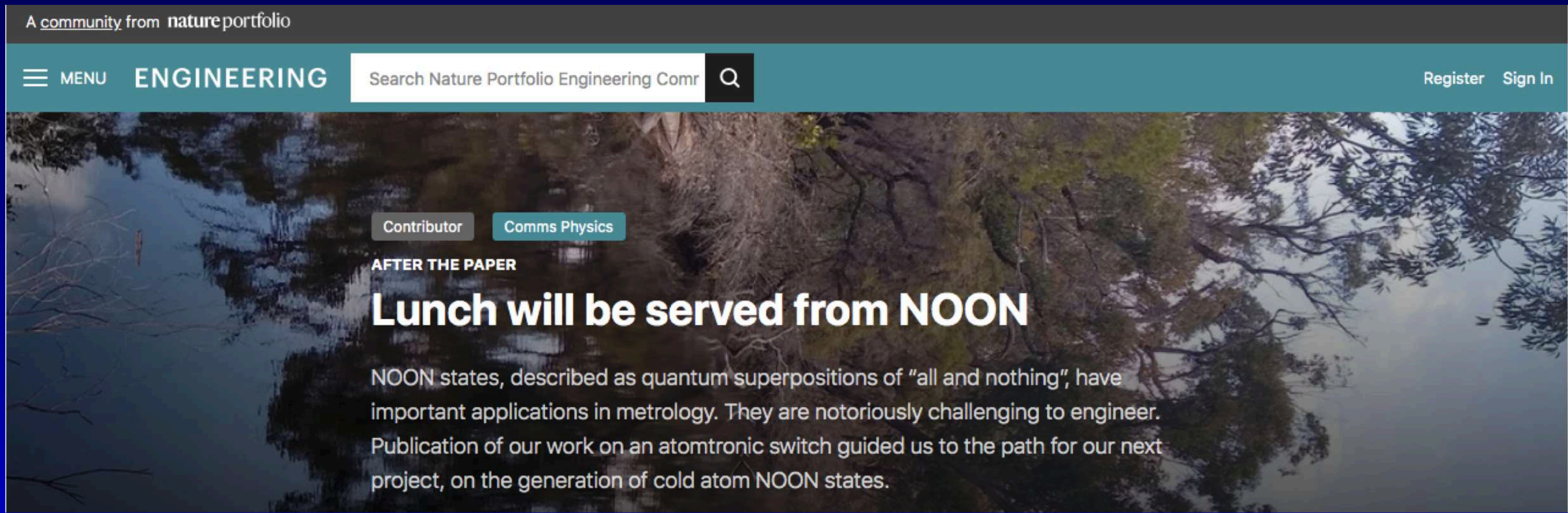
https://doi.org/10.1038/s42005-022-00812-7 OPEN

Protocol designs for NOON states

Daniel S. Grün[1], Karin Wittmann W.[1], Leandro H. Ymai[2], Jon Links[3✉] & Angela Foerster[1]

The ability to reliably prepare non-classical states will play a major role in the realization of quantum technology. NOON states, belonging to the class of Schrödinger cat states, have emerged as a leading candidate for several applications. Here we show how to generate NOON states in a model of dipolar bosons confined to a closed circuit of four sites. This is achieved by designing protocols to transform initial Fock states to NOON states through use of time evolution, application of an external field, and local projective measurements. The evolution time is independent of total particle number, offering an encouraging prospect for scalability. By variation of the external field strength, we demonstrate how the system can be controlled to encode a phase into a NOON state. We also discuss the physical feasibility, via ultracold dipolar atoms in an optical superlattice setup. Our proposal showcases the benefits of quantum integrable systems in the design of protocols.

Contribution to the blog of Nature Research Communities:



Baxter 2020: Frontiers in Integrability

It may be quite some time before we experience another conference group photo like this.

Group photo of the conference delegates attending Baxter 2020: Frontiers in Integrability. Angela is second from the left in the second row. Jon is the third person behind Angela.

Multi-well tunneling models:

- Hamiltonian for $(n+m)$ wells:

$$H_{n,m} = U(N_A - N_B)^2 + \Delta\mu(N_A - N_B) + \sum_{i=1}^{n}\sum_{j=1}^{m} t_{i,j}(a_i b_j^\dagger + a_i^\dagger b_j)$$

- in terms of sets of canonical boson operators:

$a_i,\ a_i^\dagger, N_{a,i} = a_i^\dagger a_i,\ i = 1, ..., n$

$b_j,\ b_j^\dagger,\ N_{b,j} = b_j^\dagger b_j,\ j = 1, ..., m$

$N_A = \sum_{i=1}^{n} a_i^\dagger a_i \quad N_B = \sum_{j=1}^{m} b_j^\dagger b_j \quad N = N_A + N_B$

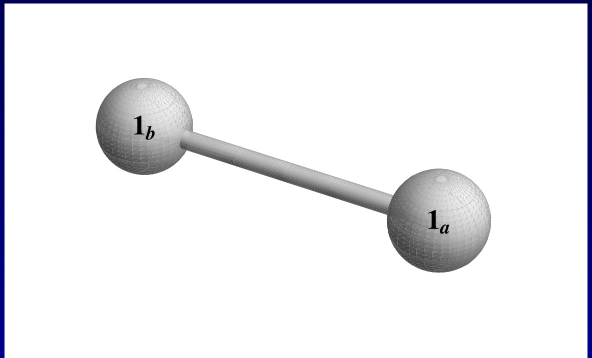
- U intra-well and inter-well interaction between bosons
- $\Delta\mu$ external potential, $t_{i,j}$ couplings for tunneling
- Models defined on complete bipartite graphs $K_{n,m}$

Some particular Hamiltonians:

For particular choices of n, m we find

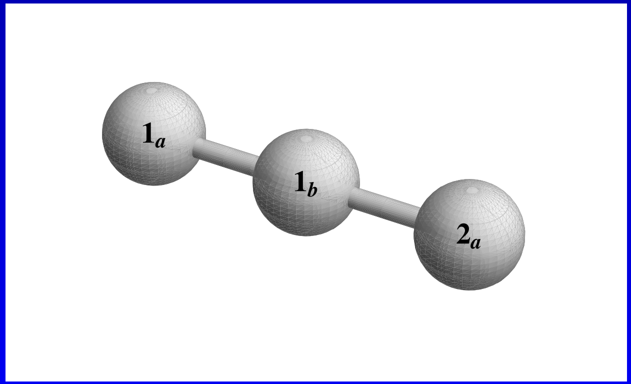
- 2 wells:

  $n = m = 1$: Two-site Bose Hubbard model



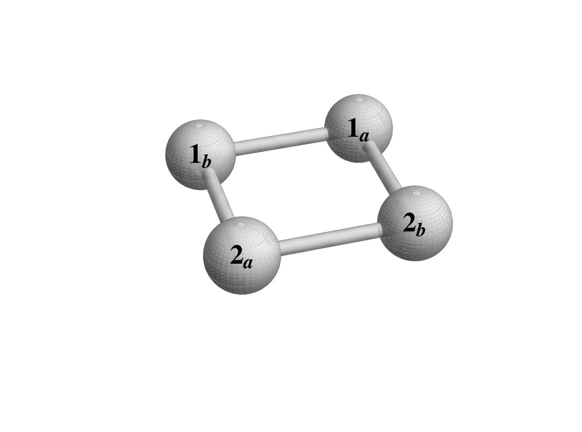
- 3 wells:

  $n = 2, m = 1$: Triple well Hamiltonian

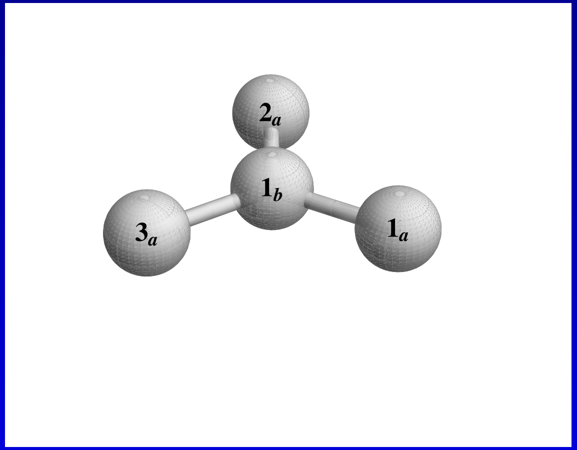


- 4 wells:

  $n = 2, m = 2$: Four-well ring model:

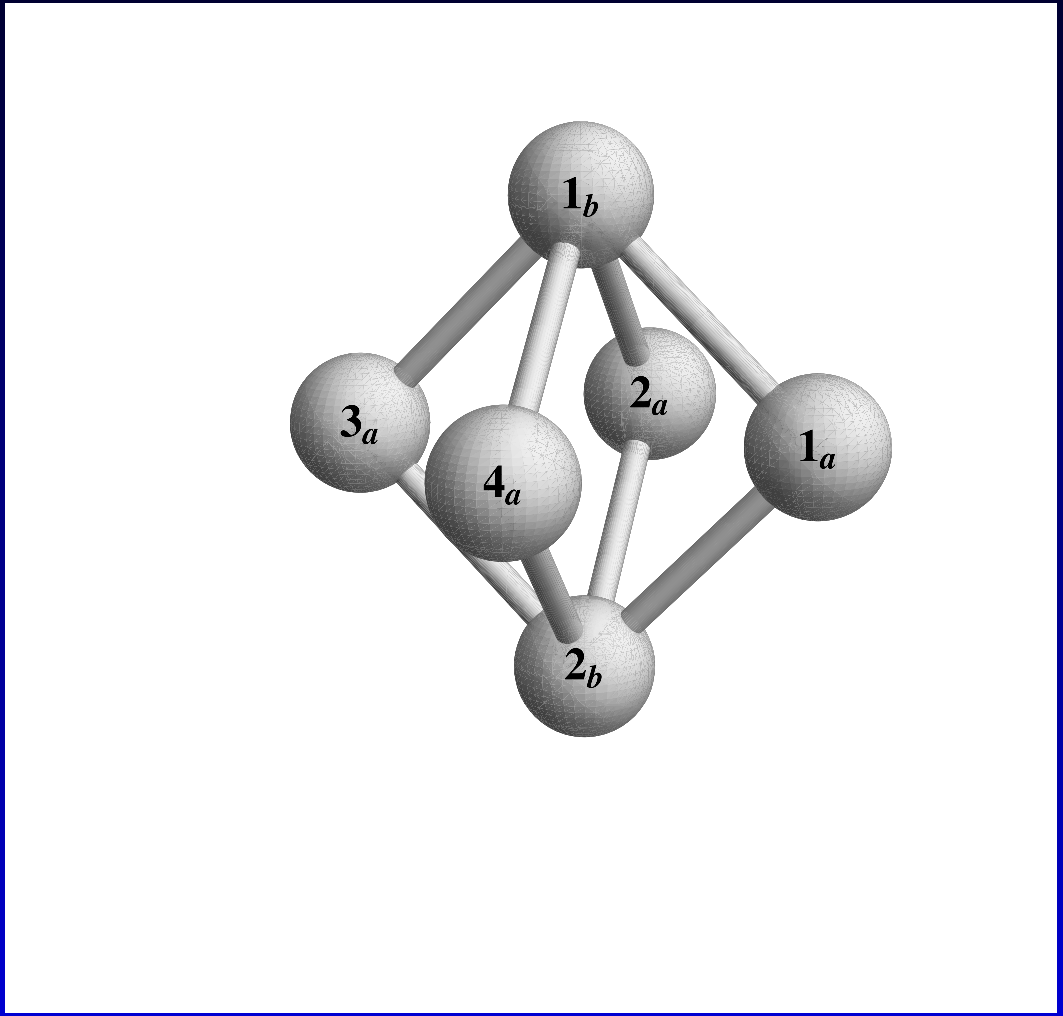


$n = 3, m = 1$: Open four well model:



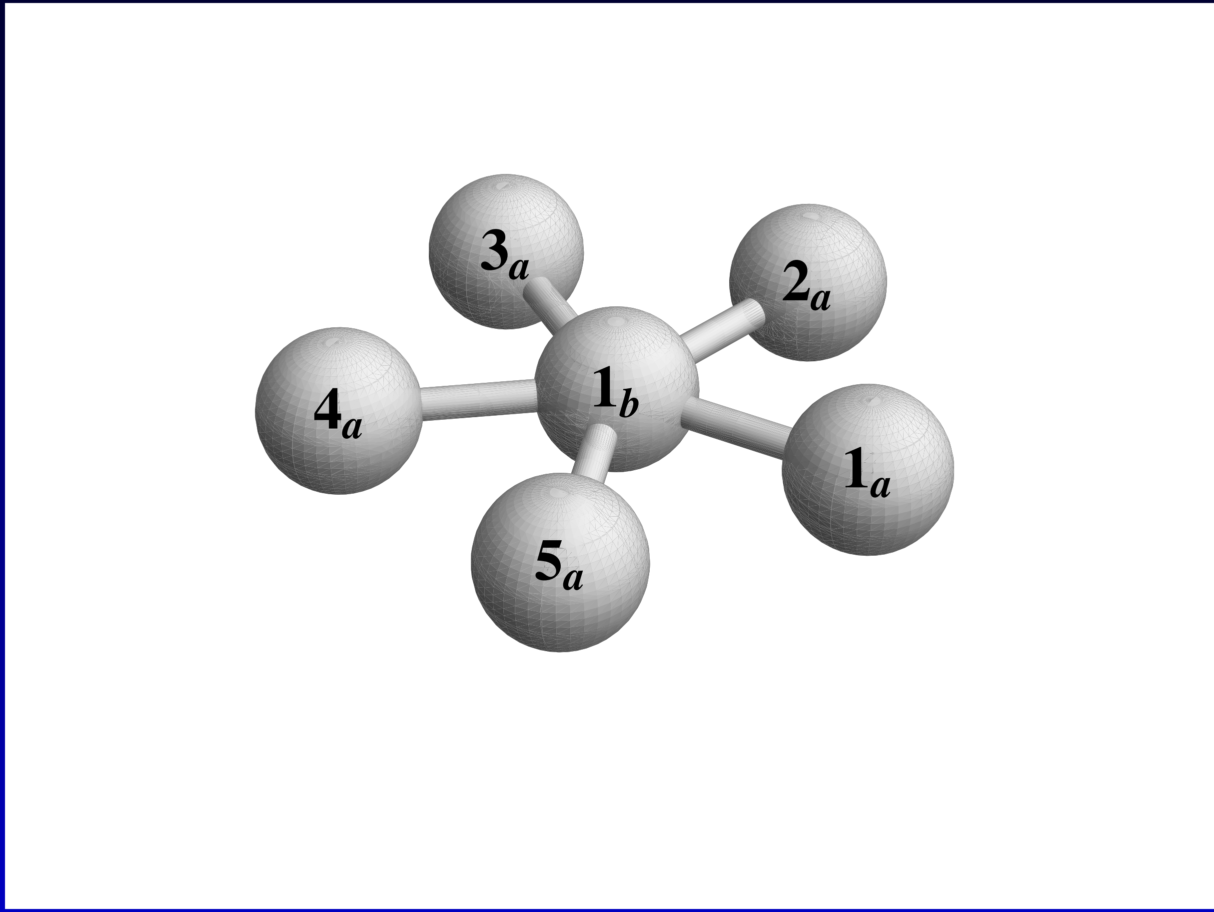
Schematic rep.: spheres represent the wells, with bonds indicating the tunneling between the wells

