Free field construction of Heterotic string compactified on Calabi-Yau manifolds of Berglund-Hubsch type in the Batyrev-Borisov combinatorial approach

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PREAMBLE, WHY HETEROTIC STRING

I. What we know from experimental and theoretical physics at the moment about the microstructure of the Universe:

a. Special and General Theories of Relativity,

b. Quantum Field Theory (fundamental object is 0-dimensional particle),

c. SU(3)xSU(2)xU(1) Standard model of Gauge Theory, Vector particles: gluons, photons, W-bosons, Spinor particles: quarks, leptons,

d. Three Generations of the quarks and leptons,

e. Quarks and leptons are chiral.

f. Dark Matter exists.

II. What don't we know and would like to know:

a. How to construct Quantum Gravity (to solve the problem of non-renomalizability),

b. How to unify 3 types of interactions of Elementary particles,

c. How to explain that the number of Generations is namely three,

d. How to unify all 4 types of interactions of the particles,

e. How to explain that the right and the left elementary fermions belong to different representations of the Gauge Group,

f. How to explain that Dark Matter exists; Why do elementary particles exist that are gauge interaction singlets? How many types of such particles are there?

Abstract

Heterotic string models in 4-dimensions are Hybrid of two theories:

of Left-moving $N = 1$ Fermionic string theory with the central charge 15 whose extra 6-dimensions are compactified on a $N = 2$ SCFT theory with the central charge 9, which is equivalent to the compactification on Calabi-Yau manifolds (Gepner)

and of Right-moving Bosonic string theory with the central charge 26, whose 6-dimensions are also compactified on $N = 2$ SCFT theory with the central charge 9,

and whose extra 13 dimensions form the torus of $E(8) \times SO(10)$ Lie algebra.

The important class of exactly solvable Heterotic string models discovered earlier (1987) by D. Gepner correspond to products of $N = 2$ minimal models with total central charge $c = 9$, which is equivalent to compactification on Calabi-Yau manifolds, which belong to the subclass of CY-manifolds of Berglund-Hubsch type.

We generalize this construction to all cases of compactifications on Calabi-Yau manifolds of general Berglund-Hubsch (BH) type.

For this purpose, we use the Batyrev-Borisov combinatorial approach for constructing Calabi-Yau manifolds of BH type and several additional reductions to the BRST reduction of left- and right-vertex algebras.

As a result these models obtain the $N = 1$ Space-Time Supersymmetry, arising from GSO reduction on Left side,

and the Gauge symmetry $E(8) \times E(6)$ Lie algebra (whose subalgebra is $SU(3) \times SU(2) \times U(1)$) arising from a similar reduction on Right side.

We show how the number of 27, $\overline{27}$ representations of $E(6)$ (including quarks and gluons)

and Singlets of $E(6)$ (i.e. Dark matter particles) are determined by the data of reflexive Batyrev polyhedra corresponding to the CY manifold under consideration.

The method uses the Batyrev-Borisov combinatorial approach, to implement a vertex algebra realized by free bosonic and fermionic fields for the states of the Calabi-Yau sector.

Our construction uses the requirement of mutual locality
of Left-moving vertices with $N = 1$ SUSY space-time generators
and of Right-moving vertices with generators of $E(8) \times E(6)$-Gauge symmetry.

After this, the requirement of mutual locality of the products of the Left and Right vertices of physical states with each other together with other requirements of the Conformal bootstrap, leads to a self-consistent result precisely for the above-chosen torus $E(8) \times SO(10)$.

Based on these requirements, we explicitly construct Vertex operators of the physical states of the theory in the following order.

The initial product of two Conformal field theories

The construction of Heterotic string starts with a theory that is the product of two Conformal Field theories:
$N = 1$ SCFT on the Left, holomorphic side and
$N = 0$ CFT on the Right, antiholomorphic side.

In Left sector we have the product of the 4-dimensional $N = 1$ CFT for the subsector of Space-Time with central charge 6, and the $N = 1$ CFT for the compact Calabi-Yau subsector with central charge 9, so that the total central charge in the left sector is $c = 15$.

$N = 1$ SCFT of Left ST subsector is the theory of 4 free bosons $x^\mu(z)$ and 4 Majorana fermions $\psi^\mu(z)$.

$N = 1$ SCFT of Left CY subsector with central charge 9 is the theory of free bosons $X_i^\pm(z)$ and Majorana fermions $\Psi_i^\pm(z)$, where $i = 1, ..., 5$.

The algebra of vertex operators in this subsector is determined by the data of a Calabi-Yau manifold of Berglund-Hubsch type.
Left-moving energy-momentum tensor is (others will be defined later)

$$T^L(z) = \frac{1}{2}\eta_{\mu\nu}\partial x^\mu \partial x^\nu + \frac{1}{2}\eta_{\mu\nu}\psi^\mu(z)\partial\psi^\nu(z) + T^L_{CY}(z).$$

The initial product of two Conformal field theories

In Right sector we have the product of four $N=0$ Conformal Field theories:

CFT Space-Time subsector with central charge 4, realized by 4 bosonic fields $\bar{x}^{\mu}(\bar{z})$,
CFT $E(8)$ subsector with the central charge 8, realized by 8 free boson fields $Y_I(\bar{z}), I=1,...,8$ compactified on the torus of the $E(8)$ algebra,
CFT $SO(10)$ subsector with the central charge 5, realized by 5 free bosonic fields $\Phi_{\alpha}(\bar{z}), \alpha=1,...,5$ compactified on the torus of the $SO(10)$ algebra,
CFT Calabi-Yau subsector with central charge 9.

So that the total central charge in the right sector is $c=26$.

The CFT of CY subsector with the central charge 9 is realized similarly to the left one by free bosons $\bar{X}_i^{\pm}(\bar{z})$ and Majorana fermions $\bar{\Psi}_i^{\pm}(\bar{z})$, where $i=1,...,5$.
Right-moving energy-momentum tensor is

$$T^R(\bar{z}) = \frac{1}{2}\eta_{\mu\nu}\bar{\partial}\bar{x}^{\mu}\bar{\partial}\bar{x}^{\nu} + \frac{1}{2}(\bar{\partial}Y_I)^2 + \frac{1}{2}(\bar{\partial}\Phi_{\alpha})^2 + T^R_{CY}(\bar{z}).$$

Following to Borisov we construct of N=2 SCFT models corresponding to Calabi-Yau models of the Berglund-Hubsch type in terms the free fields.

The $N=2$ Virasoro superalgebra generators in this construction are expressed in terms of the free bosons $X_i^{\pm}(z)$, the Majorana fermions $\Psi_i^{\pm}(z)$, where $i=1,...,5$.

