talagrand's convex distance

The weighted Hamming distance is

$$d_\alpha(x, A) = \inf_{y \in A} d_\alpha(x, y) = \inf_{y \in A} \sum_{i: x_i \neq y_i} |\alpha_i|$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$. The same argument as before gives

$$\mathbb{P}\left\{ d_\alpha(X, A) \geq t + \sqrt{\frac{\|\alpha\|^2}{2} \log \frac{1}{\mathbb{P}\{A\}}} \right\} \leq e^{-2t^2/\|\alpha\|^2} ,$$

This implies

$$\sup_{\alpha: \|\alpha\|=1} \min\left(\mathbb{P}\{A\}, \mathbb{P}\left\{d_\alpha(X, A) \geq t\right\}\right) \leq e^{-t^2/2} .$$

convex distance inequality

convex distance:

$$d_T(x, A) = \sup_{\alpha \in [0,\infty)^n : \|\alpha\|=1} d_\alpha(x, A) .$$

Talagrand's convex distance inequality:

$$\mathbb{P}\{A\}\mathbb{P}\left\{d_T(X, A) \geq t\right\} \leq e^{-t^2/4} .$$

Follows from the fact that $d_T(X, A)^2$ is $(4, 0)$ weakly self bounding (by a saddle point representation of d_T).

Talagrand's original proof was different.

convex lipschitz functions

For $A \subset [0,1]^n$ and $x \in [0,1]^n$, define

$$D(x, A) = \inf_{y \in A} \|x - y\| .$$

If A is convex, then

$$D(x, A) \leq d_T(x, A) .$$

Proof:

$$\begin{aligned} D(x,A) &= \inf_{\nu \in \mathcal{M}(A)} \|x - \mathbb{E}_\nu Y\| \quad \text{(since } A \text{ is convex)} \\ &\leq \inf_{\nu \in \mathcal{M}(A)} \sqrt{\sum_{j=1}^{n} \left(\mathbb{E}_\nu \mathbb{1}_{x_j \neq Y_j}\right)^2} \quad \text{(since } x_j, Y_j \in [0,1]) \\ &= \inf_{\nu \in \mathcal{M}(A)} \sup_{\alpha : \|\alpha\| \leq 1} \sum_{j=1}^{n} \alpha_j \mathbb{E}_\nu \mathbb{1}_{x_j \neq Y_j} \quad \text{(by Cauchy-Schwarz)} \\ &= d_T(x, A) \quad \text{(by minimax theorem)} . \end{aligned}$$

John von Neumann (1903–1957)

convex lipschitz functions

Let $X = (X_1, \ldots, X_n)$ have independent components taking values in $[0,1]$. Let $f : [0,1]^n \to \mathbb{R}$ be quasi-convex such that $|f(x) - f(y)| \leq \|x - y\|$. Then

$$\mathbb{P}\{f(X) > \mathbb{M}f(X) + t\} \leq 2e^{-t^2/4}$$

and

$$\mathbb{P}\{f(X) < \mathbb{M}f(X) - t\} \leq 2e^{-t^2/4} .$$

Proof: Let $A_s = \{x : f(x) \leq s\} \subset [0,1]^n$. A_s is convex. Since f is Lipschitz,

$$f(x) \leq s + D(x, A_s) \leq s + d_T(x, A_s) ,$$

By the convex distance inequality,

$$\mathbb{P}\{f(X) \geq s + t\}\mathbb{P}\{f(X) \leq s\} \leq e^{-t^2/4} .$$

Take $s = \mathbb{M}f(X)$ for the upper tail and $s = \mathbb{M}f(X) - t$ for the lower tail.

empirical processes

Let $\mathcal{T}$ be a countable index set.
For $i = 1, \ldots, n$, let $X_i = \left(X_{i,s}\right)_{s \in \mathcal{T}}$ be vectors of real-valued random variables. Assume that $X_1, \ldots, X_n$ are independent.

The empirical process is $\sum_{i=1}^{n} X_{i,s}$, $s \in \mathcal{T}$.

