Zonal flows in stellarators

Simon’s Summer Retreat, 2023, Canberra

Gabriel Plunk, December 13, 2023

Talk Outline

1. Background
    - What is a zonal flow?
    - What do zonal flows do?
    - How do zonal flows arise?
    - What limits their amplitude, and how do zonal flows decay?
2. The residual flow in well-optimized stellarators.
3. Optimizing stellarators for strong zonal flows.

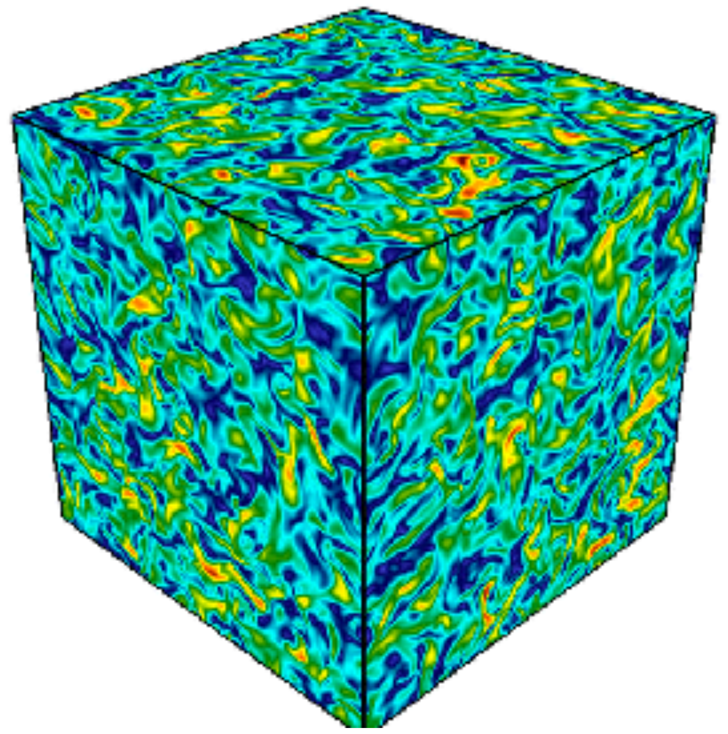
Zonal flow

Origins and definition

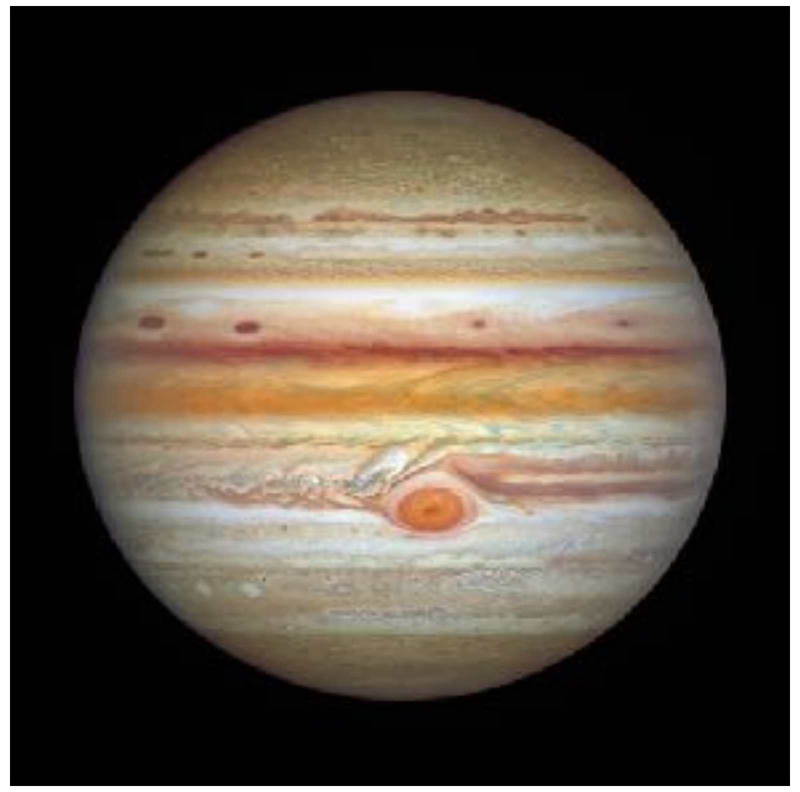
- "**Homogeneous, isotropic turbulence**": no preferred direction or location, K41, etc.
- With geometry: preferential directions, lower dimensionality (3D->2D), "self-organization".
- For nested toroidal magnetic geometries (tokamaks, stellarators...) we **define ZFs** as $E \times B$ flows due to the part of the electrostatic potential (ϕ) that is constant within flux surfaces, *e.g.*:

$$\phi = \delta\phi + \Phi, \quad \text{with} \quad \Phi(\psi) = \langle \phi \rangle$$

- Gyrokinetic theory:
  - $\mathbf{k}_\perp = k_\psi \nabla \psi + k_\alpha \nabla \alpha$ where $\mathbf{B} = \nabla \psi \times \nabla \alpha$.
  - No source of free energy for perturbations having $k_\alpha = 0$.



Box of turbulence.



Coherent structures and zonal flows in atmospheric systems.