6 wells: $n = 4, m = 2$**:**



L. Ymai, A. Tonel, A. Foerster, J. Links, JPA 2017

6 wells: $n = 5, m = 1$**:**

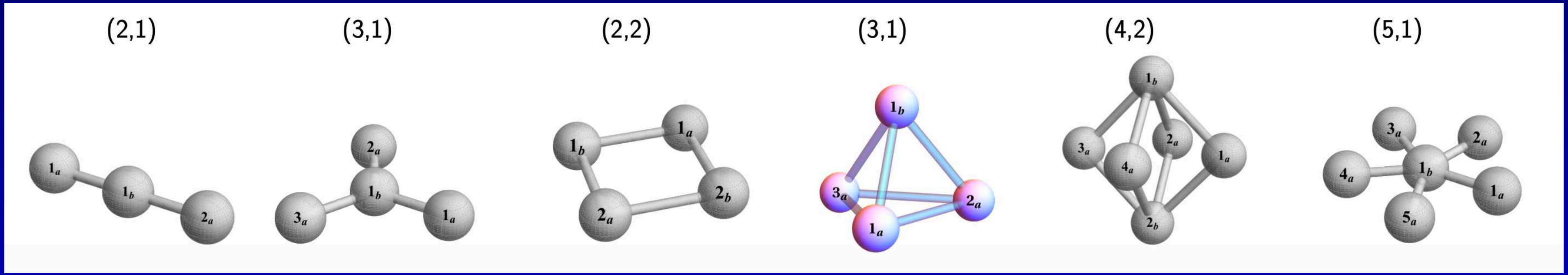


OBS: Superintegrability:

- For $n > 2$ or $m > 2$ the $(n+m)$-wells (or sites) tunneling Hamiltonian is superintegrable:
  more conserved quantities than degrees of freedom.
- The existence of superintegrability allows for the inclusion of additional interaction terms that can break the global symmetry algebra without compromising integrability.
- We can extend the models to include tunneling terms while retaining integrability

Example: $n = 3, m = 1$

- Superintegrable 4-sites in a tetrahedral geometry:



2-Application: 4-sites in a ring: Protocol designs for NOON states

What is a NOON state?

- Definition: It is an *all and nothing* superposition of two different modes. For N particles, it has the form:

$$|\mathrm{NOON}\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle + e^{i\varphi}|0,N\rangle\right)$$

where the phase φ typically records information in applications

- Applications:
  Quantum metrology and sensing
  Quantum lithography
  Quantum communication
  Quantum computing
  ......

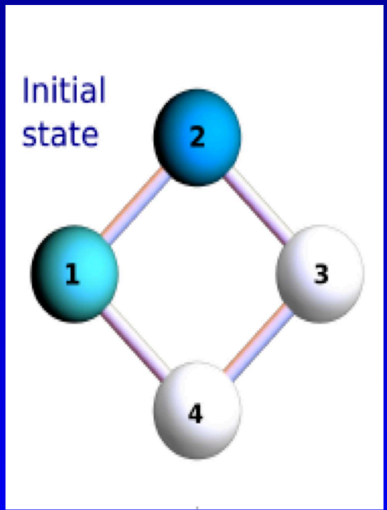
- *Here we show how to generate a NOON state and, in additon, how to encode a phase into this NOON state.*
- *Our approach is based on the formation of an uber-NOON state en route to the final NOON state.*

Uber-NOON state:

- Consider a closed-circuit of 4 sites, with a Fock-state input $|\Psi_0\rangle = |M, P, 0, 0\rangle$. The initial step is to create an *uber-NOON state*, with the general form:

$$|\text{u-NOON}\rangle = \frac{1}{2}\big(|M, P, 0, 0\rangle + e^{i\varphi_1}|M, 0, 0, P\rangle + e^{i\varphi_2}|0, P, M, 0\rangle + e^{i\varphi_3}|0, 0, M, P\rangle\big)$$

for a set of phases $\{\varphi_1, \varphi_2, \varphi_3\}$. It may be viewed as an embedding of NOON states within two-site subsystems



- *We then describe two protocols to extract a NOON state from an uber-NOON state.*
- *For this, we consider a model of dipolar bosons confined to 4 wells, more especifically, to a closed circuit of 4 sites.*
- *The system has long-range interactions and can be described by the EBHM.*

Extended 4-site Bose-Hubbard Model (EBHM)

$$H = \frac{U_0}{2}\sum_{i=1}^{4} N_i(N_i-1) + \sum_{i=1}^{4}\sum_{j=1, j\neq i}^{4} \frac{U_{ij}}{2} N_i N_j$$
$$- \frac{J}{2}\left[(a_1^\dagger + a_3^\dagger)(a_2 + a_4) + (a_1 + a_3)(a_2^\dagger + a_4^\dagger)\right],$$

- U_0: characterizes the interaction between bosons at the same site
- $U_{ij} = U_{ji}, i \neq j$: characterize DDI between bosons at different sites (i and j)
- J: couplings for the tunneling between different sites
- Integrability condition:

  The integrable closed 4-wells model can be obtained from EBHM if:

$$U_0 = U_{13} = U_{24}, \quad U_{12} = U_{23} = U_{34} = U_{14}.$$

- Conserved quantities:

The model has 4 modes, so 4 independent conserved quantities are expected:

$$[H,N]=[H,Q_k]=[N,Q_k]=[Q_1,Q_2]=0, \quad k=1,2.$$

$$Q_1=\frac{1}{2}(N_1+N_3-a_1^\dagger a_3-a_3^\dagger a_1),$$
$$Q_2=\frac{1}{2}(N_2+N_4-a_2^\dagger a_4-a_4^\dagger a_2),$$

Hereafter we consider the integrable EBHM case.

To generate NOON states we consider that:

- the system is initially in a Fock state $|\Psi_0\rangle = |M, P, 0, 0\rangle$,

  with total boson number $N = M + P$ odd

- the system is in the resonant tunneling regime

  achieved when $U|M - P| \gg J$, where $U = (U_{12} - U_0)/4$.

  In this resonant regime:

  a) the energy levels separate into distinct bands

  b) an effective Hamiltonian H_{eff}, dynamically equivalent to the integrable EBHM, can be obtained

  c) H_{eff} can be written in terms of charges Q_1, Q_2

  d) H_{eff} enables the derivation of analytical expressions for physical quantities.