We study concentration of the supremum:

$$Z = \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} X_{i,s} .$$

empirical processes–the variance

We may use Efron-Stein: let

$$Z_i = \sup_{s \in \mathcal{T}} \sum_{j : j \neq i} X_{j,s}$$

and $\widehat{s} \in \mathcal{T}$ be such that $Z = \sum_{i=1}^n X_{i,\widehat{s}}$. Then

$$(Z - Z_i)_+ \leq (X_{i,\widehat{s}})_+ \leq \sup_{s \in \mathcal{T}} |X_{i,s}|$$

so

$$\mathbf{Var}(Z) \leq \mathbb{E} \sum_{i=1}^n (Z - Z_i)^2 \leq \mathbb{E} \sum_{i=1}^n \sup_{s \in \mathcal{T}} X_{i,s}^2 .$$

empirical processes–the variance

A more clever use of Efron-Stein: suppose $\mathbb{E}X_{i,s} = 0$.
Let $Z_i' = \sup_{s\in\mathcal{T}} \left(\sum_{j\neq i} X_{j,s} + X_{i,s}'\right)$. Note that

$$(Z - Z_i')_+^2 \le \left(X_{i,\widehat{s}} - X_{i,\widehat{s}}'\right)^2 .$$

By Efron-Stein,

$$\begin{aligned}
\mathbf{Var}(Z) &\le \mathbb{E}\sum_{i=1}^n (Z - Z_i')_+^2 \\
&\le \mathbb{E}\sum_{i=1}^n \mathbb{E}'\left[\left(X_{i,\widehat{s}} - X_{i,\widehat{s}}'\right)^2\right] \\
&\le \mathbb{E}\sum_{i=1}^n \left(X_{i,\widehat{s}}^2 + \mathbb{E}'\left[X_{i,\widehat{s}}'^2\right]\right) \\
&\le \mathbb{E}\sup_{s\in\mathcal{T}}\sum_{i=1}^n X_{i,s}^2 + \sup_{s\in\mathcal{T}}\sum_{i=1}^n \mathbb{E}X_{i,s}^2 .
\end{aligned}$$

weak and strong variance

We have proved that

$$\mathbf{Var}(Z) \leq V \quad \text{and} \quad \mathbf{Var}(Z) \leq \Sigma^2 + \sigma^2$$

where

$$V = \sum_{i=1}^{n} \mathbb{E} \sup_{s \in \mathcal{T}} X_{i,s}^2 \qquad \text{strong variance}$$

$$\Sigma^2 = \mathbb{E} \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} X_{i,s}^2 \qquad \text{weak variance}$$

$$\sigma^2 = \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} \mathbb{E} X_{i,s}^2 \qquad \text{wimpy variance}$$

$$\sigma^2 \leq \Sigma^2 \leq V \ .$$

weak and strong variance

If $\mathbb{E}\boldsymbol{X_{i,s}} = 0$ and $|\boldsymbol{X_{i,s}}| \leq 1$, we also have, by symmetrization and contraction arguments,

$$\Sigma^2 \leq 8\mathbb{E}\boldsymbol{Z} + \boldsymbol{\sigma}^2$$

and therefore

$$\mathbf{Var}(\boldsymbol{Z}) \leq 8\mathbb{E}\boldsymbol{Z} + 2\boldsymbol{\sigma}^2 \ .$$

If the $\boldsymbol{X_i}$ are also identicaly distributed, then

$$\mathbf{Var}(\boldsymbol{Z}) \leq 2\mathbb{E}\boldsymbol{Z} + \boldsymbol{\sigma}^2 \ .$$

empirical processes–exponential inequalities

A Bernstein type inequality. "Talagrand's inequality".

Assume $\mathbb{E}X_{i,s} = 0$, and $|X_{i,s}| \leq 1$. For $t \geq 0$,

$$\mathbb{P}\left\{Z \geq \mathbb{E}Z + t\right\} \leq \exp\left(-\frac{t^2}{2\left(2(\Sigma^2 + \sigma^2) + t\right)}\right) .$$

proof.