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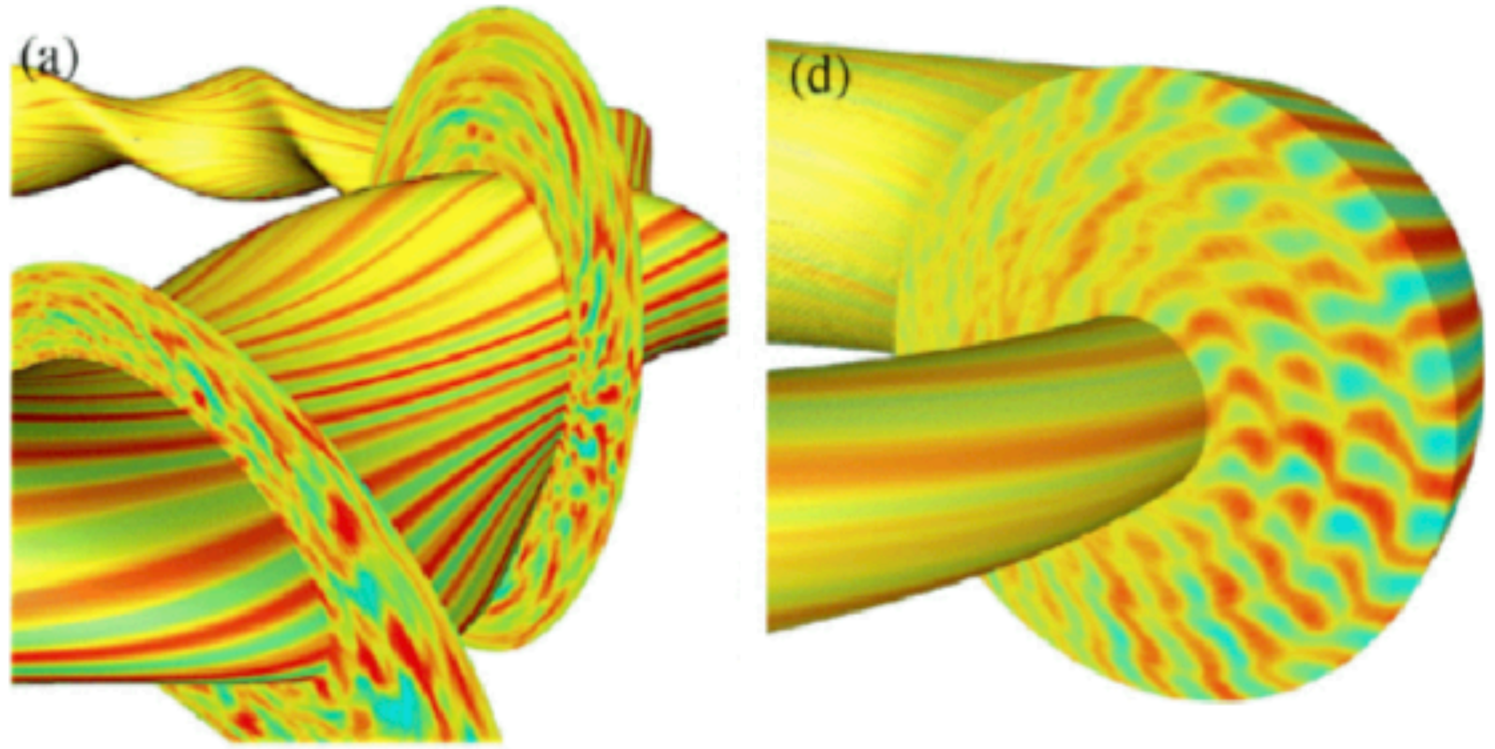
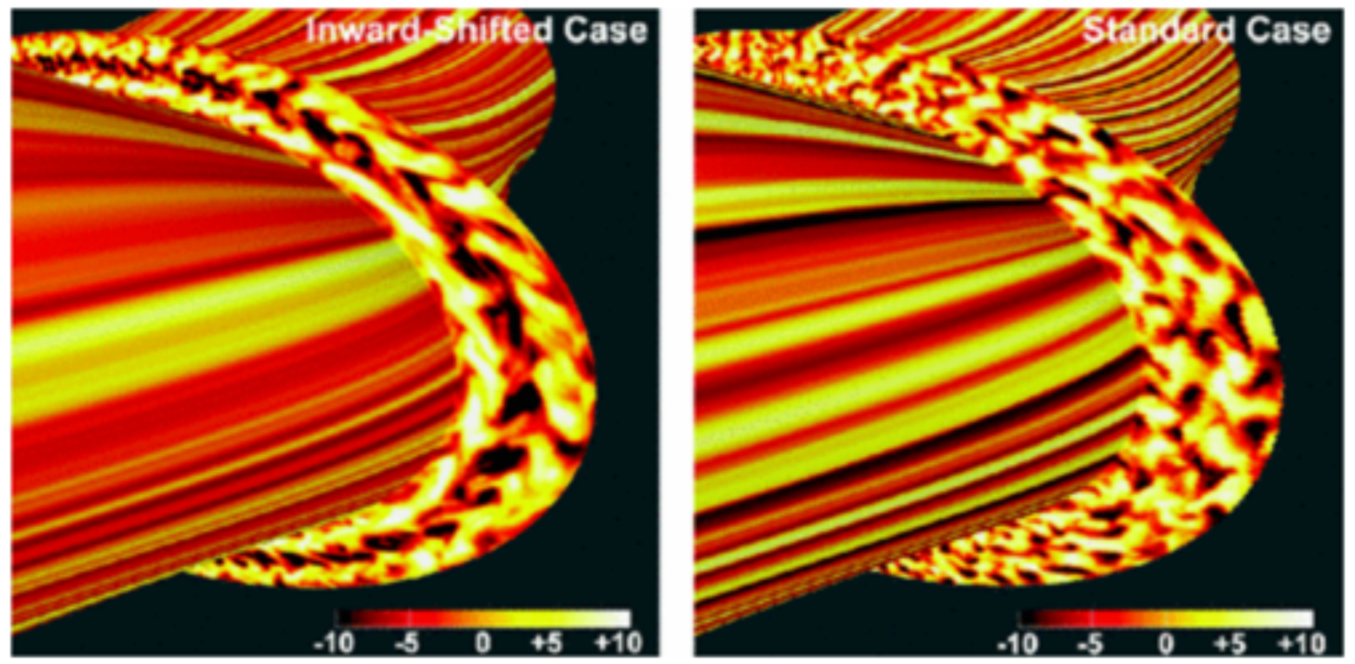


Zonal flows in toroidal plasmas.
Nakata, Nunami &Sugama, PRL (2017)

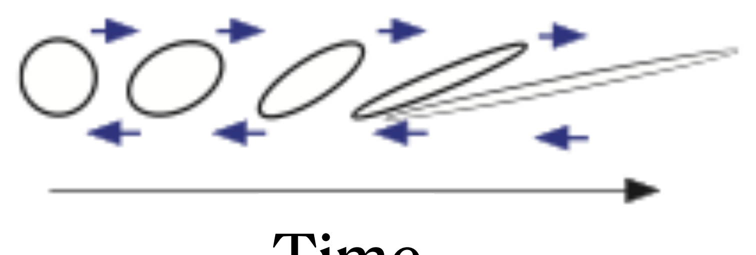


LHD shows that the shape of the magnetic field matters for ZFs!
Watanabe, Sugama & Ferrando, PRL(2008)

Zonal flows suppress turbulence

Why ZFs are "good"

- Zonal flow "shearing": one-dimensional transport of energy in k-space. Lower saturation amplitude of turbulence.



→ "eddy tilting"

- The Dimit's shift: "supercritical" turbulence.
- The Dimit's shift can be understood with "tertiary instability" theory — *Rogers, Dorland & Kotschenreuther (2000) PRL* **85** *(25), 5336*.
- The tertiary mode operates by the coupling with damped eigenmodes induced by ZF shearing — *St-Onge, J. Plasma Phys. 83, 905830504 (2017).*
- see also damped eigenmode theory of *Pueschel, Li & Terry, NF (2021).*

