Expansion coefficients and the Yang–Baxter equation

Michael Wheeler
University of Melbourne
Joint work with **Ajeeth Gunna**

Motivation

- Symmetric functions are central objects in algebraic combinatorics, representation theory and geometry.

- There are many well-studied bases of the ring of symmetric functions. We understand how to write most of them as partition functions of vertex models.

- The connection with integrability yields many nice features of the functions:

Symmetry	Commuting transfer matrices
Symmetrization identity	Coordinate Bethe Ansatz
Cauchy summation identity	Yang–Baxter algebra or RTT relation

- A much harder problem is to study the expansion coefficients from one basis to another:

$$F_\lambda(x) = \sum_\mu c_{\lambda,\mu} G_\mu(x).$$

Macdonald polynomials

- The Macdonald polynomials $P_\nu(x;q,t)$ depend on an alphabet $x = (x_1, x_2, \dots)$ and two further parameters q, t.
- They are uniquely characterized by the following two properties:

1. $$P_\nu(x;q,t) = m_\nu(x) + \sum_{\mu<\nu} C_{\mu,\nu}(q,t) m_\mu(x),$$
   where $m_\mu(x)$ denotes a monomial symmetric function
   $$m_\mu(x) = \mathrm{Sym}\left(\prod_{i\geqslant 1} x_i^{\mu_i}\right)$$
2. $$\langle P_\lambda, P_\nu \rangle = 0, \qquad \lambda \neq \nu$$
   where the scalar product $\langle \cdot, \cdot \rangle$ is defined via its action on the power sum basis:
   $$\langle p_\lambda, p_\nu \rangle = \delta_{\lambda,\nu} \cdot z_\lambda \cdot \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \qquad p_\lambda(x) = \prod_{i\geqslant 1}\left(\sum_{k\geqslant 1} x_k^{\lambda_i}\right)$$

Macdonald polynomials

- Many reductions are of interest. When $q = t$, one has the following reduction to Schur polynomials:

$$P_\nu(x;q,q) = s_\nu(x) = \det_{1\leqslant i,j\leqslant \ell(\nu)} \left(h_{\nu_i - i + j}(x)\right),$$

where $h_k(x)$ denotes a complete symmetric function

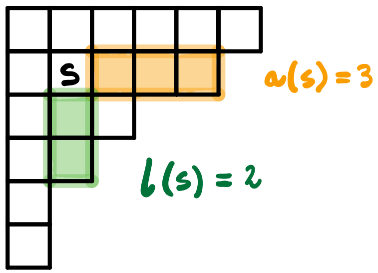
$$h_k(x) = \mathrm{Coeff}_{z^k} \left\{ \prod_{i\geqslant 1} \frac{1}{1 - x_i z} \right\}$$

- In this talk we consider the integral form of the Macdonald polynomials:

$$J_\nu(x;q,t) = c_\nu(q,t) P_\nu(x;q,t)$$

where

$$c_\nu(q,t) = \prod_{s\in\nu} \left(1 - q^{a(s)} t^{l(s)+1}\right)$$



Macdonald polynomials

- The integral Macdonald polynomials take their name from the fact that

$$J_\nu(x;q,t) = \sum_{\mu \leqslant \nu} D_{\mu,\nu}(q,t) m_\mu(x),$$

  where $D_{\mu,\nu}(q,t) \in \mathbb{Z}[q,t]$.

- This was originally conjectured by Macdonald, and later proved by [Sahi 96] and [Knop 97].

- In fact, one can get stronger results for these coefficients.

Macdonald polynomials

- For any $i, j \in \mathbb{Z}_{\geqslant 0}$ and $S \subset \mathbb{Z}^2_{\geqslant 0}$ define

$$f_S^{\pm,i,j}(q,t) = \pm q^i t^j \prod_{(\alpha,\beta)\in S} (1 - q^\alpha t^\beta).$$

Then for any $k \in \mathbb{N}$, define

$$\mathbb{F}_k^{\pm}(q,t) = \left\{ f_S^{\pm,i,j}(q,t) \middle| i,j \in \mathbb{Z}_{\geqslant 0},\ S \subset \mathbb{Z}^2_{\geqslant 0} : |S| = k \right\}.$$

Theorem (Haglund–Haiman–Loehr 06)

There exists a family of tableaux T such that

$$J_\nu(x;q,t) = \sum_{T \text{ of shape } \nu} W_T(q,t) x^T$$

where $W_T(q,t) \in \mathbb{F}^{+}_{|\nu|}(q,t)$ is explicit.