For each $i = 1, \ldots, n$, let $Z_i' = \sup_{s \in \mathcal{T}} (X_{i,s}' + \sum_{j \neq i} X_{j,s})$. We already proved that

$$\sum_{i=1}^{n} \mathbb{E}'(Z - Z_i')_+^2 \leq \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} X_{i,s}^2 + \sigma^2 \stackrel{\text{def.}}{=} W + \sigma^2 .$$

By the exponential Efron-Stein inequality, for $\lambda \in [0, 1)$,

$$\log \mathbb{E} e^{\lambda(Z - \mathbb{E}Z)} \leq \frac{\lambda}{1 - \lambda} \log \mathbb{E} e^{\lambda(W + \sigma^2)} .$$

W is a self-bounding function, so

$$\log \mathbb{E} e^{\lambda W} \leq \Sigma^2 \left(e^{\lambda} - 1 \right) .$$

Putting things together implies the inequality.

bousquet's inequality

A Bennett type inequality with the right constant.
Assume $X_1, \ldots, X_n$ are i.i.d. with $\mathbb{E}X_{i,s} = 0$ and $X_{i,s} \leq 1$.
For all $t \geq 0$,

$$\mathbb{P}\{Z \geq \mathbb{E}Z + t\} \leq e^{-vh(t/v)} .$$

where $v = 2\mathbb{E}Z + \sigma^2$ and $h(u) = (1+u)\log(1+u) - u$.
In particular,

$$\mathbb{P}\{Z \geq \mathbb{E}Z + t\} \leq \exp\left(-\frac{t^2}{2(v + t/3)}\right) .$$

ϕ entropies

For a convex function ϕ on $[0,\infty)$, the ϕ-entropy of $Z \geq 0$ is

$$H_\phi(Z) = \mathbb{E}[\phi(Z)] - \phi(\mathbb{E}[Z]) \ .$$

H_ϕ is subadditive:

$$H_\phi(Z) \leq \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}\left[\phi(Z) \mid X^{(i)}\right] - \phi\left(\mathbb{E}\left[Z \mid X^{(i)}\right]\right)\right]$$

if (and only if) ϕ is twice differentiable on $(0,\infty)$, and either ϕ is affine strictly positive and $1/\phi''$ is concave.

$\phi(x) = x^2$ corresponds to Efron-Stein.

$x \log x$ is subadditivity of entropy.

We may consider $\phi(x) = x^p$ for $p \in (1,2]$.

generalized efron-stein

Define

$$Z_i' = f(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n) \ ,$$

$$V^+ = \sum_{i=1}^{n} (Z - Z_i')_+^2 \ .$$

For $q \geq 2$ and $q/2 \leq \alpha \leq q - 1$,

$$\mathbb{E}\left[(Z - \mathbb{E}Z)_+^q\right]$$
$$\leq \mathbb{E}\left[(Z - \mathbb{E}Z)_+^\alpha\right]^{q/\alpha} + \alpha\,(q - \alpha)\,\mathbb{E}\left[V^+\,(Z - \mathbb{E}Z)_+^{q-2}\right]$$

and similarly for $\mathbb{E}\left[(Z - \mathbb{E}Z)_-^q\right]$.

moment inequalities

We may solve the recursions, for $q \geq 2$.

If $V^+ \leq c$ for some constant $c \geq 0$, then for all integers $q \geq 2$,

$$\left(\mathbb{E}\left[(Z - \mathbb{E}Z)_+^q\right]\right)^{1/q} \leq \sqrt{Kqc} \ ,$$

where $K = 1/\left(e - \sqrt{e}\right) < 0.935$.

More generally,

$$\left(\mathbb{E}\left[(Z - \mathbb{E}Z)_+^q\right]\right)^{1/q} \leq 1.6\sqrt{q}\left(\mathbb{E}\left[V^{+q/2}\right]\right)^{1/q} \ .$$

sums: khinchine's inequality

Let $X_1, \ldots, X_n$ be independent Rademacher variables and $Z = \sum_{i=1}^n a_i X_i$. For any integer $q \geq 2$,

$$\left(\mathbb{E}\left[Z_+^q\right]\right)^{1/q} \leq \sqrt{2Kq}\sqrt{\sum_{i=1}^n a_i^2}$$

Proof:

$$V^+ = \sum_{i=1}^n \mathbb{E}\left[(a_i(X_i - X_i'))_+^2 \mid X_i\right] = 2\sum_{i=1}^n a_i^2 \mathbb{1}_{a_i X_i > 0} \leq 2\sum_{i=1}^n a_i^2 \ ,$$