Significant feature:

- Under this H_{eff}, the time evolution of:
  $N_1 + N_3 = M$ is constant
  $N_2 + N_4 = P$ is constant

- In this resonant tunneling regime:
  M particles can oscillate between sites $1, 3$: subsystem A
  P particles can oscillate between sites $2, 4$: subsystem B

NOON-state protocols:

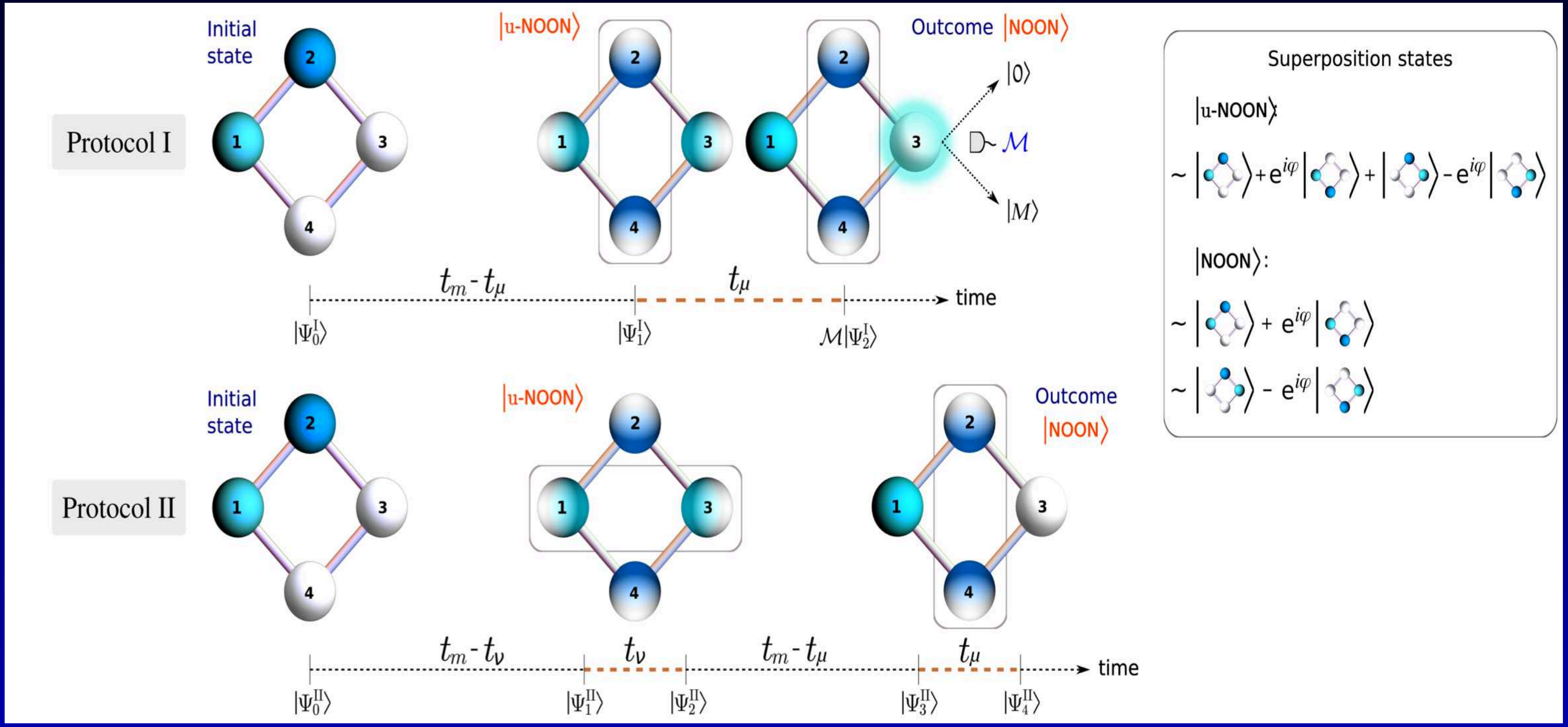
- We describe two protocols that enable the generation of NOON states. For Protocol I the outcomes are probabilistic while Protocol II are deterministic.

- Both protocols consider the initial state $|\Psi_0\rangle = |M, P, 0, 0\rangle$ and are built around a general time evolution operator:

$$\mathcal{U}(t,\mu,\nu) = \exp\left(-\frac{it}{\hbar}[H + \mu(N_2 - N_4) + \nu(N_1 - N_3)]\right)$$

The applied field strengths μ, ν implement the breaking of integrability

NOON state generation scheme:



The four spheres on the left represent the initial state, with white indicating empty site and cyan and blue M and P particles, respectively. Gradient colors are used to indicate that the state of a site is entangled with the rest of the system - superposition states for each step are shown in the framed legend. Rectangles represent applied external fields to sites 1-3 (ν) and 2-4 (μ).

Protocol I:

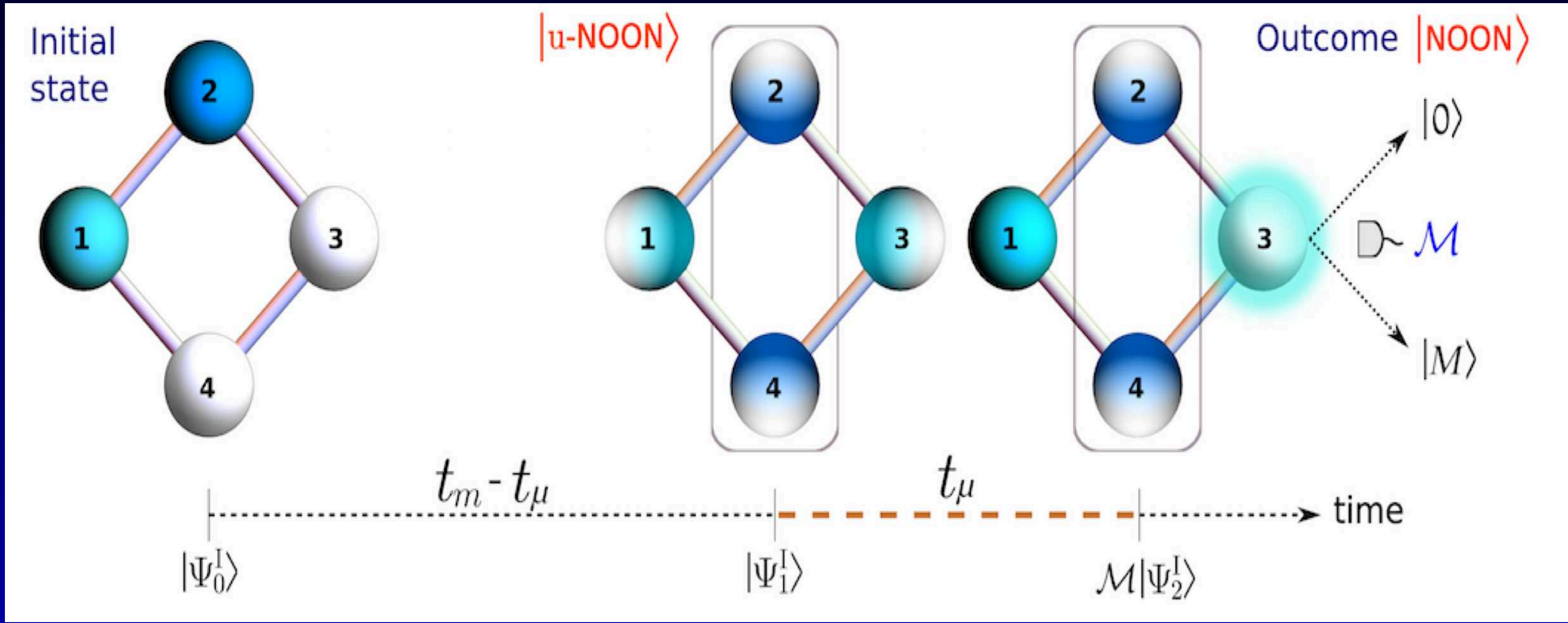
Here are the steps we implement to arrive at a NOON state in subsystem $B = \{2, 4\}$:

$$\text{(i)} \quad |\Psi_1^{\mathrm{I}}\rangle = \mathcal{U}(t_{\mathrm{m}} - t_\mu, 0, 0)\, |\Psi_0\rangle$$

$$\text{(ii)} \quad |\Psi_2^{\mathrm{I}}\rangle = \mathcal{U}(t_\mu, \mu, 0)\, |\Psi_1^{\mathrm{I}}\rangle$$

$$\text{(iii)} \quad |\Psi_3^{\mathrm{I}}\rangle = \mathcal{M}\, |\Psi_2^{\mathrm{I}}\rangle$$

Protocol I:



In Protocol I the system evolves for $t_{\mathrm{m}} - t_\mu$, towards the u-NOON state. Then, a field is applied across sites 2-4 for time t_μ, to encode a phase. The cyan halo portrays a measurement process at site 3: outcomes $|0\rangle$ and $|M\rangle$ signify which of two possible NOON states results across sites 2-4.