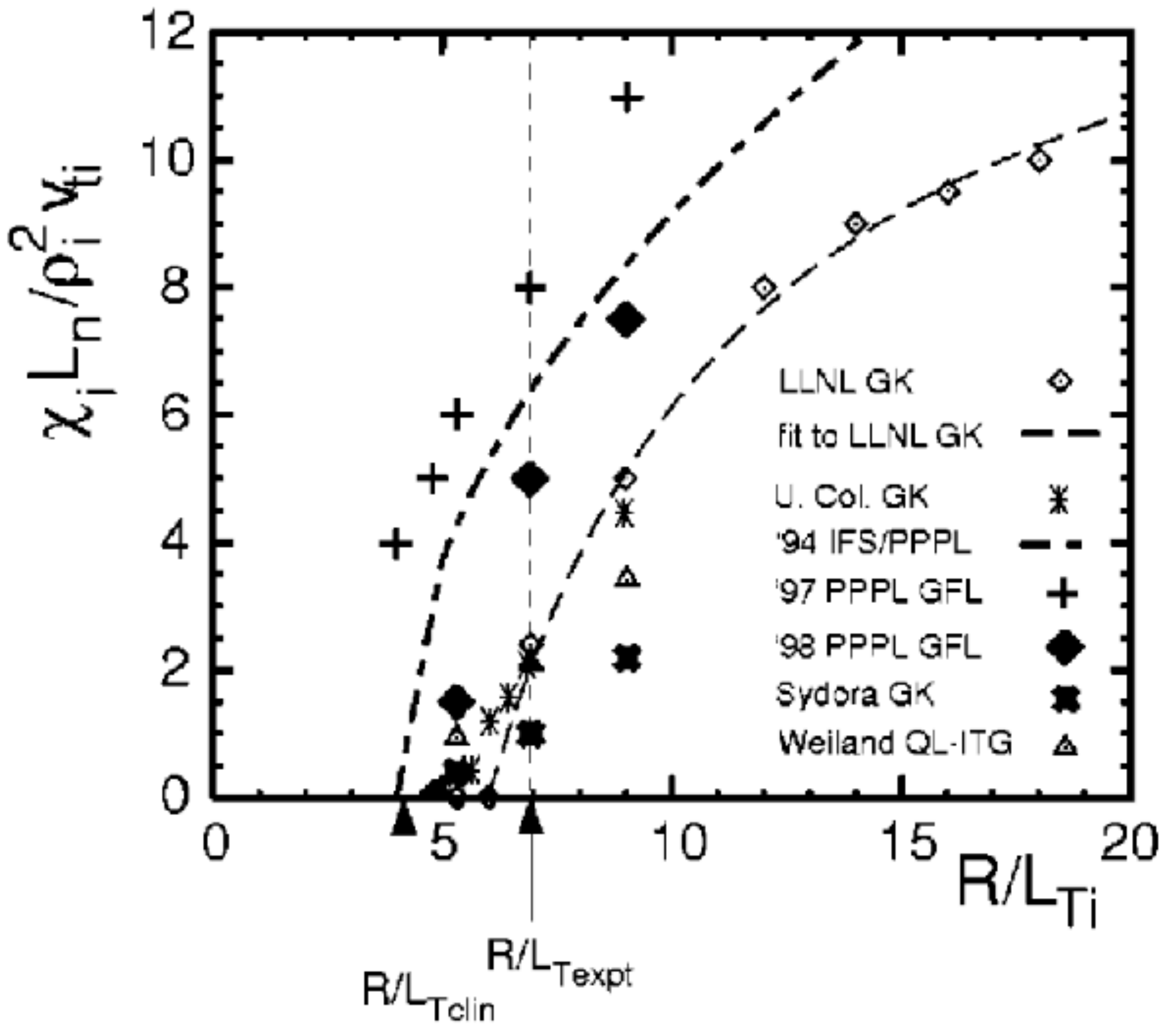


Fig: The Dimits shift in tokamaks.
Dimits, et al, Phys Plasmas Vol 7, No 3 (2000)

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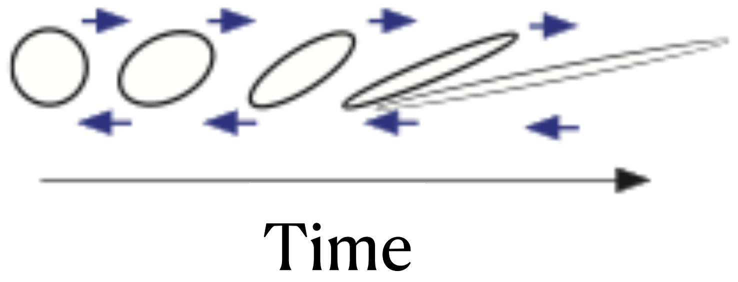

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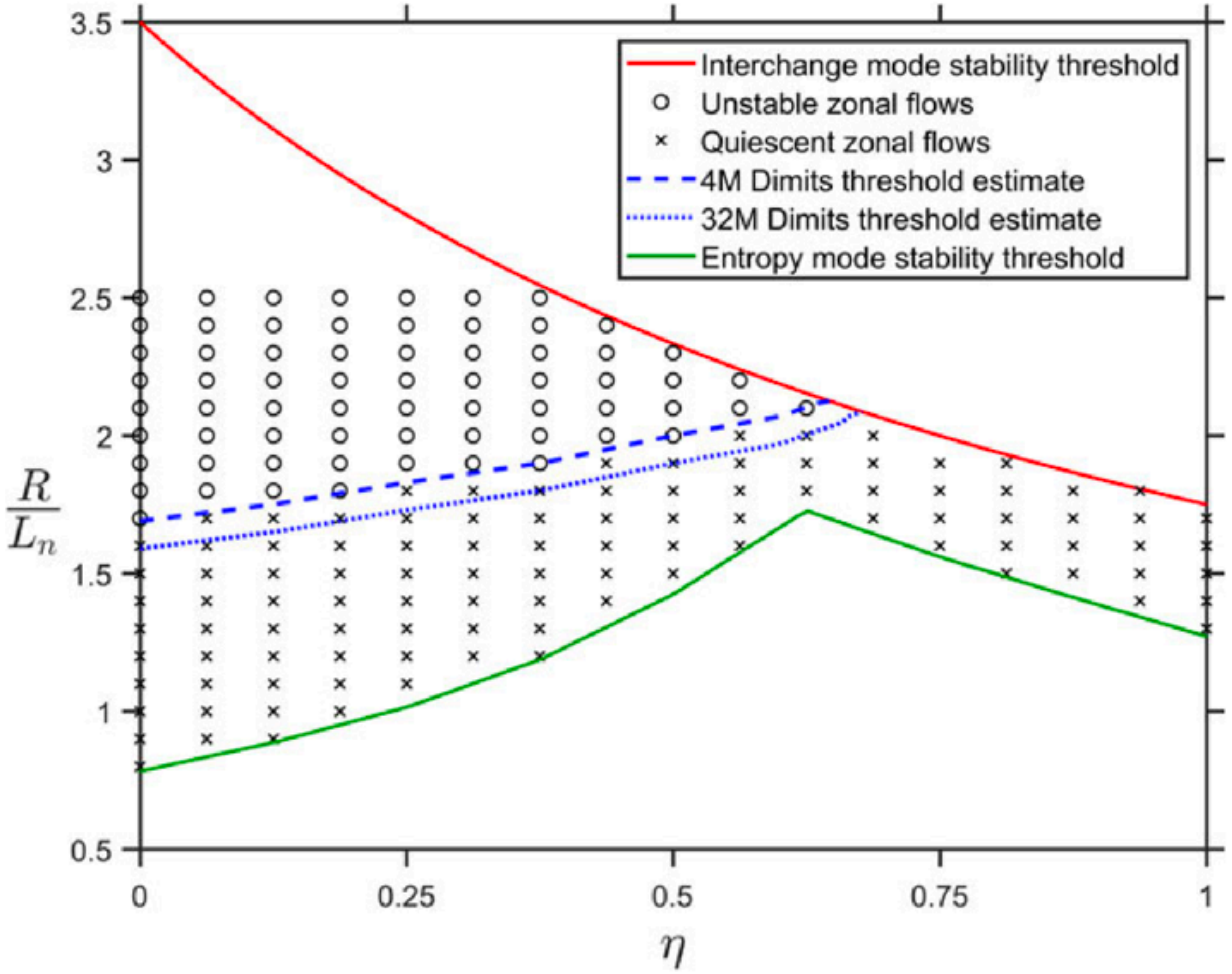


Fig: Predicting the Dimits shift through gyrokinetic tertiary instability analysis.
Hallenbert & Plunk, J. Plasma Phys 88(4), 905880402 (2022)

Zonal flow generation

Linear and nonlinear mechanisms

- ZFs are linearly stable and arise spontaneously via nonlinear mechanisms. Why?
- Modes of "minimal inertia" — see e.g. *P. Diamond, Plasma Phys. Control. Fusion 47 (2005)*:

    1. Electrons move *fast* along the field lines, exploring a flux surface (irrational ι).
    2. zero electron density response.
    3. Small ion (polarization) density gives **large electrostatic potential** for $k_\perp \rho_i \ll 1$, i.e. gyrokinetic quasi-neutrality constraint gives:

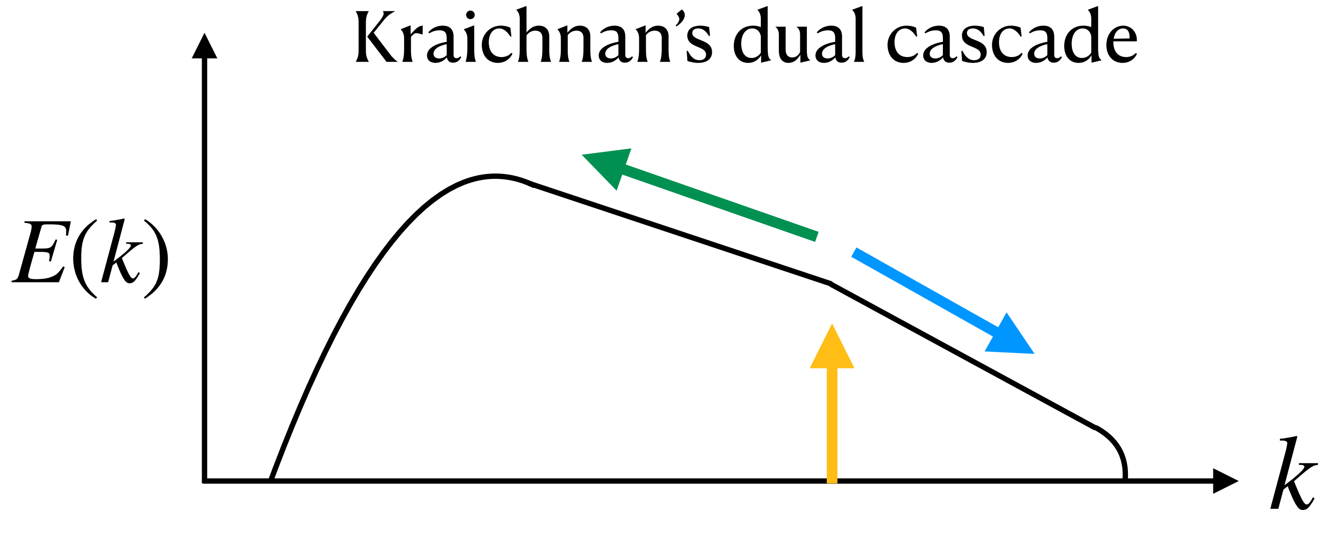
$$e_i \delta n_i = (k_\perp \rho_i)^2 n_i \frac{e_i^2 \phi}{T_i}$$

- Inverse cascade: Quasi-2D limits show conservation of two invariants.
- Secondary instability: A "primary instability" cannot grow forever — eventually the mode itself goes unstable!

ZF generation as inverse cascade

Fjørtoft's argument

- Fjørtoft (1953): Spectral redistribution of energy in 2D turbulence is constrained by the conservation of enstrophy.
- Enstrophy flows to small scales, while energy flows to large scales.



- Zonal flows are modes of maximal effective scale, *G G Plunk et al, New J. Phys. 14 103030 (2012).*

Modified Hasegawa-Mima equation:

$$\partial_t(\tau\tilde{\varphi} - \nabla^2\varphi) + \{\varphi,\ \tau\tilde{\varphi} - \nabla^2\varphi\} + v_*\partial_y\tilde{\varphi} = L_*\tilde{\varphi}$$

Energy & enstrophy spectra:

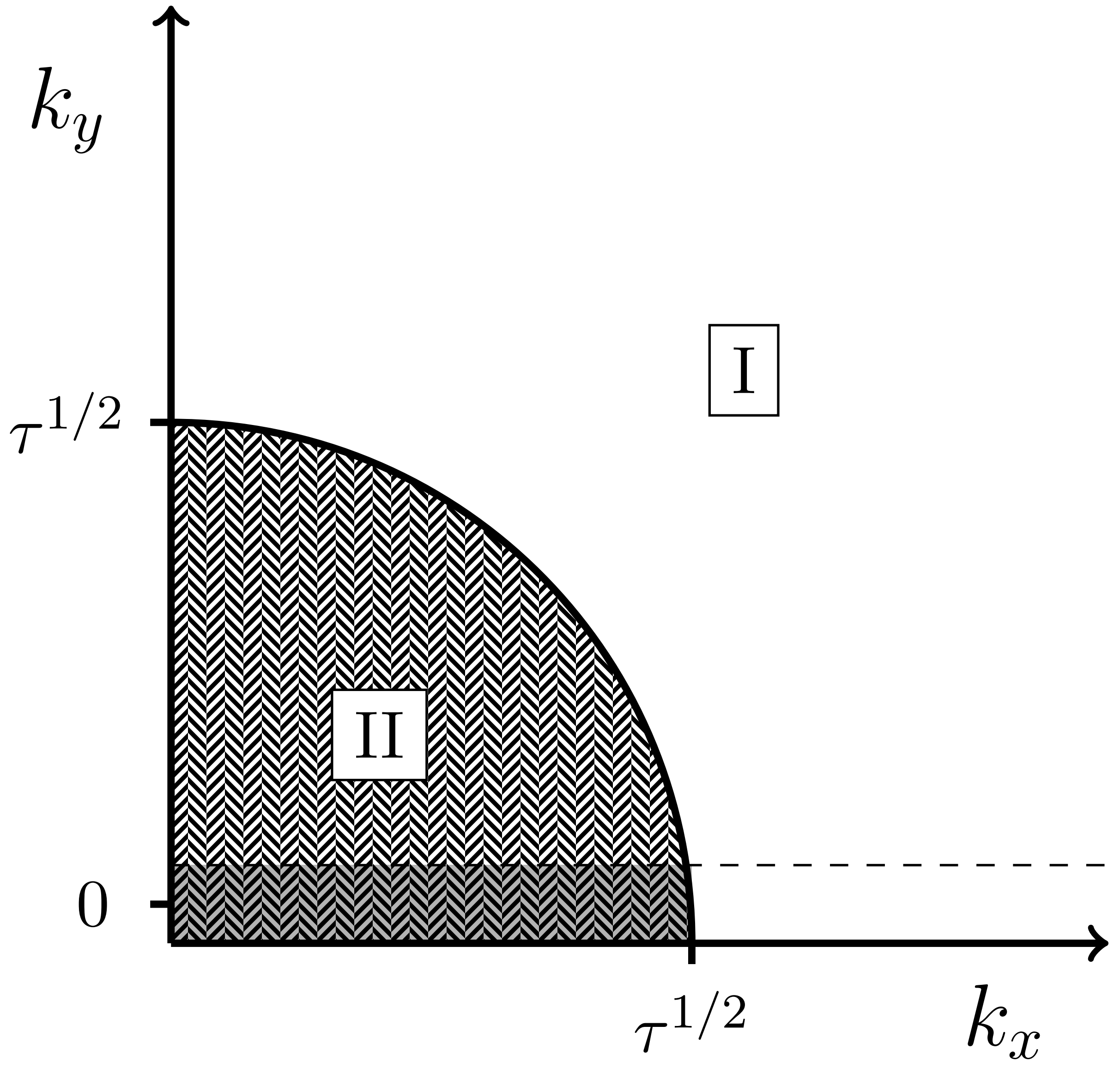
$$E(\mathbf{k}) = \frac{1}{2}\left(\tilde{\tau} + k^2\right)|\varphi(\mathbf{k})|^2$$

$$Z(\mathbf{k}) = \frac{1}{2}\left(\tilde{\tau} + k^2\right)^2|\varphi(\mathbf{k})|^2$$