Macdonald into Schur basis

- The topic of this talk is the expansion of $J_\nu(x;q,t)$ over the Schur basis:

$$J_\nu(x;q,t) = \sum_{\lambda \leqslant \nu} E_{\lambda,\nu}(q,t) s_\lambda(x)$$

where $E_{\lambda,\nu}(q,t) \in \mathbb{Z}[q,t]$.

Conjecture (Haglund 10)

For any partitions λ, ν such that $|\lambda| = |\nu|$ and $k \in \mathbb{N}$, one has

$$\frac{E_{\lambda,\nu}(q,q^k)}{(1-q)^{|\nu|}} \in \mathbb{N}[q].$$

Partial progress by [Bhattacharya 22].

Main result

Theorem (Gunna–W 25+)

There exists an explicit partition function Z such that

$$E_{\lambda,\nu}(q,t) = \sum_{\mathcal{C} \text{ with frame } (\lambda,\nu)} W_{\mathcal{C}}(q,t),$$

where $W_{\mathcal{C}}(q,t) \in \mathbb{F}^{\pm}_{|\nu|}(q,t)$ is explicit.

- This just fails, however, to prove Haglund's conjecture:

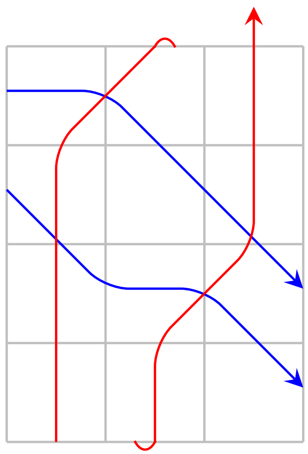
Corollary

$$\frac{E_{\lambda,\nu}(q,q^k)}{(1-q)^{|\nu|}} = \sum_{\mathcal{C} \text{ with frame } (\lambda,\nu)} \operatorname{sgn}(\mathcal{C}) \mathcal{P}_{\mathcal{C}}(q),$$

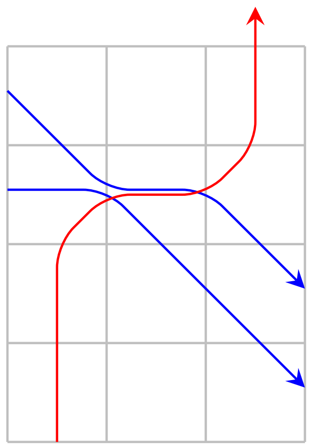
where $\operatorname{sgn}(\mathcal{C}) \in \{\pm 1\}$ may be explicitly defined, and with each $\mathcal{P}_{\mathcal{C}}(q) \in \mathbb{N}[q]$.

Examples

- Here are some examples of the combinatorial formula.
- $\nu = (2)$, $\lambda = (1,1)$