Analytical results:

Analytic results are obtained by employing the effective Hamiltonian in an extreme limit, with divergent applied fields acting for infinitesimally small times: $\mu \to \infty, t_\mu \to 0$, such that θ remains finite. Here $\beta = (-1)^{(N+1)/2}$. We find for the final state at step (iii):

$$|\Psi_3^{\mathrm{I}}\rangle = \begin{cases} \frac{1}{\sqrt{2}} \left(\beta \, |M, P, 0, 0\rangle + e^{iP\theta} \, |M, 0, 0, P\rangle\right), \; r = 0 \\ \frac{1}{\sqrt{2}} \left(|0, P, M, 0\rangle - \beta e^{iP\theta} \, |0, 0, M, P\rangle\right), \; r = M \end{cases}$$

- These states are recognized as products of a NOON state for subsystem $B = \{2, 4\}$ with Fock basis states for subsystem $A = \{1, 3\}$.

Protocol II:

Here the following steps are implemented to arrive at a NOON state in subsystem $B = \{2,4\}$:

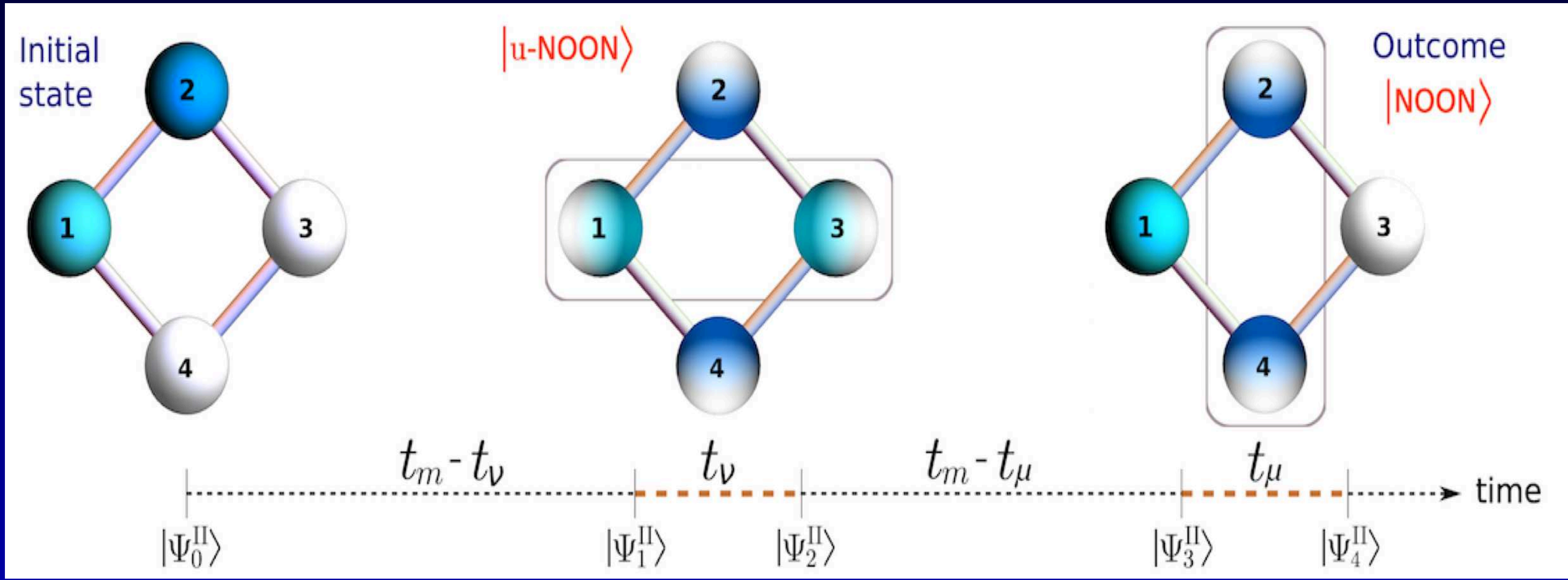
$$\text{(i)} \quad |\Psi_1^{\mathrm{II}}\rangle = \mathcal{U}(t_{\mathrm{m}} - t_\nu, 0, 0)\,|\Psi_0\rangle$$

$$\text{(ii)} \quad |\Psi_2^{\mathrm{II}}\rangle = \mathcal{U}(t_\nu, 0, \nu)\,|\Psi_1^{\mathrm{II}}\rangle$$

$$\text{(iii)} \quad |\Psi_3^{\mathrm{II}}\rangle = \mathcal{U}(t_{\mathrm{m}} - t_\mu, 0, 0)\,|\Psi_2^{\mathrm{II}}\rangle$$

$$\text{(iv)} \quad |\Psi_4^{\mathrm{II}}\rangle = \mathcal{U}(t_\mu, \mu, 0)\,|\Psi_3^{\mathrm{II}}\rangle$$

Protocol II:



In Protocol II the system first evolves for time $t_\mathrm{m} - t_\nu$, then a field is applied to sites 1-3 for time t_ν to implement the phase $\pi/2$. Next, the system evolves for time $t_\mathrm{m} - t_\mu$, after which a field is applied to sites 2-4 to encode a phase during time t_μ. This results in a NOON state across sites 2-4, without performing a measurement procedure

Analytic results:

Similar to Protocol I, analytic results are obtained by employing the effective Hamiltonian. We obtain for the final state at step (iv): (here $\Upsilon = \beta \exp(i(P\theta - \pi/2))$)

$$|\Psi_4^{\mathrm{II}}\rangle = \frac{1}{\sqrt{2}}\Big(|M, P, 0, 0\rangle + \Upsilon\, |M, 0, 0, P\rangle \Big)$$

This state is recognized as a product of a NOON state for subsystem $B = \{2, 4\}$ with a deterministic state (Fock basis state) for subsystem $A = \{1, 3\}$.

Protocol fidelities:

- We give numerical simulations of the protocols to show that, for physically realistic settings (see Set I and Set II) high-fidelity outcomes for NOON state production persist.
- The fidelities of Protocols I and II are defined as:

$$F_{\rm I} = |\,\langle \Psi^{\rm I}_3 | \Phi^{\rm I}_3 \rangle\,| \qquad F_{\rm II} = |\,\langle \Psi^{\rm II}_4 | \Phi^{\rm II}_4 \rangle\,|$$

$|\Psi\rangle$: denotes the analytical states obtained using the effective Hamiltonian
$|\Phi\rangle$: denotes the numerically calculated state obtained by EBHM time evolution.

OBS: The fidelities are computed for $P\theta$ ranging from 0 to π, achieved by varying t_μ. As the values remain almost constant for $P\theta \in [0, \pi]$, varying less than 1%, we display here only one case.