Aleksandr Khinchin
(1894–1959)

sums: rosenthal's inequality

Let $X_1, \ldots, X_n$ be independent real-valued random variables with $\mathbb{E}X_i = 0$. Define

$$Z = \sum_{i=1}^{n} X_i\,, \quad \sigma^2 = \sum_{i=1}^{n} \mathbb{E}X_i^2\,, \quad Y = \max_{i=1,\ldots,n} |X_i|\,.$$

Then for any integer $q \geq 2$,

$$\left(\mathbb{E}\left[Z_+^q\right]\right)^{1/q} \leq \sigma\sqrt{10q} + 3q\left(\mathbb{E}\left[Y_+^q\right]\right)^{1/q} \ .$$

influences

If $A \subset \{-1,1\}^n$ and $X = (X_1, \ldots, X_n)$ is uniform, the influence of the i-th variable is

$$I_i(A) = \mathbb{P}\left\{\mathbb{1}_{X\in A} \neq \mathbb{1}_{X^{(i)}\in A}\right\}$$

where $X^{(i)} = (X_1, \ldots, X_{i-1}, 1 - X_i, X_{i+1}, \ldots, X_n)$.

The total influence is

$$I(A) = \sum_{i=1}^{n} I_i(A) \ .$$

Note that

$$I(A) = 2^{-(n-1)}|\partial_E(A)| \ .$$

influences: examples

dictatorship: $\boldsymbol{A} = \{\boldsymbol{x} : \boldsymbol{x}_1 = 1\}$. $\boldsymbol{I}(\boldsymbol{A}) = 1$.

parity: $\boldsymbol{A} = \{\boldsymbol{x} : \sum_{\boldsymbol{i}} \mathbb{1}_{\boldsymbol{x}_{\boldsymbol{i}}=1} \text{ is even}\}$. $\boldsymbol{I}(\boldsymbol{A}) = \boldsymbol{n}$.

majority: $\boldsymbol{A} = \{\boldsymbol{x} : \sum_{\boldsymbol{i}} \boldsymbol{x}_{\boldsymbol{i}} > 0\}$. $\boldsymbol{I}(\boldsymbol{A}) \approx \sqrt{2\boldsymbol{n}/\pi}$.

$$\text{by Efron-Stein,} \quad \boldsymbol{P}(\boldsymbol{A})(1 - \boldsymbol{P}(\boldsymbol{A})) \leq \frac{\boldsymbol{I}(\boldsymbol{A})}{4}$$

so dictatorship has smallest total influence (if $\boldsymbol{P}(\boldsymbol{A}) = 1/2$).

improved efron-stein on the hypercube

Recall that for any $f : \{-1,1\}^n \to \mathbb{R}$ under the uniform distribution,

$$\mathbf{Ent}(f^2) \leq 2\mathcal{E}(f)$$

where $\mathbf{Ent}(f^2) = E\left[f^2 \log(f^2)\right] - E\left[f^2\right] \log E\left[f^2\right]$ and

$$\mathcal{E}(f) = \frac{1}{4}\mathbb{E}\left[\sum_{i=1}^{n}\left(f(X) - f(\overline{X}^{(i)})\right)^2\right]$$

This implies, for any non-negative $f : \{-1,1\}^n \to [0,\infty)$,

$$E\left[f^2\right] \log \frac{E\left[f^2\right]}{E\left[f\right]^2} \leq 2\mathcal{E}(f) .$$

improved efron-stein on the hypercube

Recall the Doob-martingale representation
$f(X) - Ef = \sum_{i=1}^n \Delta_i$.
One easily sees that

$$\mathcal{E}(f) = \sum_{i=1}^n \mathcal{E}(\Delta_i) \; .$$

But then, by the previous lemma,

$$\begin{aligned}
\mathcal{E}(f) &\geq \sum_{j=1}^n \mathcal{E}(|\Delta_j|) \geq \frac{1}{2}\sum_{j=1}^n E\left[\Delta_j^2\right] \log \frac{E\left[\Delta_j^2\right]}{\left(E|\Delta_j|\right)^2} \\
&= -\frac{1}{2}\mathbf{Var}(f)\sum_{j=1}^n \frac{E\left[\Delta_j^2\right]}{\mathbf{Var}(f)} \log \frac{\left(E|\Delta_j|\right)^2}{E\left[\Delta_j^2\right]} \\
&\geq -\frac{1}{2}\mathbf{Var}(f) \log \frac{\sum_{j=1}^n \left(E|\Delta_j|\right)^2}{\mathbf{Var}(f)}
\end{aligned}$$

improved efron-stein on the hypercube

We obtained that for any $f : \{-1,1\}^n \to \mathbb{R}$,

$$\mathbf{Var}(f) \log \frac{\mathbf{Var}(f)}{\sum_{j=1}^{n} \left(E|\Delta_j|\right)^2} \leq 2\mathcal{E}(f) \ .$$