We will also use the free boson fields $H_i, i=1,...,5$, in terms of which the Majorana fermions are expressed as $\Psi_i^{\pm}(z)=\exp[\pm iH_i(z)]$.
The operator product expansion (OPE) of these fields looks as follows

$$X_i^+(u)X_j^-(z)=\delta_{ij}\log(u-z)+...,$$
$$\psi_i^+(u)\psi_j^-(z)=\delta_{ij}(u-z)^{-1}+...,$$
$$H_i(u)H_j(z)=-\delta_{ij}\log(u-z)+....$$

N= 2 Super-Virasoro algebra

The currents $T_{CY}(z)$, $G^{\pm}_{CY}(z)$ and the $U(1)$ current $J_{CY}(z)$ that form $N=2$ Virasoro algebra of Calabi-Yau subsectors look as follows

$$T_{CY}(z)=\sum_{i=1}^{5}[\partial X_i^+\partial X_i^- +\frac{1}{2}(\partial\Psi_i^+\Psi_i^- +\partial\Psi_i^-\Psi_i^+)+\frac{1}{2}(a_i^+\partial^2 X_i^- +a_i^-\partial^2 X_i^+)],$$

$$G^{+}_{CY}(z)=\sqrt{2}\sum_{i=1}^{5}[(\Psi_i^+\partial X_i^- +\partial\Psi_i^+ a_i^-)],$$

$$G^{-}_{CY}(z)=\sqrt{2}\sum_{i=1}^{5}[(\Psi_i^-\partial X_i^+ +\partial\Psi_i^- a_i^+)],$$

$$J_{CY}(z)=\sum_{i=1}^{5}[\Psi_i^+\Psi_i^- +a_i^+\partial X_i^- -a_i^-\partial X_i^+].$$

It is also be useful to represent the currents $T_{CY}(z)$ and $J_{CY}(z)$ in the following equivalent bosonic form

$$T_{CY}(z)=\sum_{i=1}^{5}[\partial X_i^+\partial X_i^- -\frac{1}{2}(\partial H_i)^2+\frac{1}{2}(a_i^+\partial^2 X_i^- +a_i^-\partial^2 X_i^+)],$$

$$J_{CY}(z)=\partial\sum_{i=1}^{5}[iH_i+a_i^+X_i^- -a_i^-X_i^+]=\partial H_{CY}(z).$$

Calabi-Yau manifolds and N=2 SCFT

The vectors $\vec{a}^{+}$ and $\vec{a}^{-}$ depend on the Calabi-Yau manifold selected for compactification on it.
(These vectors change the value of the central charge, as was the case in the Dotsenko-Fateev design of the Virasoro minimal models.)

Below we will show how to define $\vec{a}^{+}$ and $\vec{a}^{-}$, and also that the scalar product of these vectors is equal 1.

The central charge c of the $N = 2$ Virasoro algebra under consideration is expressed through these vectors as follows

$$\frac{c}{3} = 5 - 2\sum_{i=1}^{5} a_i^{+} a_i^{-} .$$

These statements will be explained below.
The central charge of the left and right Calabi-Yau subsectors is 9, which is necessary for Heterotic string theory to be self-consistent.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

Calabi-Yau manifolds of BH type are defined as a hypersurface in the weighted projective space $P_{\vec{k}}$ by the equation

$$W(y_1, ..., y_5) = \sum_{i=1}^{5} \prod_{j=1}^{5} y_j^{A_{ij}} = 0.$$

Here $\vec{k} = k_1, ..., k_5$, where $k_i, i = 1, ..., 5$ are the weights of $P_{\vec{k}}$.
$W(y_1, ..., y_5)$ is a nondegenerated polynomial with invertible integer matrix A_{ij}.
It is assumed that the variables y_i have positive rational degrees $q_i = \frac{k_i}{d}$, and $d = \sum_{i=1}^{5} k_i$, such that $\sum A_{ij} q_j = 1$ for all i.
The polynomial W and the monomial $\prod_{i=1}^{5} y_i$ are invariant under the substitution $y_j \to \exp(i2\pi q_j) y_j$.

Such a symmetry group G described above always exists, call it the "minimum admissible group."The maximum allowed group can be larger.

The mirror CY-manifold is defined by the mirror polynomial W^T with the transposed matrix A^T_{ij} in the mirror projective space $P^*_{\vec{k}}$, and by the dual admissible group G^T.
This group is defined based on some duality requirements.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

From the described Berglund-Hubsch data, the potential W and the admissible group G, one can obtain the Batyrev-Borisov combinatorial data, which will then be used in constructing the Calabi-Yau sectors in the Heterotic string.

Namely, this can be done as follows. Let M_0 and N_0 be two integer 5-dimensional lattices with bases $\vec{u_i}$ and $\vec{v_j}$, whose pairing is

$$\vec{u_i} \cdot \vec{v_j} = A_{ij},$$

where A_{ij} are the exponents of the potential W.

The solution to these equations can be chosen as follows

$$(\vec{u_i})_j = A_{ij}, \quad (\vec{v_i})_j = \delta_{ij}.$$

We define vectors $\vec{a}^{+}$ and $\vec{a}^{-}$, which are related to vectors $\vec{u_i}$ and $\vec{v_j}$ as follows

$$\vec{a}^{+} = \frac{1}{d^*}\sum_i k_i^* \vec{u_i}, \quad \vec{a}^{-} = \frac{1}{d}\sum_j k_j \vec{v_j}.$$

It is easy to verify that these vectors satisfy to the following equations

$$\vec{u_i} \cdot \vec{a}^{-} = 1, \quad \vec{a}^{+} \cdot \vec{v_j} = 1.$$

Berglund-Hubsch type CY manifolds and N=2 SCFT models

The next step is to extend the lattices M_0 and N_0 to a pair of dual Batyrev lattices M and N for the case when the group G for the polynomial W is minimal.

First, we extend the lattice N_0 to the lattice $N = N_0 + \vec{a}^-$ which includes the vector $\vec{a}^-$.

In the next step we find the basis of the lattice N, we denote its elements as $\vec{e^*}_{\beta=1,\ldots,5}$.

Then we find the basis of the dual lattice M as the five vectors $\vec{e}_\alpha$ whose pairing with $\vec{e}^*_\beta$, $(\alpha, \beta = 1, ..., 5)$ is

$$\vec{e}_\alpha \cdot \vec{e^*}_\beta = \delta_{\alpha,\beta}.$$

In what follows, an important role in the construction of Heterotic strings will be played by elements $\vec{m} \in M$ and $\vec{n} \in N$ belonging to M and N, and especially those of them that belong to the reflexive Batyrev polyhedra Δ^+ and Δ^- inside these lattices.

The latter means means that $\vec{m} \in \Delta^+$ and $\vec{n} \in \Delta^-$, if

$$\vec{m} \cdot \vec{a}^- = 1, \quad \vec{a}^+ \cdot \vec{n} = 1.$$

From which it follows that the central charge in the subsector CY is equal to 9.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

In constructing the vertex algebra of Heterotic string we will use vertices corresponding to the points of the dual lattices M and N.

To construct fermion states we also need to use in both, Left and Right sectors vertices corresponding to $M \pm \frac{1}{2}\vec{a^+}$ and $N \pm \frac{1}{2}\vec{a^-}$ elements.

The reason for this extension is that we want to build Heterotic string theory that includes SCFT of CY as subsector.

Therefore, since we have a diagonal $N = 1$ SCFT in the left sector (in both the Space-Time and CY subsectors), therefore in both its subsectors the vertex

operators must simultaneously belong to either the Ramond (R) or the Neveu-Schwatz (NS) type. For the theory to be consistent, we need both options.

Moreover, in the absence of vertex algebras of both types (NS-type and R-type), we cannot obtain Space-Time SUSY.

A similar situation occurs in the right sector.