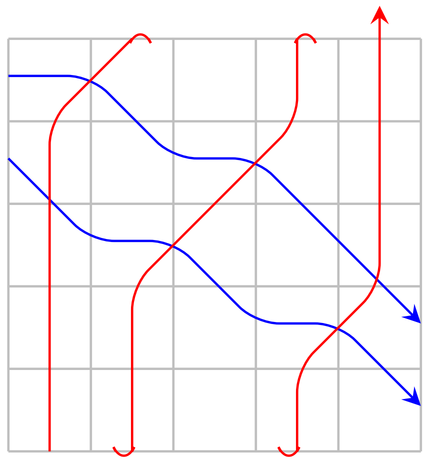
$q(1-t)^2$



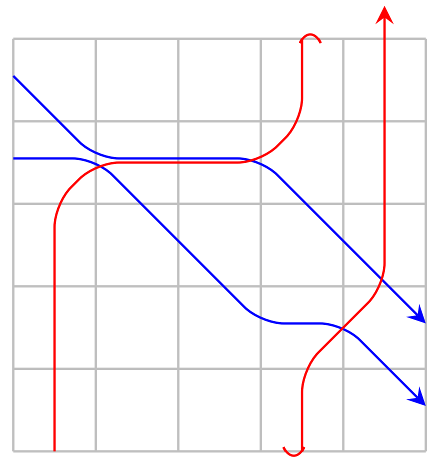
$-(1-q)(1-t)t$

Examples

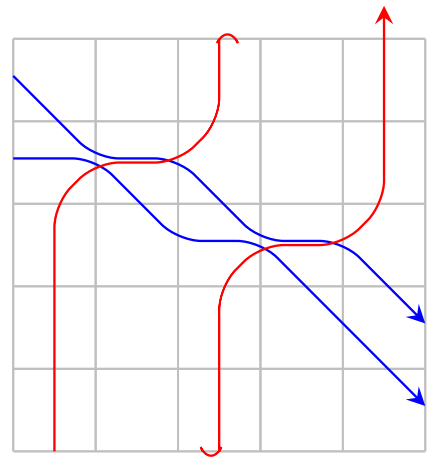
- $\nu = (4)$, $\lambda = (2,2)$



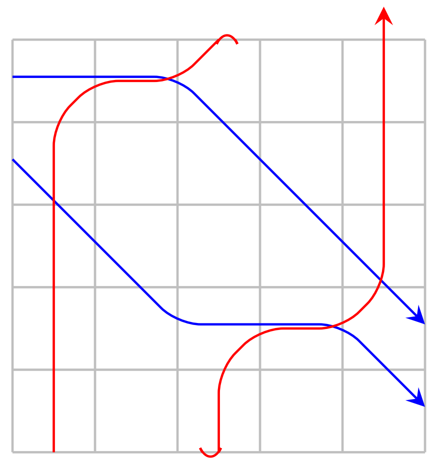
$q^4(1-t)^4$



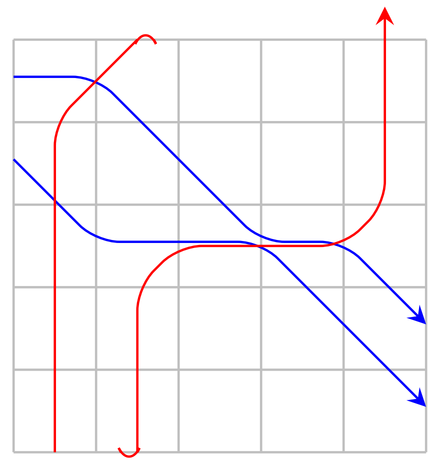
$-q\left(1-q^3\right)(1-t)^2 t\left(1-q^2 t\right)$



$(1-q)q^2\left(1-q^3\right)(1-t)^2 t^2$



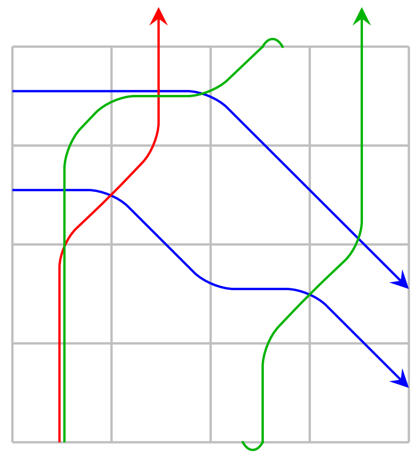
$q^2(1-t)^2(1-qt)\left(1-q^3 t\right)$



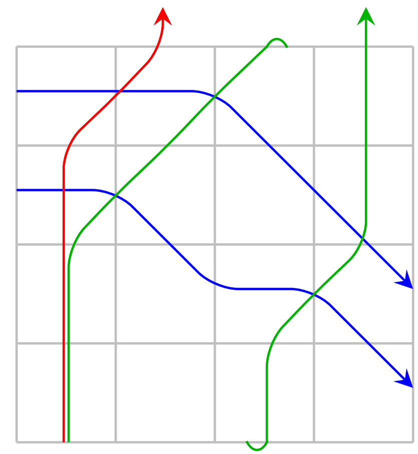
$-(1-q)q^3(1-t)^2 t\left(1-q^2 t\right)$