Protocol fidelities:

	$F_{\mathbf{I}}$		$F_{\mathbf{II}}$	$P\theta = \pi/2$		
	$r = 0$	$r = M$		t_μ	t_ν	t_m
Set 1	0.986	0.997	0.974	0.0024 s	0.0065 s	6.1639 s
Set 2	0.964	0.991	0.920	0.0026 s	0.0072 s	2.8913 s

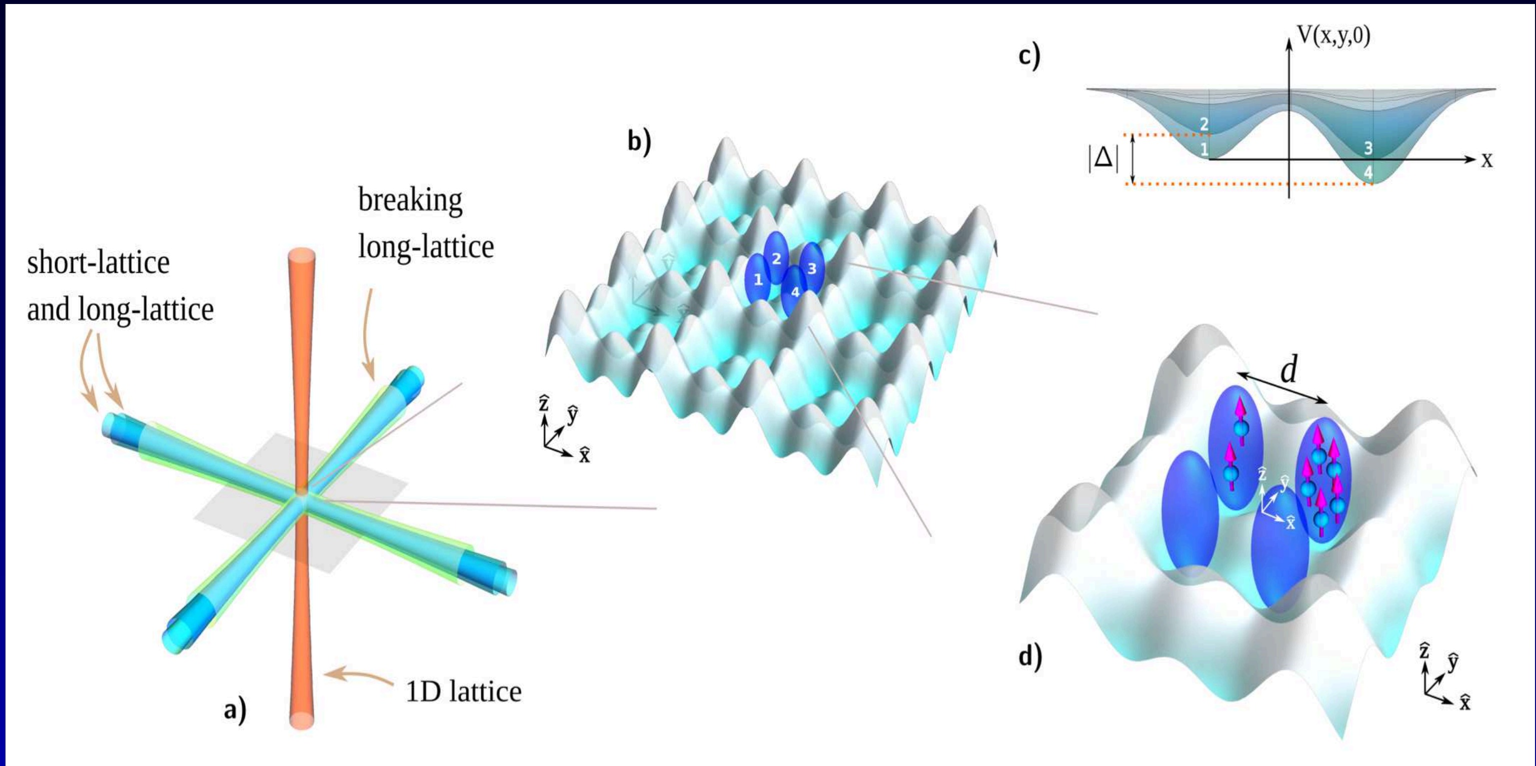
Fidelities for Protocols I (F_{I}) and II (F_{II}) and related times t_m, t_μ and t_ν (fixed), concerning to the parameters of Set 1 and Set 2, for $|\Psi_0\rangle = |4, 11, 0, 0\rangle$.

- Two realistic sets of parameters (in Hz):
  Set 1: $\{U/\hbar = 104.85,\ J/\hbar = 71.62,\ \mu/\hbar = 30.02\}$
  Set 2: $\{U/\hbar = 105.60,\ J/\hbar = 104.95,\ \mu/\hbar = 27.42\}$

Observations:

- Both protocols display high fidelity results greater than 0.9.
- The higher fidelity results associated to parameter Set 1, compared to Set 2, are produced through longer evolution time: these results reflect the trade-off between fidelity and total evolution time.
- The required times $tm, 2tm$ to produce the NOON states are comparable with typical lifetimes of optical lattice traps.
- Advantage of the protocols: the evolution time is independent of total particle number, offering an encouraging prospect for scalability.

Experimental feasibility:



a) Trapping scheme: 2D square optical lattice is generated with 2 sets of counterpropagating laser beams crossing at 90. The superlattice of 4-site model is achieved overlapping the 2D short lattice (cyan) and long-lattice (blue). The vertical lattice (orange) provides confinement in z direction. An additional 2D square long-lattice (green) is used to implement the integrability break control. b) Zoom into the region of the superlattice which contains the 4-site plaquette. c) Breaking-of-integrability scheme

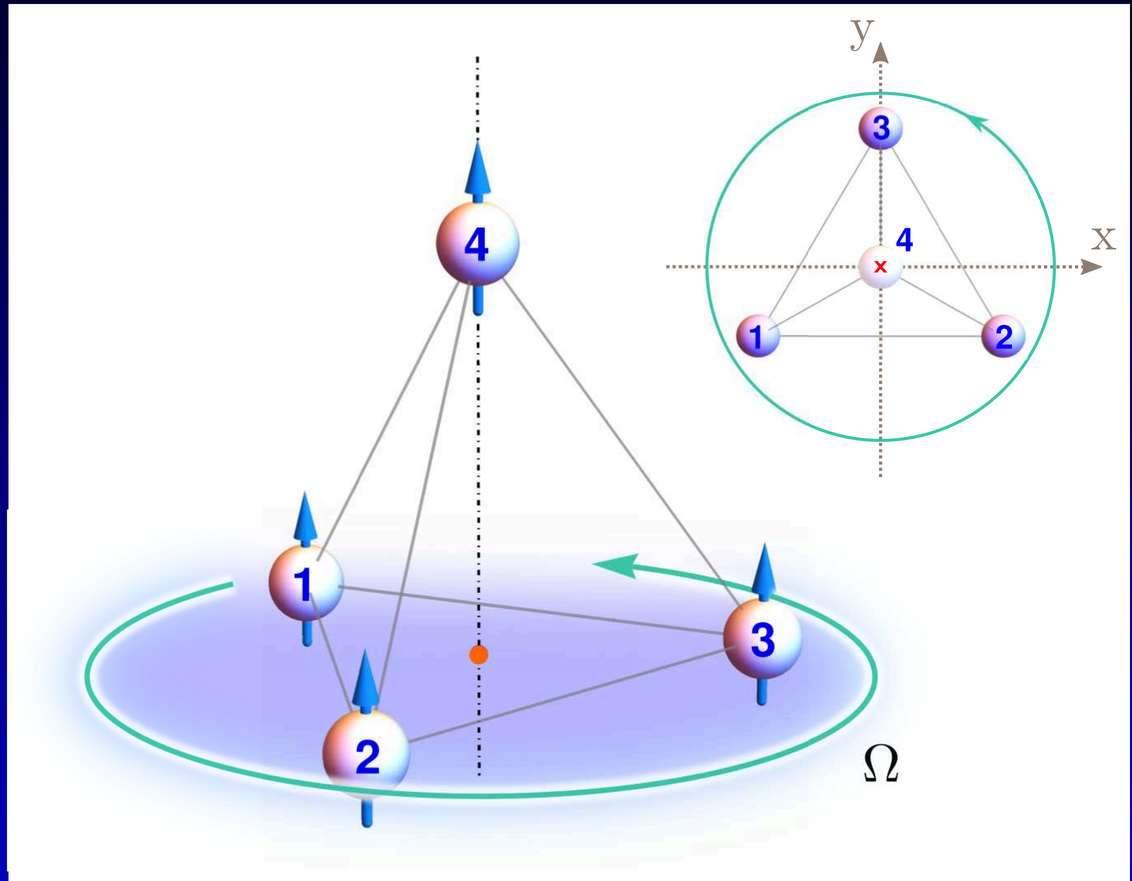
3-Application: 4-sites in a tetrahedral geometry: Quantum Turntable