Following to Borisov we define the Vertex algebra of Heterotic string as the Cohomology in respect to the set diffentials $D_{\vec{m}}$ and $D_{\vec{n}}$

$$D_{\vec{m}} = \oint du\ \vec{m}\cdot\vec{\Psi}^{-}(u)\ \exp(\vec{m}\cdot\vec{X^{-}}(u)) = \sum_{i=1}^{5} m_i \oint du \exp(-iH_i + \vec{m}\cdot\vec{X^{-}}(u)),$$

$$D_{\vec{n}} = \oint du\ \vec{n}\cdot\vec{\Psi}^{+}(u)\ \exp(\vec{n}\cdot\vec{X^{+}}(u)) = (\sum_{i=1}^{5} n_i \oint du \exp(iH_i + \vec{n}\cdot\vec{X^{+}}(u)),$$

$$D_{\vec{m}}^2 = D_{\vec{n}}^2 = \{D_{\vec{m}}, D_{\vec{m}'}\} = \{D_{\vec{n}}, D_{\vec{n}'}\} = \{D_{\vec{m}}, D_{\vec{n}}\} = 0;$$

where $\vec{m}\in\Delta^{+}$ and $\vec{n}\in\Delta^{-}$.

The integrands in the definition of the differentials $D_{\vec{m}}$ and $D_{\vec{m}}$, as can be verified, are BRST-invariant total vertices of the Heterotic string.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

Calabi-Yau multiplier of any total vertex operator of Heterotic string contains an exponential factor of the following form

$$\exp\left(i\sum_{i=1}^{5} S_i H_i + \vec{p}\cdot\vec{X}^{-} + \vec{q}\cdot\vec{X}^{+}\right),$$

whose dimensions is

$$\Delta(\vec{S},\vec{p},\vec{q}) = \frac{1}{2}\vec{S}^2 + \vec{p}\cdot\vec{q} + \frac{1}{2}\left(\vec{p}\cdot\vec{a}^{-} + \vec{q}\cdot\vec{a}^{+}\right).$$

These CY factors of the vertex operators must have the following properties.

Firstly, their vectors $\vec{S}$, $\vec{p}$, $\vec{q}$ must be elements of three 5 dimensional lattices to satisfy the OPE axioms.

Secondly, their dimensions must be equal, as we see, to 0 or 1/2 modulo an integer in the NS-case, or equal to 3/8 modulo an integer in the R-case.

Thirdly, they must be be mutually local with the differentials $D_{\vec{m}}$ and $D_{\vec{m}}$. For this, the parameters $\vec{S}$, $\vec{p}$, $\vec{q}$ must satisfy the requirement that the pairing $(\vec{p},\vec{a}^{-})$ and $(\vec{a}^{+},\vec{n})$ be integers or half -integers for all $\vec{p}$ and $\vec{q}$.

It follows that if the vertex operator belongs to the NS-sector, then all S_i are integers, $\vec{p}\in M$ and $\vec{q}\in N$, and if the vertex operator belongs to the R-sector, then all S_i are half-integers, $\vec{p}\in M\pm\frac{1}{2}\vec{a}^{+}$ and $\vec{q}\in N\pm\frac{1}{2}\vec{a}^{-}$.

Berglund-Hubsch type CY manifolds and N=2 SCFT models

We call the subset of states belonging to the intersection of the BRST cohomology set defined by the differential Q_{BRST} and Borisov cohomology set defined by differentials D_m and D_n Space of Quasi-Physical states.

Below we perform two additional reductions of this set that lead

to an extension in Left sector of Poincaré symmetry to $N = 1$ space-time Supersymmetry, and

in Right sector to an extension of $E(8) \times SO(10)$ to $E(8) \times E(6)$.

It is this set that will be the space of physical states of Heterotic string theory.

Left-moving sector, N=1 SCFT

In Left sector we have the product of 4-dimensional $N = 1$ SCFT for Left space-time subsector consisting of 4 bosons and 4 Majorano fermions with the central charge of 6 and
$N = 1$ SCFT for compact CY subsector with the central charge of 9, so that the total central charge in Left sector is $c = 15$.

The $N = 1$ SCFT of Left Space-Time subsector is a theory of 4 free bosonic fields $x^{\mu}(z)$ and 4 Majorana fermion fields $\psi^{\mu}(z)$

$$x^{\mu}(z)x^{\nu}(0) = -\eta^{\mu\nu} \log z + ...,$$
$$\psi^{\mu}(z)\psi^{\nu}(0) = \eta^{\mu\nu} z^{-1} + ...,$$

As for the $N = 1$ SCFT of Left Calabi-Yau subsector, this is the above-described theory of free bosons $X_i^{\pm}(z)$, fermions $\Psi_i^{\pm}(z)$ and free boson fields $H_i, i = 1, ..., 5$ bosonizing the fermions.

Here the $N = 1$ symmetry corresponds to the subalgebra of the $N = 2$, which was defined above when representing the CY subsectors.

Left-moving sector, N=1 SCFT

The total left-moving $N = 1$ Virasoro algebra is the diagonal subalgebra in the direct sum of the Calabi-Yau compact subsector, introduced above and of the $N = 1$ Virasoro algebras of space-time degrees of freedom

$$
\begin{aligned}
&T^L(z) = T^L_{ST}(z) + T^L_{CY}(z),\\
&G^L(z) = G^L_{ST}(z) + G^L_{CY}(z),\\
&T^L_{ST} = -\frac{1}{2}\partial x^{\mu}(z)\partial x_{\mu}(z) - \frac{1}{2}\psi^{\mu}(z)\partial\psi_{\mu}(z),\\
&G_{ST}(z) = \partial x^{\mu}\psi_{\mu}(z),\\
&GL_{CY}(z) = G^{+}_{CY} + G^{-}_{CY}(z).
\end{aligned}
$$

This $N = 1$ Virasoro superalgebra action is correctly defined on the product of only NS-representations or on the product of only R-representations.

Left-moving sector, N=1 SCFT

We use BRST approach to define the physical states.

The BRST charge is given by the integral

$$Q_{\mathrm{BRST}} = \oint dz[cT_{mat} + \gamma G_{mat} + \frac{1}{2}(cT_{gh} + \gamma G_{gh})],$$

where we introduced the ghost fields and $N = 1$ Virasoro superalgebra of the ghosts

$$\beta(z)\gamma(0) = -z^{-1} + ..., \quad b(z)c(0) = z^{-1} + ...$$

$$T_{\mathrm{gh}} = -\partial bc - 2b\partial c - \frac{1}{2}\partial\beta\gamma - \frac{3}{2}\beta\partial\gamma,$$

$$G_{\mathrm{gh}} = \partial\beta c + \frac{3}{2}\beta\partial c - 2b\gamma.$$

Left-moving sector, N=1 SCFT

The ghost space of states is characterized by the vacuum $V_q(z)$, which can be realized as a free scalar field exponent

$$V_q(z) = \exp\left(q\phi(z)\right),\ \phi(z)\phi(0) = -\log(z) + \ldots$$

The left-moving vertex can be written as

$$V^L_{\vec{\mu}} = P(\partial^k x^\mu, \partial^l \tilde{H}_a, \partial^r H_i, \partial^s X^+_i, \partial^t X^-_i)\times$$
$$\times \exp\left(q\phi + \imath\lambda^a \tilde{H}_a + i\sum_{i=1}^{5} S_i H_i + \vec{p}\vec{X}^- + \vec{q}\vec{X}^+ + \imath p_\mu x^\mu(z)\right).$$

Here P is a polynomial of the derivatives of the corresponding boson fields including the fields $\tilde{H}_a, a = 1, 2$.
These fields bosonize the Fermi fields of spacetime subsector as follows

$$\tilde{H}_a(z)\tilde{H}_b(0) = -\delta_{ab}\log(z) + \ldots,\ a, b = 1, 2.$$
$$\frac{1}{\sqrt{2}}(\pm\psi^0 + \psi^1) = \exp[\pm\imath\tilde{H}_1],\ \frac{1}{\sqrt{2}}(\psi^2 \pm \imath\psi^3) = \exp[\pm\imath\tilde{H}_2].$$