Examples

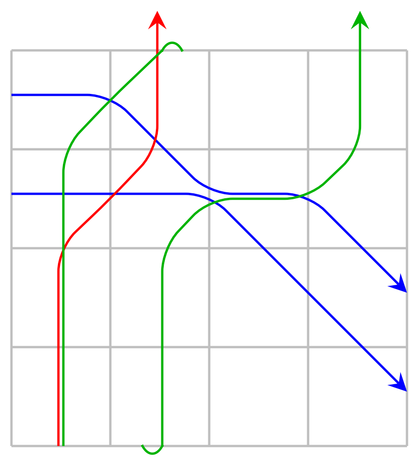
- $\nu = (3,1)$, $\lambda = (2,2)$



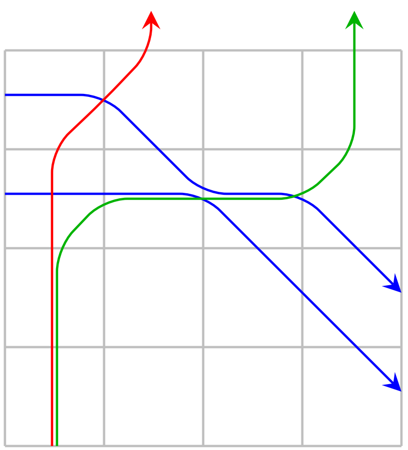
$q(1-t)^3 t\left(1-q^2 t\right)$



$q(1-t)^4$



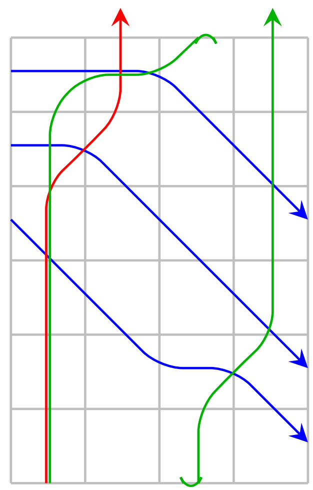
$-(1-q)q^2(1-t)^3 t^2$



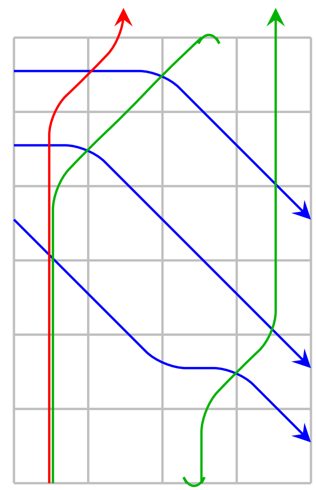
$-(1-q)(1-t)^2 t\left(1-q^2 t\right)$

Examples

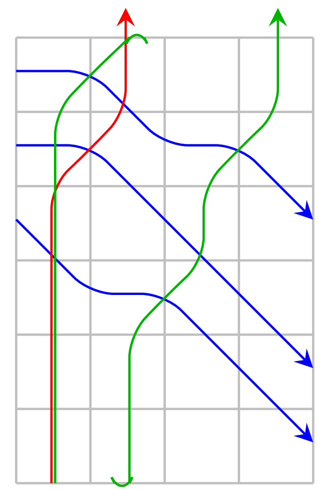
- $\nu = (3,1),\ \lambda = (2,1,1)$



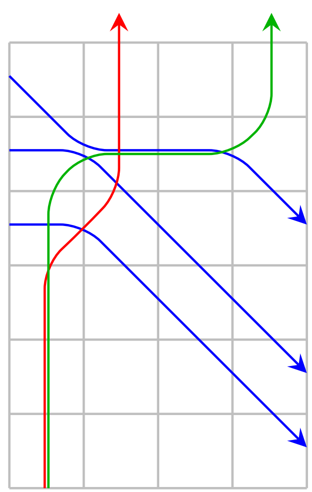
$q(1-t)^3 t\left(1-q^2 t\right)$



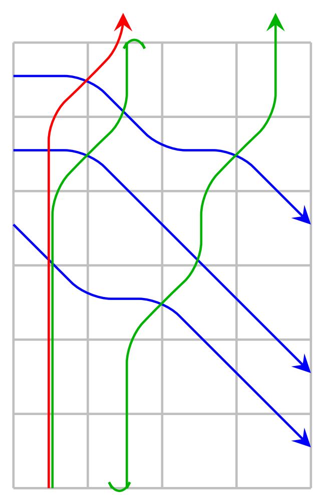