Ratio of invariants defines an effective "scale" q^{-1} for the inverse cascade:

$$\frac{Z(\mathbf{k})}{E(\mathbf{k})} = q^2 = \tilde{\tau} + k^2$$

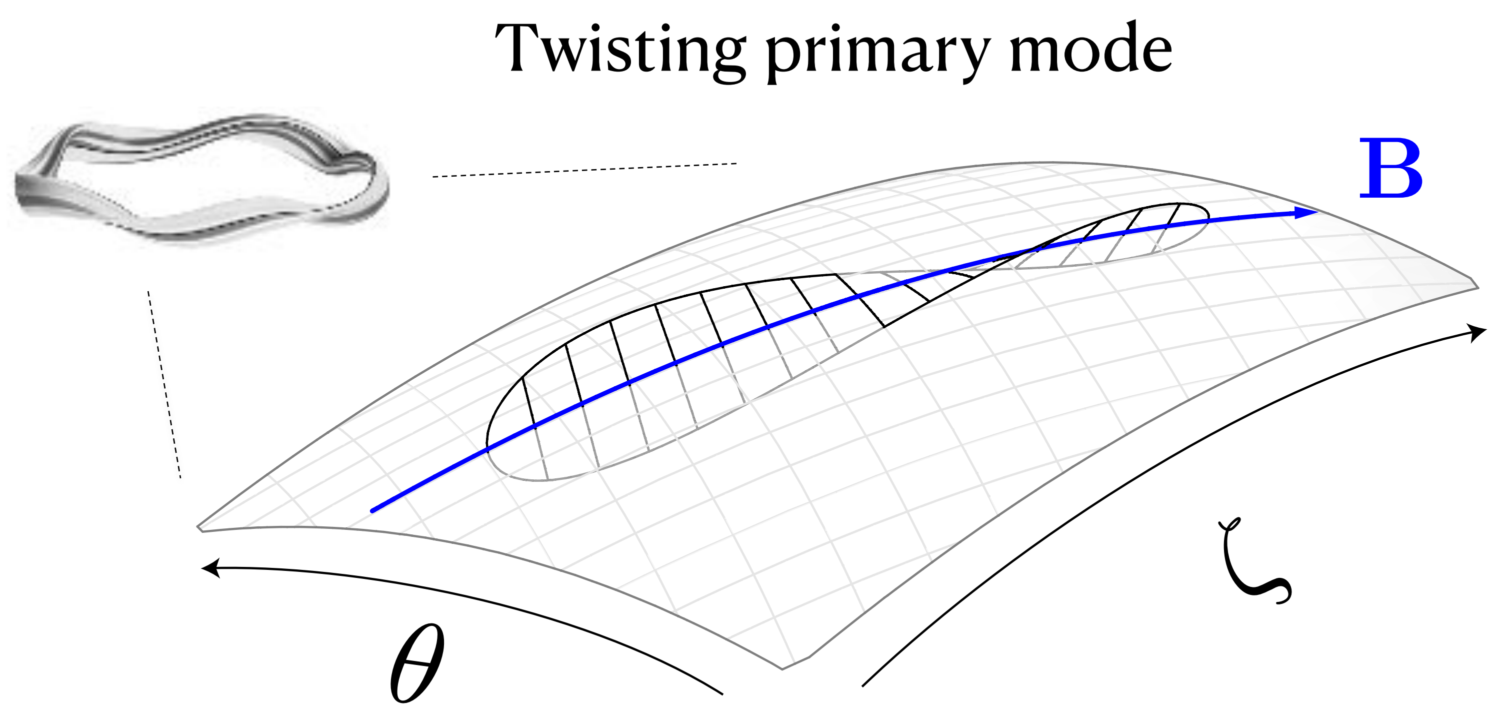
$$\tilde{\tau} = \tau(1 - \delta(k_y))$$

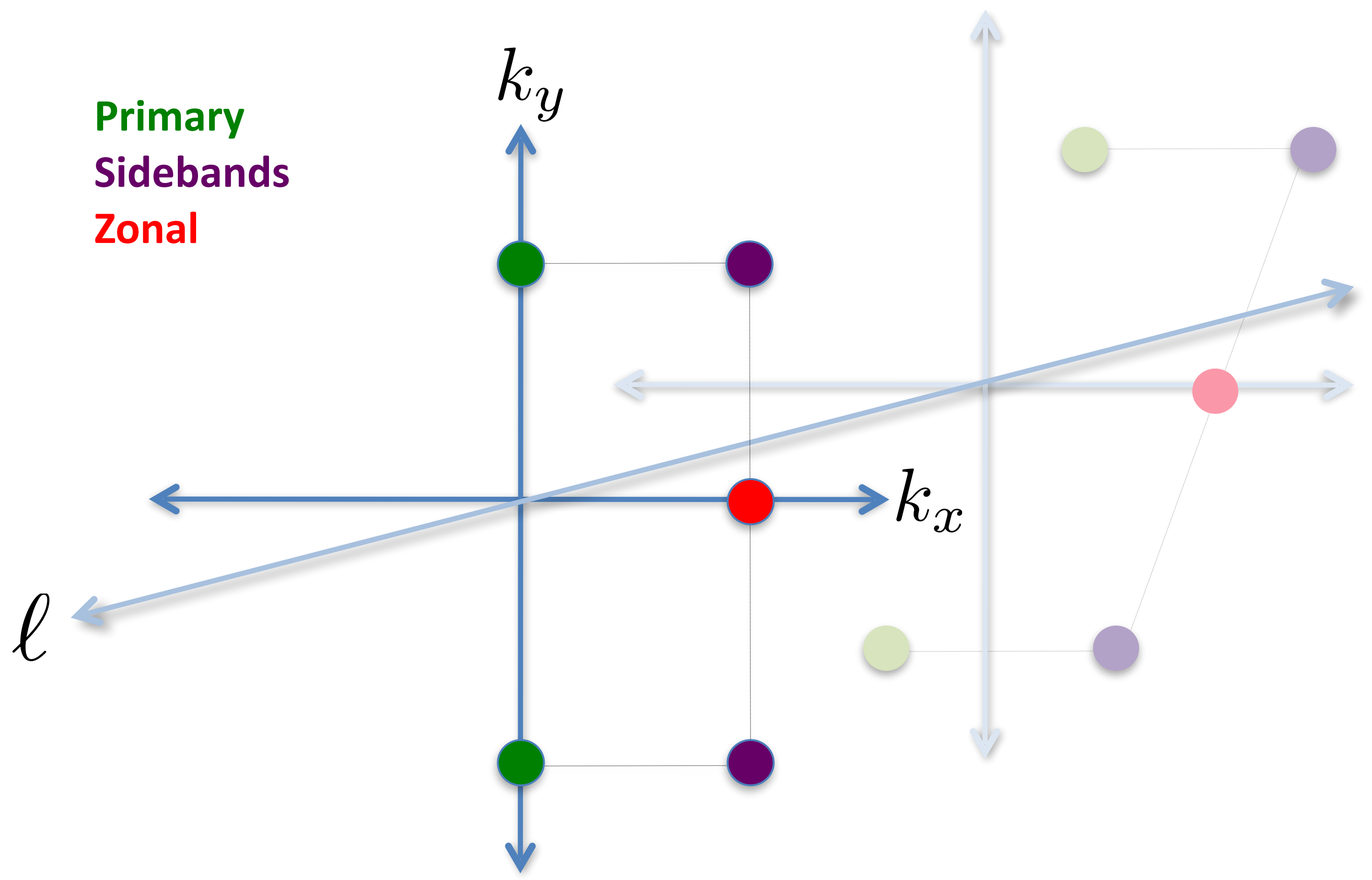


Secondary instability in stellarator geometry

(The geometry doesn’t change much)

- If a linearly unstable mode in the plasma (*primary mode* ϕ_p) grows to sufficiently large amplitude, another mode arises that grows much faster than the primary mode (*secondary mode* $\phi_s \ll \phi_p$).
- This can be described by a linear problem, involving 3-mode coupling — *Plunk et al, New J. Phys. 19 025009 (2017)*.





$$\gamma_s \sim \sqrt{2\langle \Omega_\phi^2 b_r \rangle / \langle b_r \rangle}$$

$$\Omega_\phi = k_\alpha k_\psi |\phi_{p0}| \qquad b_r = \rho_i^2 k_\psi^2 |\nabla \psi|^2$$

- Geometry affects ZF growth via flux-surface averaging.
- Space-filling turbulence couples more strongly to ZFs.