Left-moving sector, N=1 SCFT

The dimension of the left-moving vertex $V^L_{\vec{\mu}}$

$$\Delta^L(\vec{\mu}) = \Delta_{gh}(q) + \Delta_{ST} + \Delta^L_{CY},$$

where

$$\Delta_{gh}(q) = -\frac{q(q+2)}{2},$$
$$\Delta^L_{ST} = \frac{1}{2}\vec{\lambda}^2,$$
$$\Delta^L_{CY} = \frac{1}{2}\vec{S}^2 + \vec{p}\cdot\vec{q} + \frac{1}{2}\vec{p}\cdot\vec{a^-} + \frac{1}{2}\vec{q}\cdot\vec{a^+}.$$

The phase of the vertices $V^L_{\vec{\mu}}(u)$ and $V^L_{\vec{\mu'}}(z)$

$$2\pi i\ \vec{\mu}\cdot\vec{\mu} = \Delta^L(\vec{\mu}+\vec{\mu'}) - \Delta^L(\vec{\mu}) - \Delta^L(\vec{\mu'}).$$

It follows that

$$\vec{\mu}\cdot\vec{\mu}' = -qq' + \lambda\cdot\lambda' + \vec{S}\cdot\vec{S'} + \vec{p}\cdot\vec{q'} + \vec{p'}\cdot\vec{q}.$$

The vector λ^a in $\exp(\imath\lambda^a\tilde{H}_a)$ must satisfy the requirement of consistency with the structure $N=1$ on the left side.
Therefore, in NS sector the vectors $\vec{\lambda}$ fall into the classes [0] and [V], and in R sector the vectors $\vec{\lambda}$ fall into the classes [S] and [C] of $SO(1,3)$ lattice.

The integrality of $\vec{\mu}\cdot\vec{\mu}'$ is the condition of mutual locality of two vertices.

Massless Left movers and N=1 Space-Time supersymmetry

The vertices of massless physical states play two roles:

The first is that it is the states of this set that must correspond to the observed elementary particles.
The first is that it is the such states must correspond to the observed elementary particles.

The second is that some of such vertices can be used to extend the symmetry of the theory.
Namely, they can be taken as currents whose integrals become additional generators for this extension.

Requiring the vertices to be Cohomologies of Q_{BRST}, $D_{\vec{m}}$ and $D_{\vec{n}}$ simultaneously in NS-subsector we find the left-moving vertex of massless vector boson

$$\exp(-\phi(z))\psi^{\nu}(z)\exp\left(\imath p_{\mu}x^{\mu}(z)\right),$$

which can be rewritten using bosonization as

$$\exp\left(-\phi(z)+\imath\lambda^{a}\tilde{H}_{a}+\imath p_{\mu}x^{\mu}(z)\right),\vec{\lambda}=(\pm1,0),\vec{\lambda}=(0,\pm1).$$

Also we find the left-moving vertex of massless scalar bosons

$$V_{\vec{m}}=\exp\left(-\phi+\vec{m}\cdot\vec{X}^{-}+\imath p_{\mu}x^{\mu}(z)\right),$$

$$V_{\vec{n}}=\exp\left(-\phi(z)+\vec{n}\cdot\vec{X}^{+}+\imath p_{\mu}x^{\mu}(z)\right),$$

where $\vec{m}\in\Delta^{+}$ and $\vec{n}\in\Delta^{-}$.

Massless Left movers and N=1 Space-Time supersymmetry

In Ramond subsector with canonical picture number $(-\frac{1}{2})$ we find the vertices of the massless spinors

$$J^{\pm}(\vec{\sigma},\vec{S}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} + \imath\vec{S}\cdot\vec{H} \pm \frac{1}{2}(\vec{X}^{+}\cdot\vec{a}^{-} - \vec{X}^{-}\cdot\vec{a}^{+})\right)\exp(\imath p_{\mu}x^{\mu}(z)),$$

$$J^{\pm}(\dot{\vec{\sigma}},\dot{\vec{S}}) = \exp\left(-\frac{1}{2}\phi + \imath\dot{\vec{\sigma}}\cdot\vec{\tilde{H}} + \imath\dot{\vec{S}}\cdot\vec{H} \pm \frac{1}{2}(\vec{X}^{+}\cdot\vec{a}^{-} - \vec{X}^{-}\cdot\vec{a}^{+})\right)\exp(\imath p_{\mu}x^{\mu}(z)),$$

$$J^{\pm}(\dot{\vec{\sigma}},\vec{S}) = \exp\left(-\frac{1}{2}\phi + \imath\dot{\vec{\sigma}}\cdot\vec{\tilde{H}} + \imath\vec{S}\cdot\vec{H} \pm \frac{1}{2}(\vec{X}^{+}\cdot\vec{a}^{-} - \vec{X}^{-}\cdot\vec{a}^{+})\right)\exp(\imath p_{\mu}x^{\mu}(z)),$$

$$J^{\pm}(\vec{\sigma},\dot{\vec{S}}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} + \imath\dot{\vec{S}}\cdot\vec{H} \pm \frac{1}{2}(\vec{X}^{+}\cdot\vec{a}^{-} - \vec{X}^{-}\cdot\vec{a}^{+})\right)\exp(\imath p_{\mu}x^{\mu}(z)),$$

where

$$\sigma^{a} = \pm\frac{1}{2},\ \sum_{a=1}^{2}\sigma^{a} = \pm 1, \qquad \dot{\sigma}^{a} = \pm\frac{1}{2},\ \sum_{a=1}^{2}\dot{\sigma}^{a} = 0,$$

and

$$S_i = \pm\frac{1}{2},\ \sum_{i=1}^{5} S_i = \frac{1}{2}, \quad \text{mod}\ \ 2 \quad \dot{S}_i = \pm\frac{1}{2},\ \sum_{i=1}^{5}\dot{S}_i = -\frac{1}{2}, \quad \text{mod}\ \ 2\ \ .$$

Massless Left movers and N=1 Space-Time supersymmetry

All of these vertices are BRST cohomology, but the ones that are also cohomology of $D_{\vec{m}}$ and $D_{\vec{n}}$ are only the following vertices

$$J_{\sigma}^{+} = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} + \frac{1}{2}H_{CY}^{L}\right),$$
$$J_{\dot{\sigma}}^{-} = \exp\left(-\frac{1}{2}\phi + \imath\dot{\vec{\sigma}}\cdot\vec{\tilde{H}} - \frac{1}{2}H_{CY}^{L}\right),$$
$$J_{\dot{\sigma}}^{+} = \exp\left(-\frac{1}{2}\phi + \imath\dot{\vec{\sigma}}\cdot\vec{\tilde{H}} + \frac{1}{2}H_{CY}^{L}\right),$$
$$J_{\sigma}^{-} = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} - \frac{1}{2}H_{CY}^{L}\right),$$

where $H_{CY}^{L}(z) = \sum_{i=1}(iH_i + a_i^{+}X_i^{-} - a_i^{-}X_i^{+})$ and we have omitted the factors $\exp(\imath p_{\mu}x^{\mu}(z))$.
The first two currents J_{σ}^{+} and $J_{\dot{\sigma}}^{-}$ are mutually local, as are the other two currents.
We will use the first pair to extend Poincaré symmetry to $N = 1$ spacetime supersymmetry.