$q(1-t)^4$



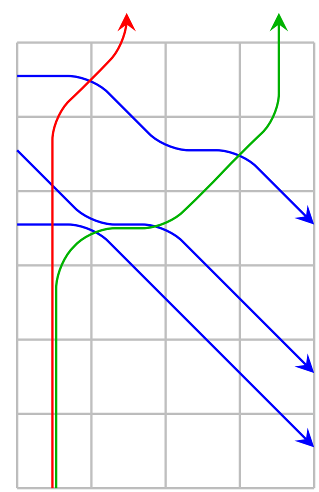
$q^2(1-t)^4 t$



$-\left(1-q^2\right)(1-t)^2 t^2 (1-qt)$



$q^2(1-t)^4$



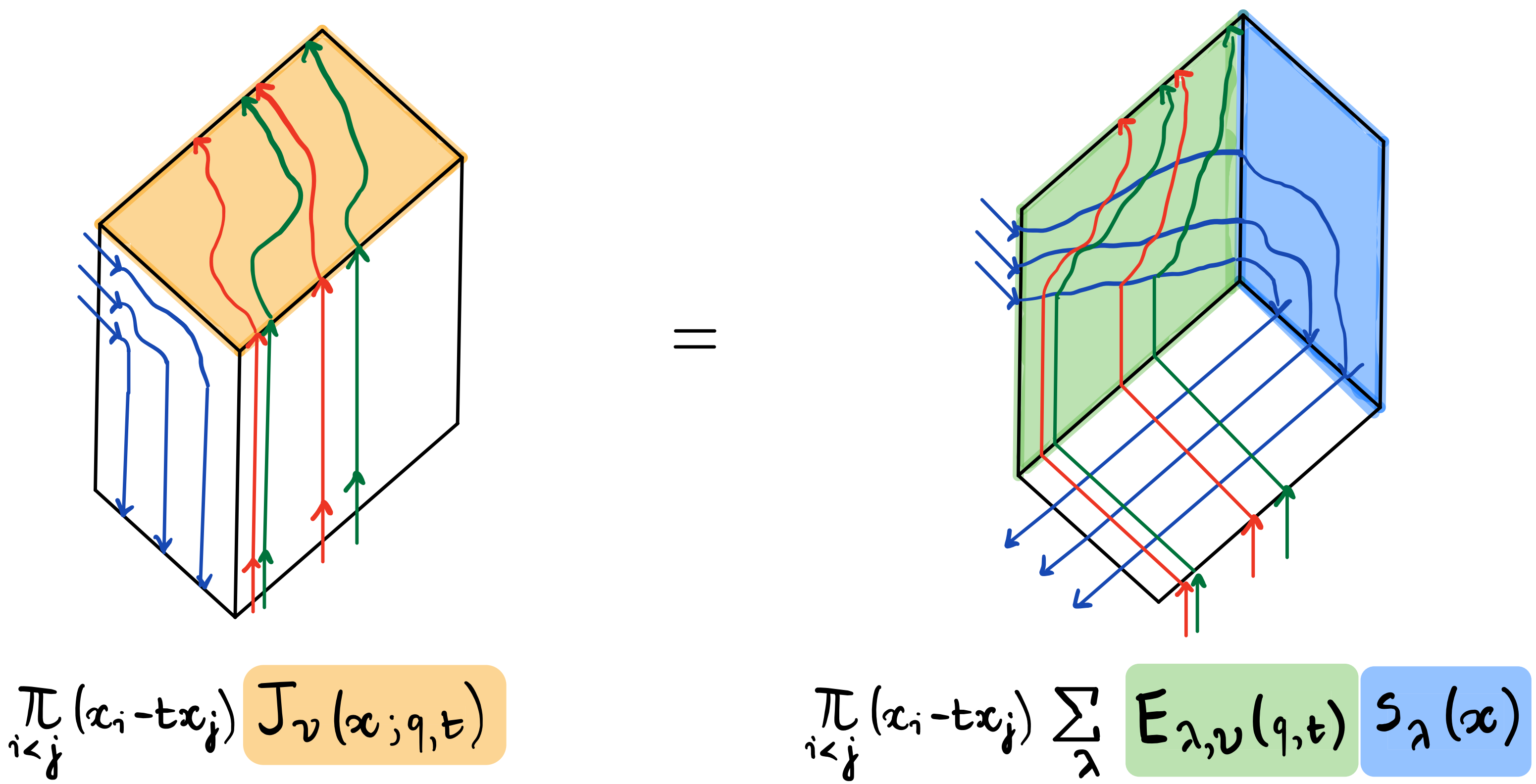
$-\left(1-q^2\right)(1-t)^3 t$

Proof via the Yang–Baxter equation

- The vertex weights used in the construction of Z satisfy a Yang–Baxter equation.
- The verification of the identity

$$J_\nu(x;q,t) = \sum_{\lambda \leqslant \nu} E_{\lambda,\nu}(q,t) s_\lambda(x)$$

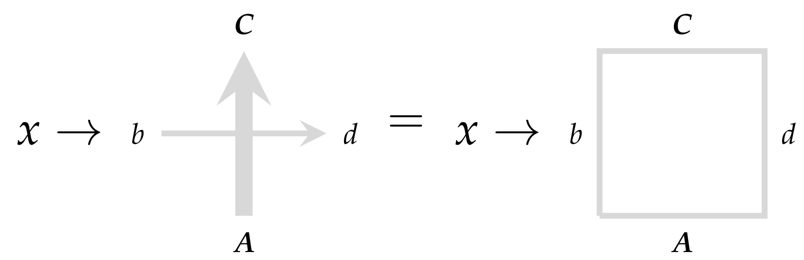
proceeds by analysis of the left and right hand sides of our Yang–Baxter equation:



Macdonald polynomials from vertex models

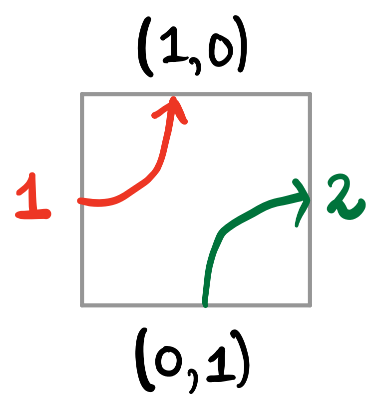
- Consider the following vertex model:

$$w_x(\boldsymbol{A}, b; \boldsymbol{C}, d) =$$



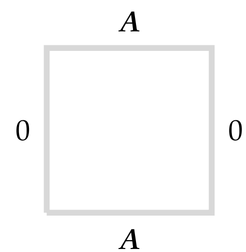
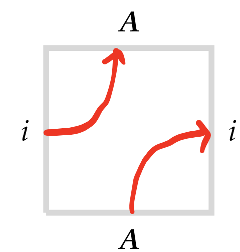
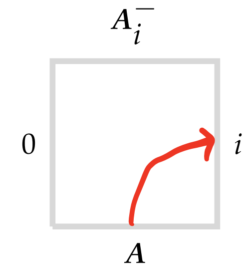
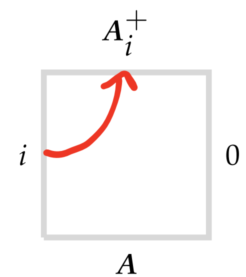
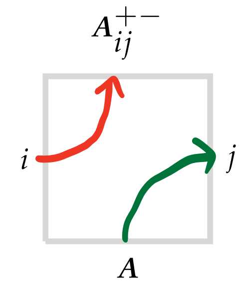
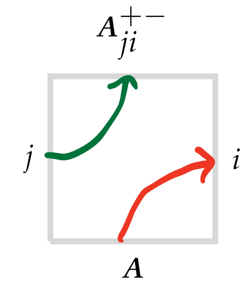
$$\boldsymbol{A}, \boldsymbol{C} \in \{0,1\}^n, \qquad b, d \in \{0,1,\ldots,n\}.$$