Zonal flow decay

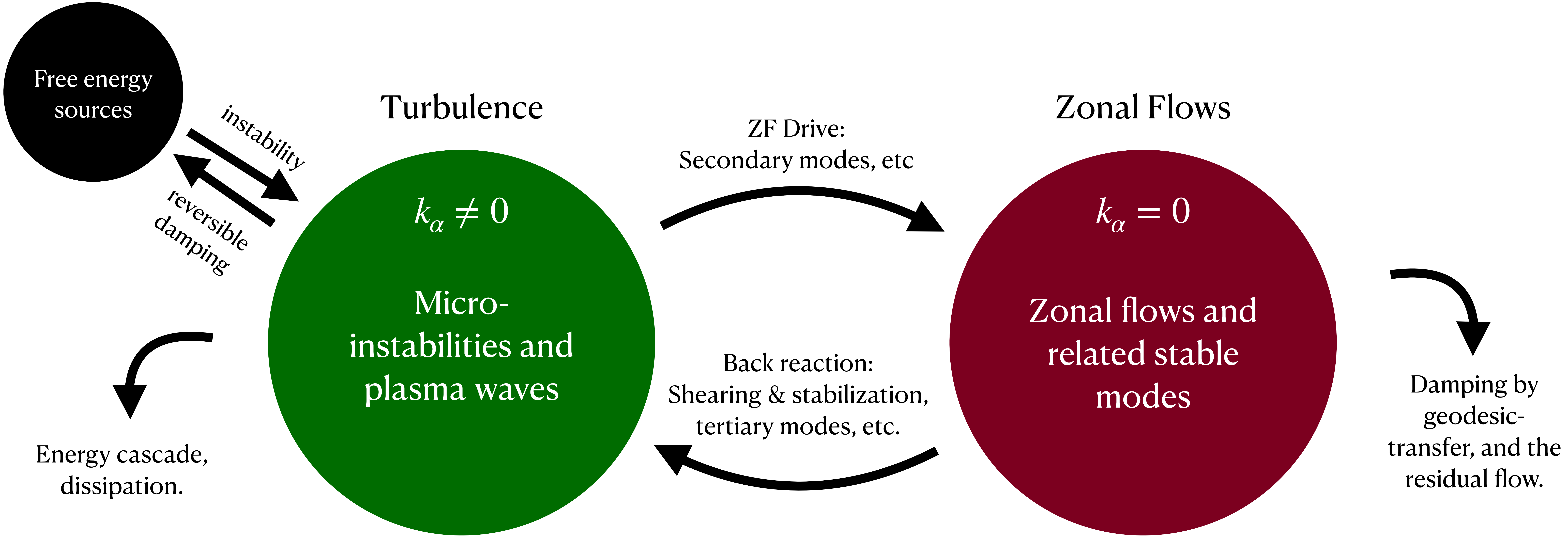
Are zonal flows forever?

- Nonlinear decay mechanisms: “turbulent viscosity”, tertiary modes — generalized Kelvin-Helmholtz limit, …
- Linear collisionless decay: geodesic transfer mechanism to damped acoustic modes: “GAMs”.
  - In tokamaks GAM damping goes as $\gamma \sim \exp(-q^2)$ — this comes from the ratio between the parallel connection length $L_{\|} = qR_c$ and the curvature scale length $R_c \approx R$.
  - The naive expectation is that $q_{\text{eff}} = L_{\|}/R_c$ is smaller in stellarators, and so damping is strong, but this is probably too simplistic.*
  - After the GAMs decay away something remains — the “**residual flow**” of *Rosenbluth & Hinton, Phys. Rev. Lett. 80, 724 (1998).*

*[E. Rodríguez, G. G. Plunk, *in prepration* (2024)]

Interaction of Turbulence and Zonal Flows

Trying to put it all together



Where do the opportunities for stellarator optimization lie?

Residual zonal flow*

Revisiting the problem with optimized stellarators in mind

- Linear gyrokinetic system ($k_\alpha = 0$):

$$\frac{\partial g_a}{\partial t} + v_\parallel \nabla_\parallel g_a + i\omega_{da} g_a = \frac{e_a F_{a0}}{T_a} \frac{\partial \phi}{\partial t} J_{0a}$$

$$\sum_a n_a \frac{e_a^2}{T_a} \phi = \sum_a e_a \int g_a J_{0a} d^3 v$$

- Undamped solutions exist for $k_\alpha = 0$, *e.g.* the residual flow of Rosenbluth and Hinton.
  - When are these solutions actually important?
  - When are they important in stellarators? Can they be optimized?
  - More basic questions remain — are such solutions necessarily "zonal"?

*[Plunk & Helander, *submitted to* J. Plasma Phys (2023); arXiv:2310.14218]

Problem setup

Electrostatic collisionless linearized gyrokinetics with $k_\alpha = 0$

Electrostatic gyrokinetic system ($k_\alpha = 0$):

$$\frac{\partial g_a}{\partial t} + v_\parallel \nabla_\parallel g_a + i\omega_{da} g_a = \frac{e_a F_{a0}}{T_a} \frac{\partial \phi}{\partial t} J_{0a}$$

Quasi-neutrality constraint:

$$\sum_a n_a \frac{e_a^2}{T_a} \phi = \sum_a e_a \int g_a J_{0a} d^3 v$$

where $J_{na} = J_n(k_\perp v_\perp / \Omega_a)$, and $\omega_{da} = k_r \mathbf{v}_{da} \cdot \nabla r = k_r v_{ra}$. Define the **orbit width** δ_r:

$$v_{ra} = \bar{v}_{ra} + v_\parallel \nabla_\parallel \delta_{ra}$$

Note that $\bar{v}_{ra} = 0$ here (*omnigenity*) and we define the transit/orbit average

$$\bar{f} = \frac{1}{2\tau_b} \sum_\sigma \int_{l_1}^{l_2} \frac{f}{\sqrt{1 - \lambda B(l)}} dl, \text{ where } \tau_b = \int_{l_1}^{l_2} \frac{1}{\sqrt{1 - \lambda B(l)}} dl.$$

where l_1 and l_2 are the bounce points such that $B(l_1) = B(l_2) = 1/\lambda$ where $\lambda = \mu / E$. (Defined in limiting sense for passing particles.)