- *Here we design a quantum turntable that spatially transfers entangled states with high fidelity in a rotating frame.*

- For this, we study a system of ultracold dipolar atoms in a tetrahedral geometry, that is set to rotate at constant angular frequency Ω.

- The system can be described by the Extended Bose-Hubbard Model in a rotating frame.

Geometric representation of rotating tetrahedral system



Sites 1, 2, 3 form the triangular base. Blue arrows show dipoles along z-axis. The system rotates around z with angular freq. Ω.

Extended Bose-Hubbard model in a rotating frame:

$$\mathcal{H} = \frac{U_0}{2}\sum_{i=1}^{4} N_i(N_i - 1) + \sum_{i=1}^{4}\sum_{j=1;j\neq i}^{4} \frac{U_{ij}}{2} N_i N_j - J \sum_{i<j=1}^{4} (a_i^\dagger a_j + a_j^\dagger a_i) + \mathcal{H}_{\text{rot}}$$

- U_0: characterizes the interaction between bosons at the same site
- $U_{ij} = U_{ji}, i \neq j$: characterize DDI between bosons at different sites (i and j)
- J: couplings for the tunneling between different sites
- $\mathcal{H}_{\text{rot}}$: characterizes the rotation of the system.

Our approach combines various ingredients:

- *Long-range interactions via dipolar atoms.*
- *Non-equilibrium phenomena via use of a rotating frame.*
- *Property of superintegrability guiding the proposal.*

Cases Considered:

- Non-rotating case: *Superintegrable.*
- Rotating case: *Integrability persists.*

  Bethe Ansatz applied as analytical basis for the quantum turntable.

Superintegrability in a Static Frame ($H_{\text{rot}} = 0$)

- Geometric symmetry: $U_{14} = U_{24} = U_{34}$, $U_{12} = U_{23} = U_{13}$
- Superintegrability condition: $U_{12} = U_0 = 2U$
- $\mathcal{H}$ reduces to:
  $\mathcal{H} = \mathcal{H}_0 = U(N_1 + N_2 + N_3 - N_4)^2 - J \sum_{i<j}(a_i^\dagger a_j + a_j^\dagger a_i)$

◇ It has 4 degrees of freedom.

◇ It is superintegrable: has more conserved quantities than degrees of freedom.

Then the system is set to rotate.

Integrability in a Rotating Frame:

- Rotation term: $H_{\text{rot}} = -\zeta \mathcal{J}$

$$\mathcal{J} = i\sqrt{3}\Big[(a_2^\dagger a_1 + a_3^\dagger a_2 + a_1^\dagger a_3) - (a_1^\dagger a_2 + a_3^\dagger a_1 + a_2^\dagger a_3)\Big]$$

- Total Hamiltonian: $\mathcal{H}(\zeta) = \mathcal{H}_0 - \zeta \mathcal{J}$
- Effect: superintegrability is broken, but...
- Integrability preserved: 4 conserved operators: $\{\mathcal{H}(\zeta), N, Q_2, Q_3\}$
- These conserved operators can define H_{eff}, enabling the derivation of analytical expressions for physical quantities.

Quantum turntable for different initial states:

- Expectation values of populations at sites $k = 1, 2, 3$

$$\langle N_k \rangle = \sum_{i=1}^{3} \frac{n_i}{9} \Big\{ 1 + 4\cos\theta_{k,i}(t)[\cos(\xi t) + \cos\theta_{k,i}(t)] \Big\},$$

where $\theta_{k,i}(t) = t\zeta + 2\pi(k-i)/3$
$\xi = 3J - \frac{3J\gamma}{2}\left[\frac{N+1-\gamma}{(N-2n_4-\gamma)^2-1}\right], \quad \gamma = \frac{J}{2U}.$

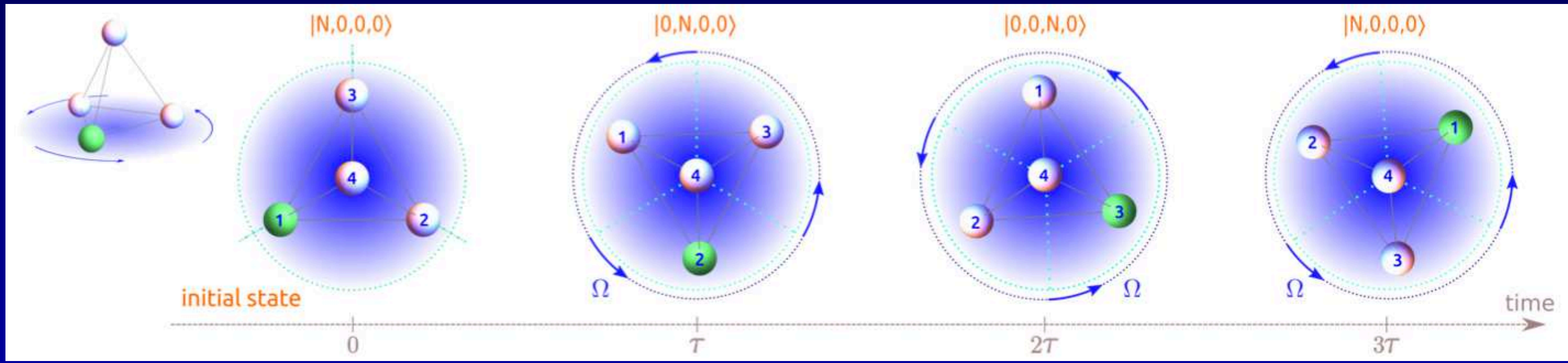
- At $\zeta = \xi/3$, this eq. yields the time interval required for particle transfer between consecutive sites:

$$\tau = \pi/\xi$$

- The system can act as a quantum turntable, capable of spatially transferring different initial states.