- We demand that $w_x(\boldsymbol{A}, b; \boldsymbol{C}, d) = 0$ unless $\boldsymbol{A} + \boldsymbol{e}_b = \boldsymbol{C} + \boldsymbol{e}_d$.



Macdonald polynomials from vertex models

- The nonzero vertices of the model are given as follows:

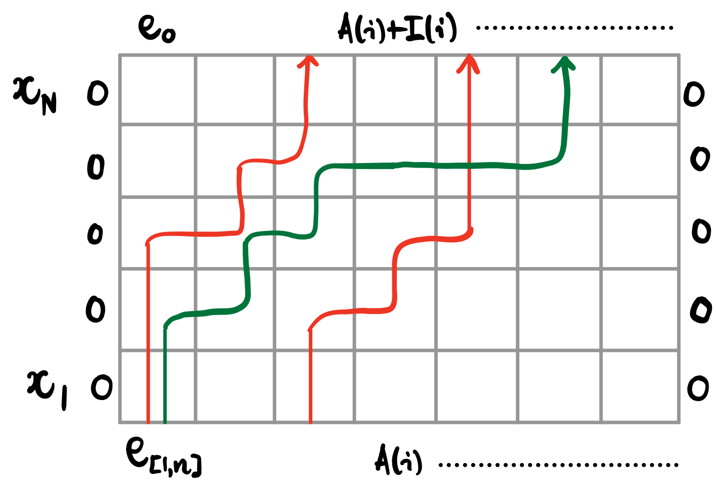
		
1	$(-1)^{A_i} t^{A_{[i,n]}} x$	$(1 - t^{A_i}) t^{A_{(i,n]}} x$
		
1	$(1 - t^{A_j}) t^{A_{(j,n]}} x$	0

where it is assumed that $0 \leqslant i < j \leqslant n$ in this tabulation.

Macdonald polynomials from vertex models

- Fix a composition $\nu = (1 \leqslant \nu_1 \leqslant \cdots \leqslant \nu_n \leqslant M)$. Construct the partition function

$$\mathcal{P}_\nu(x;q,t) = \sum_{\boldsymbol{A}(1),\ldots,\boldsymbol{A}(M)} \prod_{i=1}^{n} \prod_{j=1}^{M} w_{i,j}^{A(j)_i}$$



where

$$\boldsymbol{I}(j) = \sum_{i=1}^{n} \mathbf{1}_{\nu_i = j} \cdot \boldsymbol{e}_i$$

and

$$w_{i,j} = \mathbf{1}_{\nu_i > j} \cdot q^{\nu_i - j}$$

for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant M$.

Macdonald polynomials from vertex models

- Let ν^+ denote the partition obtained by sorting ν.

Theorem (Aggarwal–Borodin–W 21)

$$J_{\nu^+}(x;q,t) = \mathcal{P}_\nu(x;q,t).$$

- This result is motivated by, and proved in completely analogous fashion to an earlier formula for Macdonald polynomials in [Cantini–de Gier–W 15].
- By a relatively simple bijection, one may show that this expression is equivalent to the tableau formula of [Haglund–Haiman–Loehr 06].