Massless Left movers and N=1 Space-Time supersymmetry

We select the pair of currents $J^{+}(\vec{\sigma}, \vec{S})$ to determine $N = 1$ super-Poincaré supercharges as follows

$$\mathcal{Q}_{\sigma} = \oint du\ J^{+}_{\sigma}(u) = \oint du \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} + \frac{1}{2}H^{L}_{CY}\right),$$

$$\mathcal{Q}_{\dot{\sigma}} = \oint du\ J^{-}_{\dot{\sigma}}(u) = \oint du \exp\left(-\frac{1}{2}\phi + \imath\dot{\vec{\sigma}}\cdot\vec{\tilde{H}} - \frac{1}{2}H^{L}_{CY}\right).$$

The supercharges $\mathcal{Q}_{\sigma}$ and $\mathcal{Q}_{\dot{\sigma}}$ are spinors with respect to the Poincaré algebra and, together with the generators of this algebra, P_{μ} and $J_{\mu\nu}$, form $N = 1$ Poincaré superalgebra.

In order to obtain $N = 1$ Space-Time supersymmetry in the theory, we must leave from the vertices of $V^{\vec{L}}_{\vec{\mu}_L}$, where $\vec{\mu}_L = (q,\ \vec{\lambda},\ Q^{L}_{CY})$, which are the cohomologies of Q_{BRST}, D_m, D_n only those vertices that are mutually local with J^{+}_{σ} and $J^{-}_{\dot{\sigma}}$.

These vertices are mutually local with the currents J^{+}_{σ} and $J^{-}_{\dot{\sigma}}$ if

$$q + \sum_{a}\lambda^{a} + Q^{L}_{CY} \in 2\ \mathbb{Z}.$$

where $Q^{L}_{CY} = \sum_i S_i + \vec{p}\cdot\vec{a}^{-} - \vec{q}\cdot\vec{a}^{+}$ is $U(1)$ charge of Calabi-Yau subsector. This equation is nothing more than GSO condition for the vertices in the Left sector.From GSO equation it follows that the total internal charges Q^{L}_{CY} of the vertices are integers or half-integers.

Massless Left movers and N=1 Space-Time supersymmetry

The vectors $\vec{\lambda}$ are weights of the algebra $SO(1,3)$ (and $SO(2n)$), belonging to one of the four conjugacy classes of the weight lattice.

$$(0) : (0,0,0,...,0) + \text{any root};$$
$$(V) : (1,0,0,...,0) + \text{any root};$$
$$(S) : (\frac{1}{2},\frac{1}{2},\frac{1}{2},...,\frac{1}{2}) + \text{any root};$$
$$(C) : (-\frac{1}{2},\frac{1}{2},\frac{1}{2},...,\frac{1}{2}) + \text{any root}.$$

From GSO equation it follows that in NS sector the vectors $\vec{\lambda}$ fall into classes [0] and [V] and in R sector the vector $\vec{\lambda}$ fall into classes[S] and [C] of $SO(1,3)$. Therefore, we obtain the connection between the conjugacy class of $\vec{\lambda}$ and the $U(1)$ charge of the CY factor for all 4 cases of left vertices.

In the NS sector we get

$$Q^L_{CY} \in 2\mathbb{Z}+1, \quad \vec{\lambda} \in [0],$$
$$Q^L_{CY} \in 2\mathbb{Z}, \quad \vec{\lambda} \in [V].$$

In the R sector we get

$$Q^L_{CY} \in 2\mathbb{Z}+\frac{1}{2}, \quad \vec{\lambda} \in [S],$$
$$Q^L_{CY} \in 2\mathbb{Z}-\frac{1}{2}, \quad \vec{\lambda} \in [C].$$

Right-moving sector, N=0 CFT

The space-time subsector of Right-moving sector with the central charge 4 contains boson fields $\bar{X}^{\mu}(\bar{z})$.

To obtain Right sector of Bosonic string with the total central charge 26, we add bosonic fields $Y_I(\bar{z})$, $I = 1, ..., 8$ which are compactified on the torus of the algebra $E(8)$ with the central charge 8,

we also add the bosonic fields $\Phi_{\alpha}(\bar{z})$, compactified on the torus of the algebra $SO(10)$ with the central charge 5.

The final contribution is given by the right-moving part of the compact Calabi-Yau subsector with the central charge 9.

Then the energy-momentum tensor and general right-moving vertex look like

$$\bar{T}_{mat}(\bar{z}) = \frac{1}{2}(\eta_{\mu\nu}\bar{\partial}\bar{X}^{\mu}\bar{\partial}\bar{X}^{\nu} + (\bar{\partial}Y_I)^2 + (\bar{\partial}\Phi_{\alpha})^2 + \times$$

$$\times + \sum_{i=1}^{5}(\partial\bar{X}_i^{+}\partial\bar{X}_i^{-} + \frac{1}{2}(\partial\Psi_i^{+}\Psi_i^{-} + \partial\Psi_i^{-}\Psi_i^{+}) + \frac{1}{2}(a_i^{+}\partial^2\bar{X}_i^{-} + a_i^{-}\partial^2\bar{X}_i^{+})).$$

$$\bar{V}_{\vec{\mu}_R}^{\vec{I}}(\bar{z}) = P_{gh}(\bar{b},\bar{c})P_{st}(\bar{\partial}\bar{X}^{\mu})P_{int}(\bar{\partial}\bar{Y}^{I},\bar{\partial}\Phi_{\alpha},\bar{H}_i,\bar{X}_i^{+},\bar{X}_i^{-})$$

$$\exp\,(\imath\eta_I\bar{Y}^I + \imath\Lambda_{\alpha}\Phi^{\alpha} + i\sum_{i=1}^{5}S_i\bar{H}_i + \vec{p}\cdot\vec{\bar{X}}^{-} + \vec{q}\cdot\vec{\bar{X}}^{+} + \imath p_{\mu}x^{\mu}(\bar{z})),$$

$\vec{\eta}$ is vector of $E(8)$ root lattice, and $\vec{\Lambda}$ is vector of the $SO(10)$ weight lattice.

Right-moving sector, N=0 CFT

Among the vertices we find BRST-invariant massless ones.

First of all this is $SO(1,3)$ vector $V^{\mu}(\bar{z}) = \imath\bar{\partial}\bar{X}^{\mu}(\bar{z})$.

We also find the currents of $E(8)$ algebra

$$J^{I}(\bar{z}) = \imath\bar{\partial}\bar{Y}^{I}(\bar{z}),\ \ I = 1, ..., 8,\ \ J_{\vec{\epsilon}}(\bar{z}) = \exp\left[\imath\epsilon_{I}\bar{Y}^{I}\right](\bar{z}),\ \ \vec{\epsilon}^{\,2} = 2,$$

$$\vec{\epsilon} = \begin{cases} (\pm 1, \pm 1, 0, 0, 0, 0, 0, 0) + \text{permutations}, \\ (\pm\frac{1}{2}, ..., \pm\frac{1}{2}) + \text{permutations, even number of} + \frac{1}{2}, \end{cases}$$

where the vectors $\vec{\epsilon}$ are the roots of $E(8)$ algebra.