(Falik and Samorodnitsky, 2007; Rossignol, 2006; Talagrand (1994)).

"Slightly" better than Efron-Stein.

Use this for $f(x) = \mathbb{1}_{x \in A}$ for $A \subset \{-1,1\}^n$:

$$P(A)(1-P(A)) \log \frac{4P(A)(1-P(A))}{\sum_i I_i(A)^2} \leq \frac{I(A)}{4}$$

kahn, kalai, linial

Corollary: (Kahn, Kalai, Linial, 1988).

$$\max_i I_i(A) \geq \frac{P(A)(1 - P(A)) \log n}{n}$$

If the influences are equal,

$$I(A) \geq P(A)(1 - P(A)) \log n$$

Another corollary: (Friedgut, 1998).
If $I(A) \leq c$, A (basically) depends on a bounded number of variables. A is a "junta."

threshold phenomena

Let $A \subset \{-1,1\}^n$ be a monotone set and let $X = (X_1, \ldots, X_n)$ be such that

$$\mathbb{P}\{X_i = 1\} = p \qquad \mathbb{P}\{X_i = -1\} = 1 - p$$

$$P_p(A) = \sum_{x \in A} p^{\|x\|}(1-p)^{n-\|x\|}$$

is an increasing function of $p \in [0,1]$.

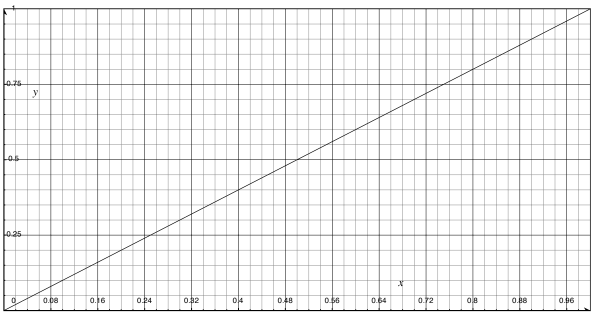
Let p_a be such that $P_{p_a}(A) = a$.

Critical value $= p_{1/2}$

Threshold width: $p_{1-\varepsilon} - p_\varepsilon$

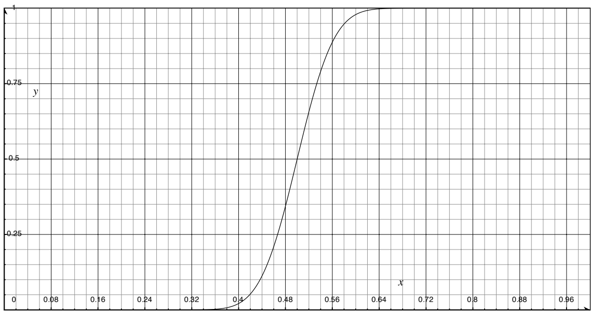
two (extreme) examples

dictatorship



threshold width $= 1 - 2\varepsilon$

majority (with $\boldsymbol{n} = 101$)



$\leq \sqrt{\log(1/\varepsilon)/(2\boldsymbol{n})}$

In what cases do we have a quick transition?

russo's lemma

If A is monotone,

$$\frac{dP_p(A)}{dp} = I^{(p)}(A)$$

The Kahn, Kalai, Linial result, generalized for $p \neq 1/2$, implies that
if A is such that $I_1^{(p)} = I_2^{(p)} = \cdots = I_n^{(p)}$, then

$$p_{1-\varepsilon} - p_{\varepsilon} = O\left(\frac{\log \frac{1}{\varepsilon}}{\log n}\right)$$

On the other hand, if $p_{3/4} - p_{1/4} \geq c$ then A is (basically) a junta.

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