Bounce point:
$v_\parallel = 0$ and $\delta_r = 0$

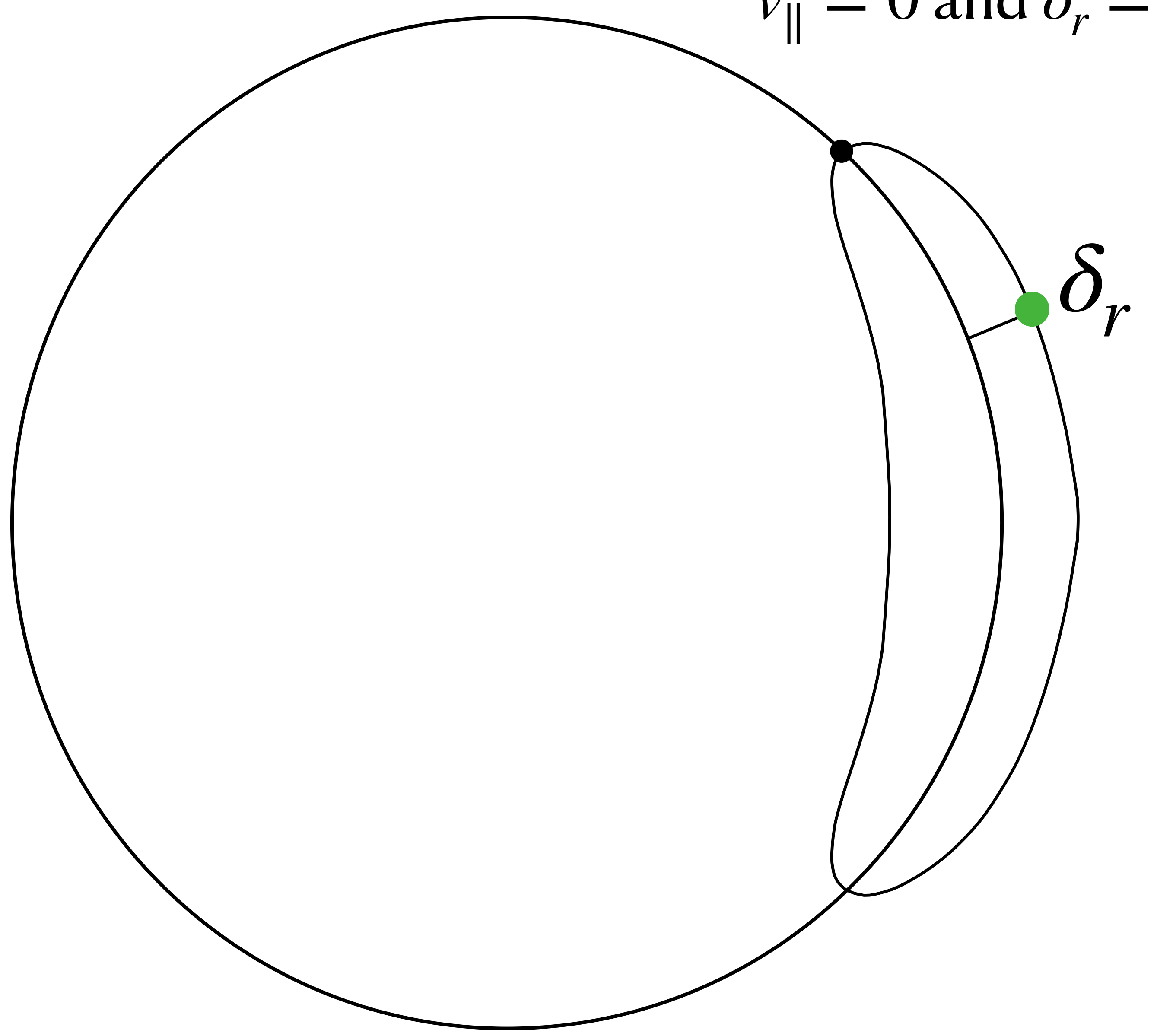


Fig: Orbit width δ_r measures the *radial excursion* from flux surfaces of particles. It is a function of phase space variables (arc length, λ, *etc.*).

Initial value problem

- Laplace transform, introduce integrating factor $h_a = \exp(ik_r\delta_{ra})g_a$

$$(p + ik_r\bar{v}_{ra})\hat{h}_a + v_\parallel \nabla_\parallel \hat{h}_a = \left[p\frac{e_a\hat{\phi}}{T_a}J_{0a}F_{a0} + \delta F_a(0)\right]e^{ik_r\delta_{ra}}$$

where $\delta F_a = g_a - \dfrac{e_a\phi}{T_a}J_{0a}F_{a0}$, and $\delta F_a(0) = \delta F_a|_{t=0}$, i.e. the initial condition.

- Take long-time limit, times much longer than transit/bounce time, $p \ll \omega_b \sim k_r v_\parallel \nabla_\parallel \delta_r \sim k_\parallel v_{th} \sim \gamma_{\mathrm{GAM}}$, i.e. long after GAMs have damped away.

- At dominant order we find $\nabla_\parallel \hat{h} = 0$. At next order we apply the transit average and obtain the solution

$$\hat{g}_a = \frac{1}{p + ik_r\bar{v}_{ra}}\left(p\overline{\frac{e_a\hat{\phi}}{T_a}J_{0a}e^{ik_r\delta_{ra}}}F_{a0} + \overline{\delta F_a(0)e^{ik_r\delta_{ra}}}\right)e^{-ik_r\delta_{ra}}.$$

- This is all standard except we *do not assume* $\overline{\hat{\phi}} = \hat{\phi}$. This means we cannot easily solve for $\hat{\phi}$, indeed using quasi-neutrality we obtain

$$\sum_a \frac{e_a^2}{T_a}\left(n_a\hat{\phi} - \int d^3v J_{0a}F_{a0}\frac{p}{p + ik_r\bar{v}_{ra}}\overline{\hat{\phi}J_{0a}e^{ik_r\delta_{ra}}}e^{-ik_r\delta_{ra}}\right) = \sum_a e_a \int d^3v J_{0a}\frac{1}{p + ik_r\bar{v}_{ra}}\overline{\delta F_a(0)e^{ik_r\delta_{ra}}}e^{-ik_r\delta_{ra}}.$$

Intermediate Residual

Assessing ZFs in a well-optimized stellarators

- Compared to tokamaks, additional solutions are found in stellarators that arise because of unconfined orbits: "Mishchenko oscillations".*
- These are associated with a slow decay ($\gamma_M \sim k_r \bar{v}_{ra}$)
- Unfortunately, they strongly deplete zonal flows even in the limit that $\bar{v}_{ra}$ is small but non-zero.**
  - The "true" long-time residual is therefore (negligibly) small in stellarators.
  - But turbulence has a much shorter intrinsic timescale $\tau_{\mathrm{NL}} \sim a/v_{th}$.
- Define the "intermediate" residual potential,

$$\phi_{\mathrm{res}} \equiv \lim_{\gamma_M t \to 0} \left(\lim_{\gamma_{\mathrm{GAM}} t \to \infty} \phi(t) \right),$$

i.e., we take $\bar{v}_{ra} = 0$!