Also there are the currents of $SO(10)$ algebra:

$$J_{\alpha}(\bar{z}) = \imath\bar{\partial}\Phi_{\alpha}(\bar{z}),\ \ \alpha = 1, ..., 5,$$

$$J_{\vec{\rho}}(\bar{z}) = \exp\left[\imath\rho_{\alpha}\Phi_{\alpha}\right](\bar{z}),\ \ \rho_{\alpha} = \pm 1,\ \ \sum(\rho_{\alpha})^{2} = 2,$$

where the vectors $\vec{\rho}$ are the roots of $SO(10)$

$$\vec{\rho} = (\pm 1, \pm 1, 0, 0, 0) + \text{permutations},$$

and the current of $U(1)$ algebra of the right Calabi-Yau sector

$$J_{CY}(\bar{z}) = \bar{\partial}\bar{H}_{CY}(\bar{z}),\ \ \ \bar{H}_{CY}(\bar{z}) = \sum_{i=1}(i\bar{H}_{i} + a_{i}^{+}\bar{X}_{i}^{-} - a_{i}^{-}\bar{X}_{i}^{+}).$$

Right-moving sector, N=0 CFT

In the Right sector we also find massless $SO(10)$ spinors

$$J^{\pm}_{\omega}(\bar{z}) = \exp\left(\imath\omega_\alpha\Phi_\alpha \pm \frac{1}{2}\bar{H}_{CY}\right)(\bar{z})$$
$$\omega_\alpha = \pm\frac{1}{2},\ \sum\omega_\alpha = \frac{1}{2} \mod 2\mathbb{Z},$$
$$J^{\pm}_{\dot\omega}(\bar{z}) = \exp\left(\imath\dot\omega_\alpha\Phi_\alpha \pm \frac{1}{2}\bar{H}_{CY}\right)(\bar{z}),$$
$$\dot\omega_\alpha = \pm\frac{1}{2},\ \sum\dot\omega_\alpha = -\frac{1}{2} \mod 2\mathbb{Z}.$$

The currents J^{+}_{ω} is mutually local with $J^{-}_{\dot\omega}$, and J^{-}_{ω} is mutually local with $J^{+}_{\dot\omega}$. All these currents are also cohomologies of the differentials $D_{\vec{m}}$ and $D_{\vec{n}}$.

By choosing one of these pairs (we choose J^{+}_{ω}, $J^{-}_{\dot\omega}$)
and adding them to the currents of the algebra $SO(10)$ and the algebra $U(1)$ we extend the product of the algebra $SO(10)\times U(1)$ to the algebra $E(6)$.

Right-moving sector, N=0 CFT

One can express the E_6 currents in terms of simple roots of E_6

$$\vec{\alpha}_i = e_1 - e_2\,,\ \ldots,\ \vec{\alpha}_4 = e_4 - e_5\,,\ \vec{\alpha}_5 = e_4 + e_5\,,$$

$$\vec{\alpha}_6 = -\frac{1}{2}(e_1 + \ldots + e_5) + \frac{\sqrt{3}}{2}e_6\,,$$

where e_i are the orthonormal basic vectors in $\mathbb{R}^6$.
So that the Cartan subalgebra currents are

$$J_j(\bar{z}) = \imath\vec{\alpha}_j \cdot \bar{\partial}\vec{\bar{H}}(\bar{z}),\ j = 1, \ldots, 6, \text{where}$$

$$\vec{\bar{H}}(\bar{z}) = (\bar{H}_1(\bar{z}), \ldots, \bar{H}_5(\bar{z}), \bar{H}_6(\bar{z})),$$

$$\bar{H}_6(\bar{z}) = \bar{H}_{CY}(\bar{z}) = \sum_{i=1}(i\bar{H}_i + a_i^+\bar{X}_i^- - a_i^-\bar{X}_i^+).$$

The currents of ladder $E(6)$ operators are given by

$$E_j(\bar{z}) = \exp\,[\imath\vec{\alpha}_j\vec{\bar{H}}](\bar{z}),$$

$$F_j(\bar{z}) = \exp\,[-\imath\vec{\alpha}_j\vec{\bar{H}}](\bar{z}),\ j = 1, \ldots, 6.$$

$E(6)$ gauge symmetry (and not any other) in Right sector ensures self-consistency of the theory whose Left sector has $N = 1$ spacetime SUSY.

It is precisely these symmetries that are necessary for the Grand Unification of the theory.

Right-moving sector, N=0 CFT

In Right sector we also find a massless boson vertex that belongs to the one-dimensional representation of $SO(10)$

$$V_{\vec{m}}^{[1]} = \exp\left(\bar{H}_{CY} + \vec{m}\cdot\vec{X}^{-}\right);$$

Another massless boson vertex that belongs to the 10-dimensional representation of $SO(10)$

$$V_{\vec{m}}^{[10]} = \exp\left(i\vec{\Lambda}\cdot\vec{\Phi} + \vec{m}\cdot\vec{X}^{-}\right);$$

One more massless boson vertex that belongs to the 16-dimensional spinor representation of $SO(10)$

$$V_{\vec{m}}^{[16]} = \exp\left(i(\vec{\Lambda}+\vec{\omega})\cdot\vec{\Phi} + \frac{1}{2}\bar{H}_{CY} + \vec{m}\cdot\vec{X}^{-}\right),$$

where $\vec{m}\in\Delta^{+}$, $\vec{\Lambda} = (\pm 1, 0, 0, 0, 0)$ + permutations and $\Lambda + \vec{\omega} = \vec{\dot{\omega}}$.

These 3 types of vertices are cohomologies of the differentials $D_{\vec{m}}$ and $D_{\vec{n}}$ and form 27-dimensional representation of E(6).

Right massless sector also contains $\bar{27}$ representations of E(6), which are constructed from representations of $SO(10)$ whose vertices depend on $\vec{n}\in\Delta^{-}$.

The numbers of representations 27 and $\bar{27}$ are equal to the number of points in the reflexive Batyrev polytopes for a given CY-manifold.

Right-moving sector, N=0 CFT

The fourth class of massless right vertices, which are also BRST, $D_{\vec{m}}$, $D_{\vec{n}}$ cohomology, consists of $E(8) \times E(6)$ singlets

$$V(\vec{S}, \vec{m}, \vec{n}) = \exp\big(i \sum_{i=1}^{5} S_i H_i + \vec{m} \cdot \vec{X}^{-} + \vec{n} \cdot \vec{X}^{+} + \imath p_\mu \bar{X}^\mu\big)(\bar{z}),$$

whose dimension

$$\Delta(\vec{S}, \vec{m}, \vec{n}) = \frac{\vec{S}^2}{2} + \vec{m} \cdot \vec{n} + \frac{\vec{m} \cdot \vec{a}^{-} + \vec{\ } \cdot \vec{a}^{-}}{2} = 1,$$

and $U(1)$ charges

$$Q_{CY}(\vec{S}, \vec{m}, \vec{n}) = \sum_i S_i + \vec{n} \cdot \vec{a} - \vec{m} \cdot \vec{a}^{-} = 0,$$

where $\vec{m} \cdot \vec{a}^{-} = \vec{n} \cdot \vec{a}^{+} = 1$, that is $\vec{m} \in \Delta^+$ and $\vec{n} \in \Delta^-$.

Right-moving sector, N=0 CFT

There are three types of solutions to all these requirements that are the $E(8) \times E(6)$ singlets.

The first type of the singlet vertices is

$$V(\vec{S}, \vec{m}, \vec{n}) = \exp\left(\vec{m} \cdot \vec{X}^{-} + \vec{n} \cdot \vec{X}^{+} + \imath p_{\mu} \bar{X}^{\mu}\right)(\bar{z}),$$

where the vectors $\vec{m} \in \Delta^{+}$, $\vec{n} \in \Delta^{-}$ and their product $\vec{m} \cdot \vec{n}$ is equal to 0.