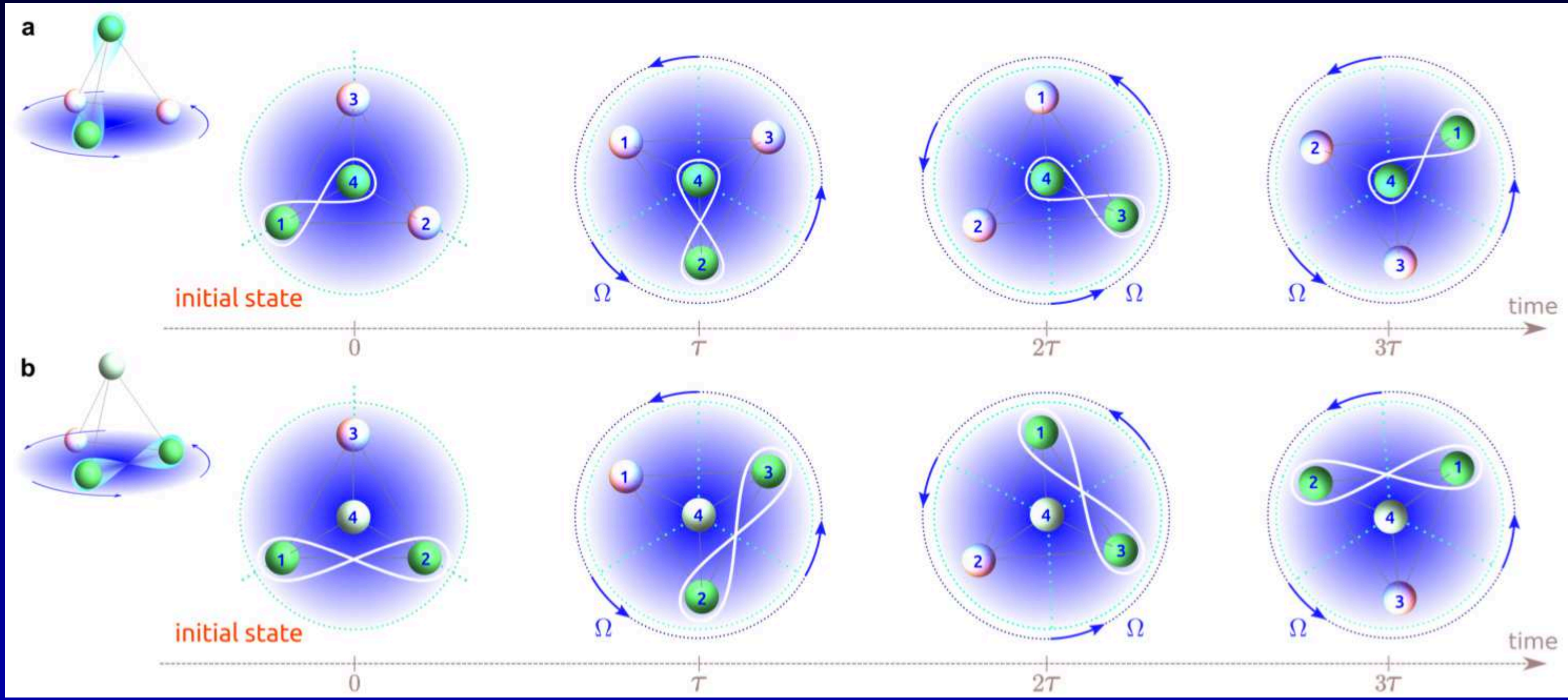
Transfer of Fock states:

- Schematic of Fock state dynamics on a quantum turntable rotating at Ω. Colored spheres represent sites with particles.



For a particular Ω, the initial state $|N,0,0,0\rangle$ evolves to $|0,N,0,0\rangle$ over time interval τ.

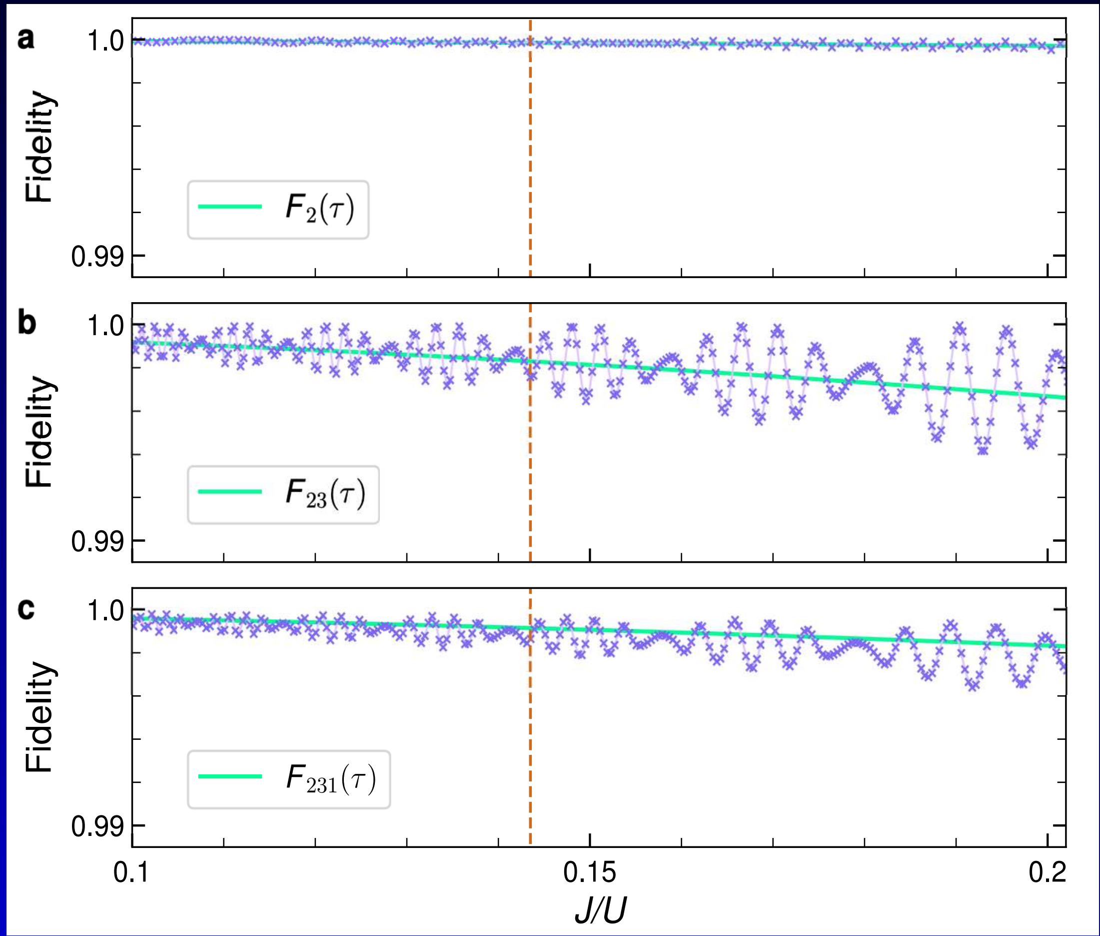
Transfer of NOON states:



a) One of the components of the NOON state remains fixed at site 4, while the other is transferred across sites 1, 2 and 3.

b) The NOON state prepared in sites 1 and 2 is transferred along the neighbouring two-site subsystems.

Fidelity:



Fidelity computations for N = 19 - numerical versus analytic expressions from Bethe Ansatz results. All cases display high-fidelity results greater than 0.99.

4- CONCLUDING REMARKS

Concluding remarks:

- We discussed integrable multi-sites qtum tunneling models, which may be formulated in different configurations and geometries;
- As an application, we showed how to generate and encode a phase into a NOON state by exploring the 4-sites in a ring;
- We showed how to use 4-sites in a tetrahedral geometry to design quantum turntable that spatially transfers entangled states with high fidelity in a rotating frame.

- *Integrable models continue to evolve, showing potential for quantum technologies.*
- *Exactly solved models - past, present and future - are windows into the deeper structure of nature.*
- *Rodney Baxter's clarity, elegance, and curiosity remain timeless.*

Collaborators

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Team in Porto Alegre:

Photo taken in Porto Alegre, September, 2018

Team in Porto Alegre:

Photo taken in Porto Alegre, June, 2025

THANK YOU !!!