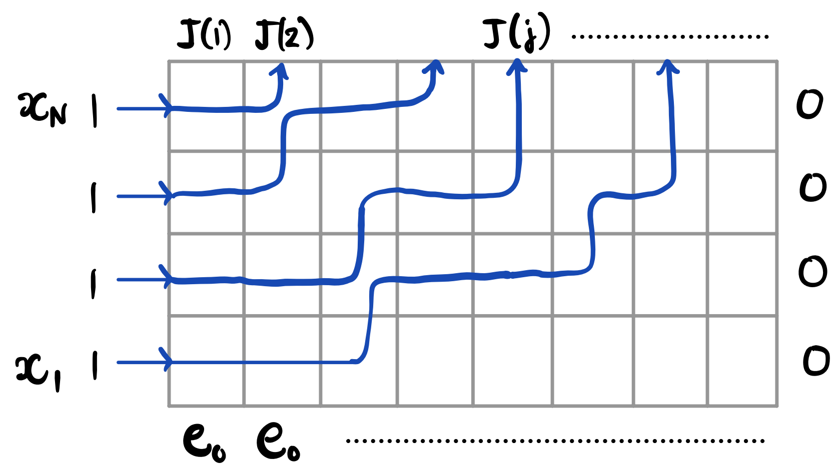
Schur polynomials from vertex models

- Fix a composition $\lambda = (0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_N)$. Define a shifted version of λ as follows:

$$\tilde{\lambda} = \{\lambda_1 + 1, \lambda_2 + 2, \ldots, \lambda_N + N\} \subset \mathbb{N}.$$

We define

$$\mathcal{Q}_\lambda(x;t) =$$



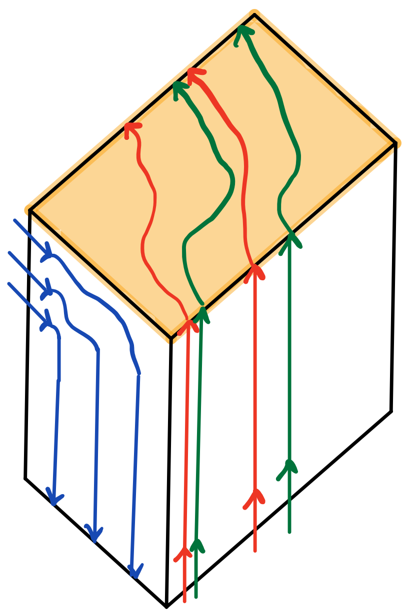
where $\boldsymbol{J}(j) = \mathbf{1}_{j \in \tilde{\lambda}} \cdot e_1$ for all $j \geqslant 1$.

Theorem (Aggarwal–Borodin–W 21)

$$s_{\lambda^+}(x) \prod_{1 \leqslant i < j \leqslant N} (x_i - t x_j) = \mathcal{Q}_\lambda(x;t).$$

Thanks for listening!

$$J_\nu(x;q,t) = \sum_{\lambda \leqslant \nu} E_{\lambda,\nu}(q,t) s_\lambda(x)$$



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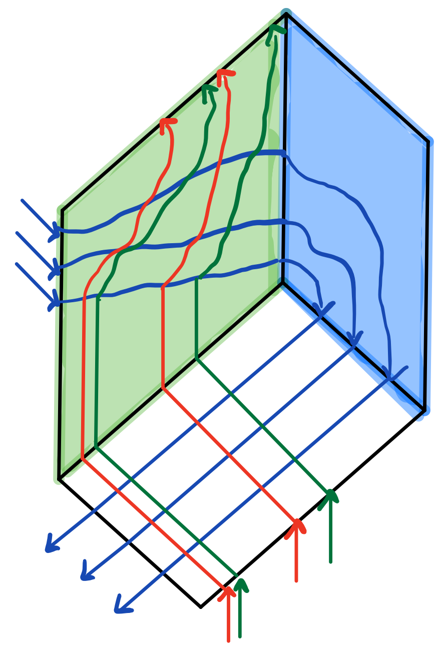

$\prod_{i<j}(x_i - t x_j)\, J_\nu(x;q,t)$

$\prod_{i<j}(x_i - t x_j) \sum_\lambda E_{\lambda,\nu}(q,t)\, s_\lambda(x)$