The second type of the singlet vertices, where $\vec{m} \in \Delta^{+}$, is

$$V(\vec{S}, \vec{m}, \vec{n}) = \sum_{i} m_i \exp\left(-iH_i + \vec{m} \cdot \vec{X}^{-} + \imath p_{\mu} \bar{X}^{\mu}\right)(\bar{z})).$$

.

The third type of the singlet vertices, where $\vec{n} \in \Delta^{-}$, is

$$V(\vec{S}, \vec{m}, \vec{n}) = \sum_{i} n_i \exp\left(iH_i + \vec{n} \cdot \vec{X}^{+} + \imath p_{\mu} \bar{X}^{\mu}\right)(\bar{z})).$$

Verified that the total number of singlet massless states determined by these conditions in Quintic case coincides with the result of D. Gepner.

Massless right movers and E(6) gauge symmetry

To obtain the $E(8) \times E(6)$ gauge symmetry we need to restrict the space of states in the right moving sector to a set of states compatible with the action of the $E(8) \times E(6)$ generators.

This means that we have to choose among the vertices that are BRST $D_{\vec{m}}$ and $D_{\vec{n}}$ cohomology, those that are mutually local with the currents $E(8)$ and $E(6)$.

These requirements are met if the following "GSO"equations are satisfied

$$\vec{\rho} \cdot \vec{\Lambda} \in \mathbb{Z}, \quad \vec{\epsilon} \cdot \vec{\eta} \in \mathbb{Z},$$
$$\omega \cdot \Lambda + \frac{1}{2} Q^R_{CY} \in \mathbb{Z}.$$

From these equations we get that the $SO(10)$ parts of the right vertices, that fall into one of the four conjugacy classes of the $SO(10)$ weight lattice, determine the sixth Calabi-Yau component Q^R_{CY} as follows

$$\vec{\Lambda} \in [0] \Rightarrow Q^R_{CY} \in 2\mathbb{Z},$$
$$\vec{\Lambda} \in [V] \Rightarrow Q^R_{CY} \in 2\mathbb{Z} + 1,$$
$$\vec{\Lambda} \in [S] \Rightarrow Q^R_{CY} \in 2\mathbb{Z} - \frac{1}{2},$$
$$\vec{\Lambda} \in [C] \Rightarrow Q^R_{CY} \in 2\mathbb{Z} + \frac{1}{2}.$$

The data of the above-considered vertices satisfy these conditions.

Mutual locality of Full physical vertices

The complete vertices of a Heterotic string must be left and right BRST invariant, obey the left *GSO* equation and the right "*GSO*" equation, be cohomologies of D_m and D_n, and be mutually local with respect to each other.

We begin the search for complete mutually local vertices among the so-called "quasi-diagonal" vertices, which are a special case of complete vertices.

The "quasi-diagonal" full vertices are given by the following product of GSO-invariant left-moving and "GSO"-invariant right-moving factors

$$\exp\left(q\phi + \imath\vec{\lambda}\vec{\tilde{H}} + i\vec{S}_L \cdot \vec{H}_L + \vec{p}_L\vec{X}_L^- + \vec{q}_L\vec{X}_L^+\right)(z)\times$$
$$\times \exp\left(\imath\eta_I \bar{Y}^I + \imath\Lambda_\alpha\Phi^\alpha + i\vec{S}_R\vec{H}_R + \vec{p}_R \cdot \vec{X}_R^- + \vec{q}_R \cdot \vec{X}_R^+\right)(\bar{z}),$$

where

$$\vec{S}_L = \vec{S}_R,\ \vec{p}_L = \vec{p}_R,\ \vec{q}_L = \vec{q}_R.$$

The product of two such vertices after moving one around the other receives a complex factor, the phase of which has the following form

$$2\pi\imath(\vec{\mu}_L \cdot \vec{\mu}'_L - \vec{\mu}_R \cdot \vec{\mu}'_R) = 2\pi\imath(-qq' + \vec{\lambda}\cdot\vec{\lambda}' - \vec{\Lambda}\cdot\vec{\Lambda}').$$

Mutual locality of the full physical vertices

The mutually locality requirement

$$\vec{\mu}_L \cdot \vec{\mu}'_L - \vec{\mu}_R \cdot \vec{\mu}'_R \in \mathbb{Z}$$

imposes certain correlations between the classes $\vec{\lambda}$ and $\vec{\Lambda}$.

The reason for this is the requirements of compatibility formulated above lead to a correlation between the picture number q, the conjugacy classes of $\vec{\lambda}$ and the CY $U(1)$ charge in the left sector.

The same is true for the correlation between the $\vec{\Lambda}$ classes and the CY charge $U(1)$ in the right sector, which are required for compatibility with the $E(6)$ symmetry.

Taking into account also that for quasi-diagonal full vertices with $Q^L_{CY} - Q^R_{CY} \in \mathbb{Z}$, we get four types of them, which satisfy the following requirements

$$\begin{aligned}
Q^L_{CY},\ Q^R_{CY} &\in 2\mathbb{Z} &&\Rightarrow \lambda \in [V],\ \Lambda \in [0],\\
Q^L_{CY},\ Q^R_{CY} &\in 2\mathbb{Z}+1 &&\Rightarrow \lambda \in [0],\ \Lambda \in [V],\\
Q^L_{CY},\ Q^R_{CY} &\in 2\mathbb{Z}+\frac{1}{2} &&\Rightarrow \lambda \in [C],\ \Lambda \in [C],\\
Q^L_{CY},\ Q^R_{CY} &\in 2\mathbb{Z}-\frac{1}{2} &&\Rightarrow \lambda \in [S],\ \Lambda \in [S].
\end{aligned}$$

Mutual locality of the full physical vertices

These vertices are mutually local due to the proper correlation of the internal charges Q^L_{CY}, $SO(1,3)$ and the weights λ and $SO(10)$ weights Λ.

However, this set of the full mutually “quasi-diagonal” vertices does not satisfy the requirement of Space-Time supersymmetry.

To solve this problem we use the fact that superpartners in the left sector obtained by action the $N = 1$ supercharges Q_σ and $Q_{\dot\sigma}$ also satisfy the all requirement on the left vertices defined above, including the requirement of space-time supersymmetry.

So we solve the problem by simply adding these superpartners to the set of "quasi-diagonal"vertices.

A similar technique can be used on the right side of full vertices, where instead of $N = 1$ Poincaré supergenerators, the action of the $E(6)$ algebra generators is used to obtain $E(6)$ partners.

The result of these operations leads to the fulfillment of the requirement of $N = 1$ spacetime supersymmetry, $E(8) \times E(6)$ gauge symmetry and preservation of mutual locality of complete vertex operators.

For phenomenological applications, the most important are the massless states of the Heterotic string.

We explicitly represent complete vertices for massless physical states as products of suitable left and right vertices
(we omit the factor $\exp(\imath p_\mu x^\mu)$ in order to shorten notations).

We obtain expressions for the vertices of the gravitational supermultiplet starting from the graviton vertex

$$\exp(-\phi(z))\ \psi^\mu(z) \times \imath\bar{\partial}x^\nu(\bar{z})$$

and using the actions of Poincaré supergenerators.

We also get the following expressions for the vertices of vector gauge supermultiplets.

The currents of $E(8)$ are

$$V_\mu^I(z,\bar{z}) = \exp(-\phi(z))\ \psi^\mu(z) \times \imath\bar{\partial}\bar{Y}^I(\bar{z}),\ \ I = 1, ..., 8,$$
$$V_\mu^{\vec{\epsilon}}(z,\bar{z}) = \exp(-\phi(z))\ \psi^\mu(z) \times \exp\left[\imath\epsilon_I \bar{Y}^I\right](\bar{z}),$$

where the vectors $\vec{\epsilon}$ are the roots of $E(8)$ algebra;

The vertices belonging to adjoint representation of $SO(10)$ algebra

$$V_{\mu,\alpha}(z,\bar{z}) = \exp(-\phi(z)\ \psi^\mu(z) \times \imath\bar{\partial}\Phi_\alpha(\bar{z}),\ \ \alpha = 1, ..., 5,$$
$$V_{\mu,\vec{\rho}}(z,\bar{z}) = \exp(-\phi(z)\ \psi^\mu(z) \times \exp\left[\imath\rho_\alpha\Phi_\alpha\right](\bar{z}),$$

where the vectors $\vec{\rho}$ are the roots of $SO(10)$;

The gauge supermultiplets

The vertex of $U(1)$ algebra of the Calabi-Yau sector

$$V_\mu^{CY}(z,\bar z) = \exp(-\phi(z)\ \psi^\mu(z) \times \bar\partial \bar H_{CY}(\bar z),$$
$$\bar H_{CY}(\bar z) = \sum_{i=1} (i\bar H_i + a_i^+ \bar X_i^- - a_i^- \bar X_i^+).$$

We then extend the $E_8 \times SO(10) \times U(1)$ symmetry to $E_8 \times E_6$ using 32 spinor currents of $SO(10)$ algebra J_ω^+ and $J_{\dot\omega}^-$ given by (30)

$$J_\omega^\pm(\bar z) = \exp\left(\imath\omega_\alpha \Phi_\alpha \pm \frac{1}{2}\bar H_{CY}\right)(\bar z)$$
$$\omega_\alpha = \pm\frac{1}{2},\ \sum \omega_\alpha = \frac{1}{2} \mod 2\mathbb{Z},$$
$$J_{\dot\omega}^\pm(\bar z) = \exp\left(\imath\dot\omega_\alpha \Phi_\alpha \pm \frac{1}{2}\bar H_{CY}\right)(\bar z),$$
$$\dot\omega_\alpha = \pm\frac{1}{2},\ \sum \dot\omega_\alpha = -\frac{1}{2} \mod 2\mathbb{Z}.$$

As a result we obtain the 32 additional currents of $E(6)$ algebra

$$V_\omega(z,\bar z) = \exp(-\phi(z)\ \psi^\mu(z) \times J_\omega^+(\bar z),$$
$$V_{\dot\omega}(z,\bar z) = \exp(-\phi(z)\ \psi^\mu(z) \times J_{\dot\omega}^-(\bar z),$$

that together with the other 46 currents of $SO(10)$ and $U(1)$ form the 78-dimensional adjoint representation of E(6) algebra.

27 and $\overline{27}$ supermultiplets

The third type are the vertex operators of 27 supermultiplet of $E(6)$ algebra whose right-moving factors belong to the 1-dimensional, 10-dimensional and 16- dimensional representations of $S(10)$ algebra.

The left-moving factors of all these vertex operators belong to spinor reppresentation of $N = 1$ Super Poincare algebra.

Thus, the vertex operators of the 27 supermultiplet include a 1-dimensional representation of $SO(10)$

$$V_{\vec{\sigma},\vec{m}}(z,\bar{z}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} + \frac{1}{2}H^L_{CY}\right) \times \exp\left(\bar{H}^R{}_{CY} + \vec{m}\cdot\vec{X}^-\right),$$

the 10-dimensional representation of $SO(10)$

$$V^{\vec{\Lambda}}_{\vec{\sigma},\vec{m}}(z,\bar{z}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} + \frac{1}{2}H^L_{CY}\right) \times \exp\left(i\vec{\Lambda}\cdot\vec{\Phi} + \vec{m}\cdot\vec{X}^-\right),$$

and the 16-dimensional spinor representation of $SO(10)$

$$V^{\dot{\vec{\omega}}}_{\vec{\sigma},\vec{m}}(z,\bar{z}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} + \frac{1}{2}H^L_{CY}\right)(z) \times \exp\left(i\dot{\vec{\omega}}\cdot\vec{\Phi} + \frac{1}{2}\bar{H}^R{}_{CY} + \vec{m}\cdot\vec{X}^-\right),$$

where $\vec{m} \in \Delta^+$, $\vec{\Lambda} = (\pm 1, 0, 0, 0, 0) +$ permutations.

27 and $\overline{27}$ supermultiplets

The condition $\vec{m} \in \Delta^+$ means that the number 27 of supermultiplets is equal to the number of dots $\vec{m}$ of the Batyrev polyhedron corresponding to the Calabi-Yau manifold of the Heterotic string model under consideration.

Explicit expressions for the other vertices of the 27 supermultiplet can be obtained by acting through OPE on these vertices by the generators of the $N = 1$ Poincaré superalgebra.
The similar actions we obtain the set of vertices of the $\bar{27}$ supermultiplet.

Note that this operation does not break the mutual locality between the extended set of vertex operators that was between the original ones.

Singlet supermultiplets

The final, fourth class of vertices that satisfies all the requirements of our construction is the $N=1$ supermultiplets, which are $E(8)\times E(6)$ singlets.

The first type of the singlet vertices is

$$V_{\vec{\sigma},\vec{m},\vec{n}}(z,\bar{z}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} + \frac{1}{2}H^L_{CY}\right) \times \exp(\vec{m}\cdot\vec{X}^- + \vec{n}\cdot\vec{X}^+),$$

where the vectors $\vec{m}\in\Delta^+$, $\vec{n}\in\Delta^-$ and their product $\vec{m}\cdot\vec{n}=0$.
The second type of the singlet vertices is

$$V_{\vec{\sigma},\vec{m}}(z,\bar{z}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} + \frac{1}{2}H^L_{CY}\right) \times \sum_i m_i \exp\left(-iH_i + \vec{m}\cdot\vec{X}^-\right)(\bar{z}),$$

where $\vec{m}\in\Delta^+$.
The third type of the singlet vertices is

$$V_{\vec{\sigma},\vec{n}}(z,\bar{z}) = \exp\left(-\frac{1}{2}\phi + \imath\vec{\sigma}\cdot\vec{\tilde{H}} + \frac{1}{2}H^L_{CY}\right) \times \sum_i n_i \exp\left(iH_i + \vec{n}\cdot\vec{X}^+\right),$$

where $\vec{n}\in\Delta^-$.

For the case when Calabi-Yau sector is defined by Quintic polynomial we find a total number of $E(8)\times E(6)$ which is coincide with the result of Gepner.

Conclusion

In this paper we developed a method for explicitly constructing the models of Heterotic string compactified on the product of a torus of the Lie algebra $E(8) \times SO(10)$ and general Calabi-Yau manifolds of Berglund-Hubsch type.

The construction uses Batyrev-Borisov combinatorial approach to construct the Vertex algebra of the Calabi-Yau sector.

We used the requirement for the simultaneous fulfillment of mutual locality of the left-moving vertices with the space-time symmetry generators
and of right-moving vertices with generators of $E(8) \times E(6)$ gauge symmetry together with the requirement of mutual locality of complete (left-right) vertices of physical states.