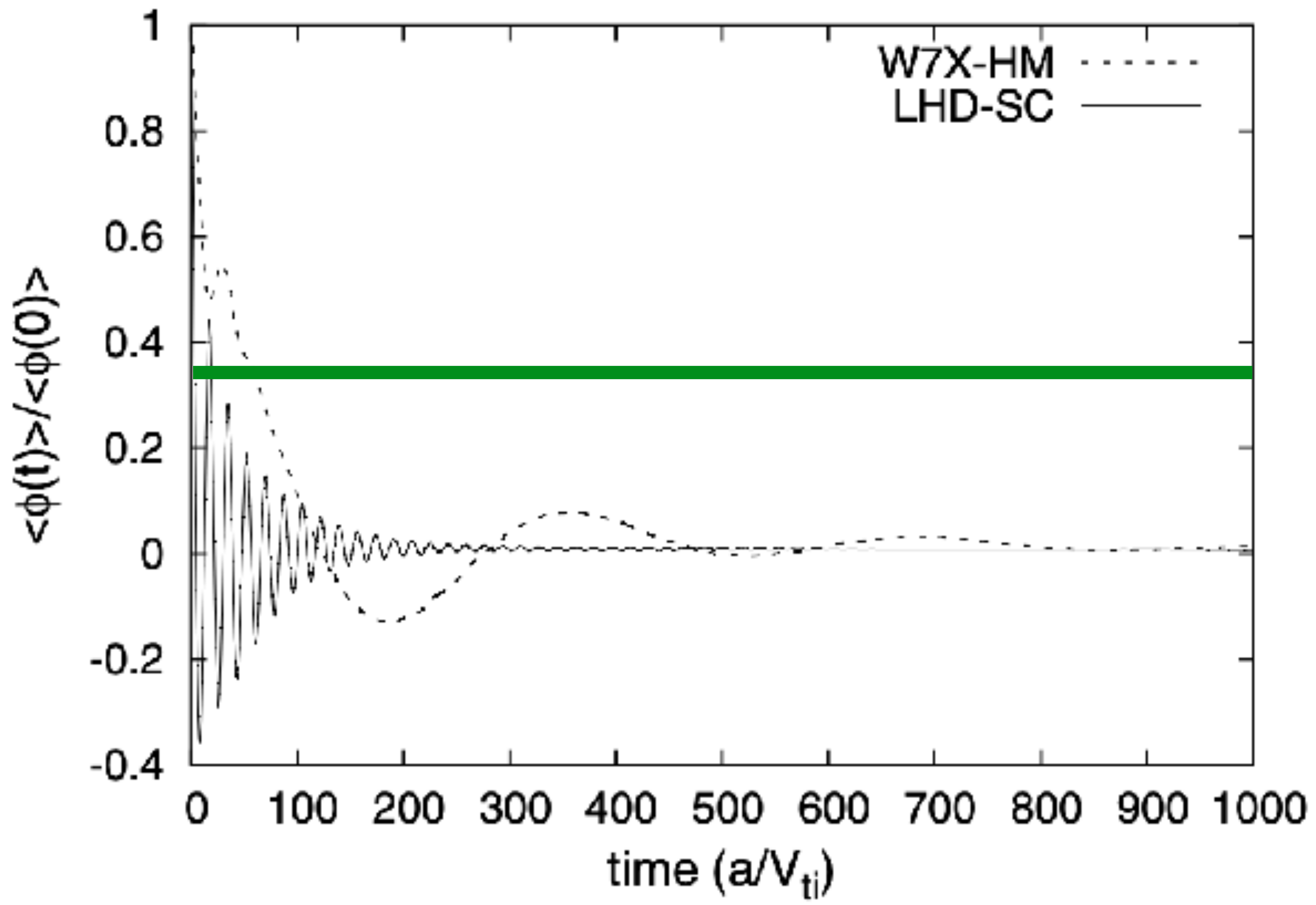


Fig.: Gyrokinetic simulations of linear zonal flow response.** Green line added by me: "intermediate residual".

*[Mishchenko, Helander & Könies, Phys. Plasmas, 15(7):072309 (2008)]
**[Helander, Mishchenko, Kleiber & Xanthopoulos, Plasma Phys. Control. Fusion, 53(5):054006, (2011)]

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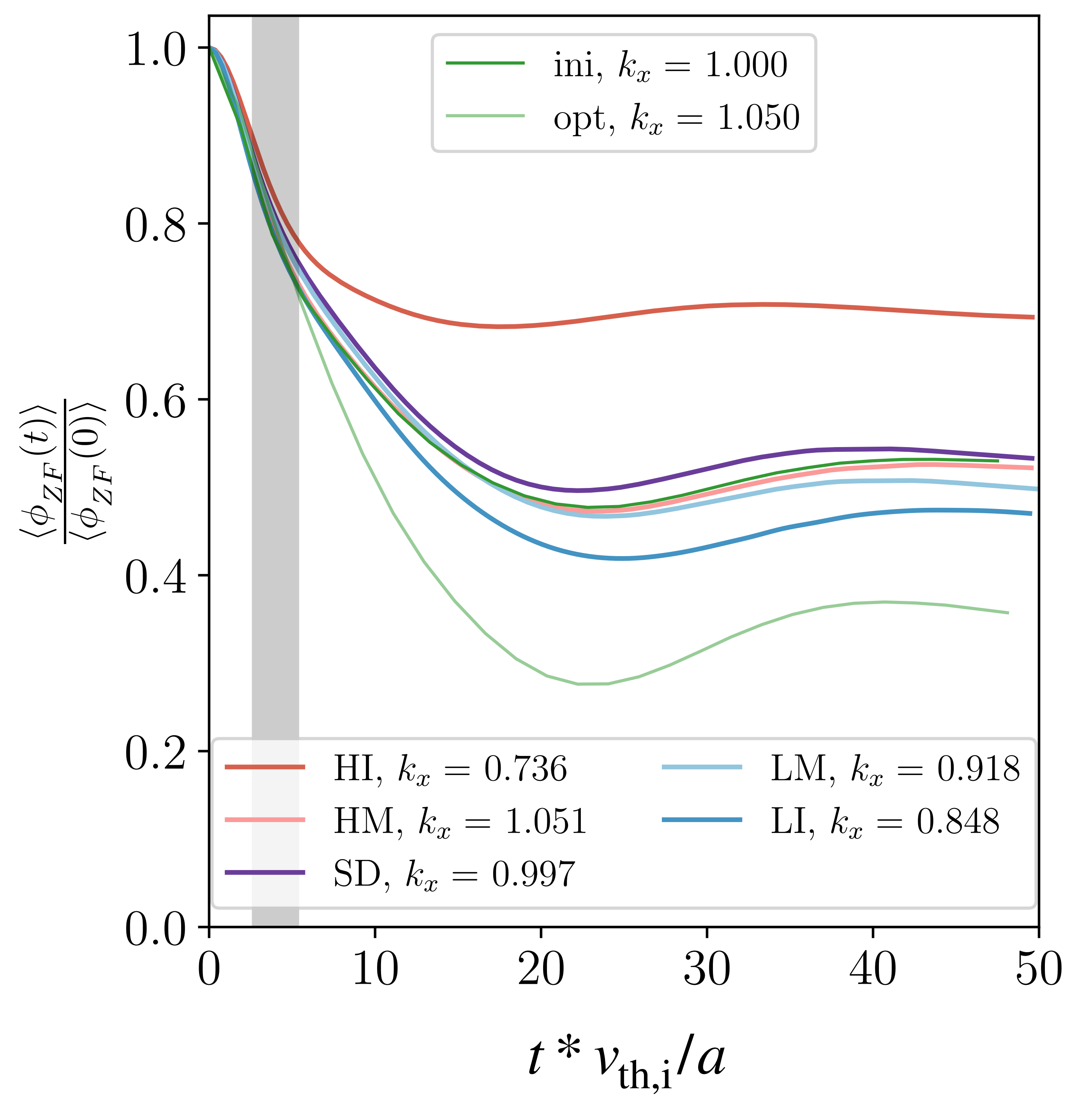

Fig.: Linear zonal flow response in different W7X configurations.
Mora Moreno, et al, Phys Plasmas (submitted)

*[Mishchenko, Helander & Könies, Phys. Plasmas, 15(7):072309 (2008)]
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The residual *zonal flow*

Simplifying assumptions

- The general solution for the residual is complicated, *non-zonal* and dependent on details of the initial condition — interesting theoretically, but hardly the clean outcome of Rosenbluth & Hinton.
- Must make some simplifying assumptions
    1. Well-optimized stellarator (intermediate residual): $k_r \bar{v}_{ra} \ll p$.
    2. Small orbit width & Larmor radius: $k_r \rho_i \sim k_r \delta_{ri} \ll 1$.
    3. Sensible initial condition: $(v_{tha}^3 / n_a)\ \delta F_a(0) \sim b_i e_i \phi(0)/T_i$.
- Item (3) is not obvious, but recall GK-QN equation:

$$\sum_a e_a \int \delta F_a J_{0a} d^3 v = b_i n_i \frac{e_i^2 \phi}{T_i}$$

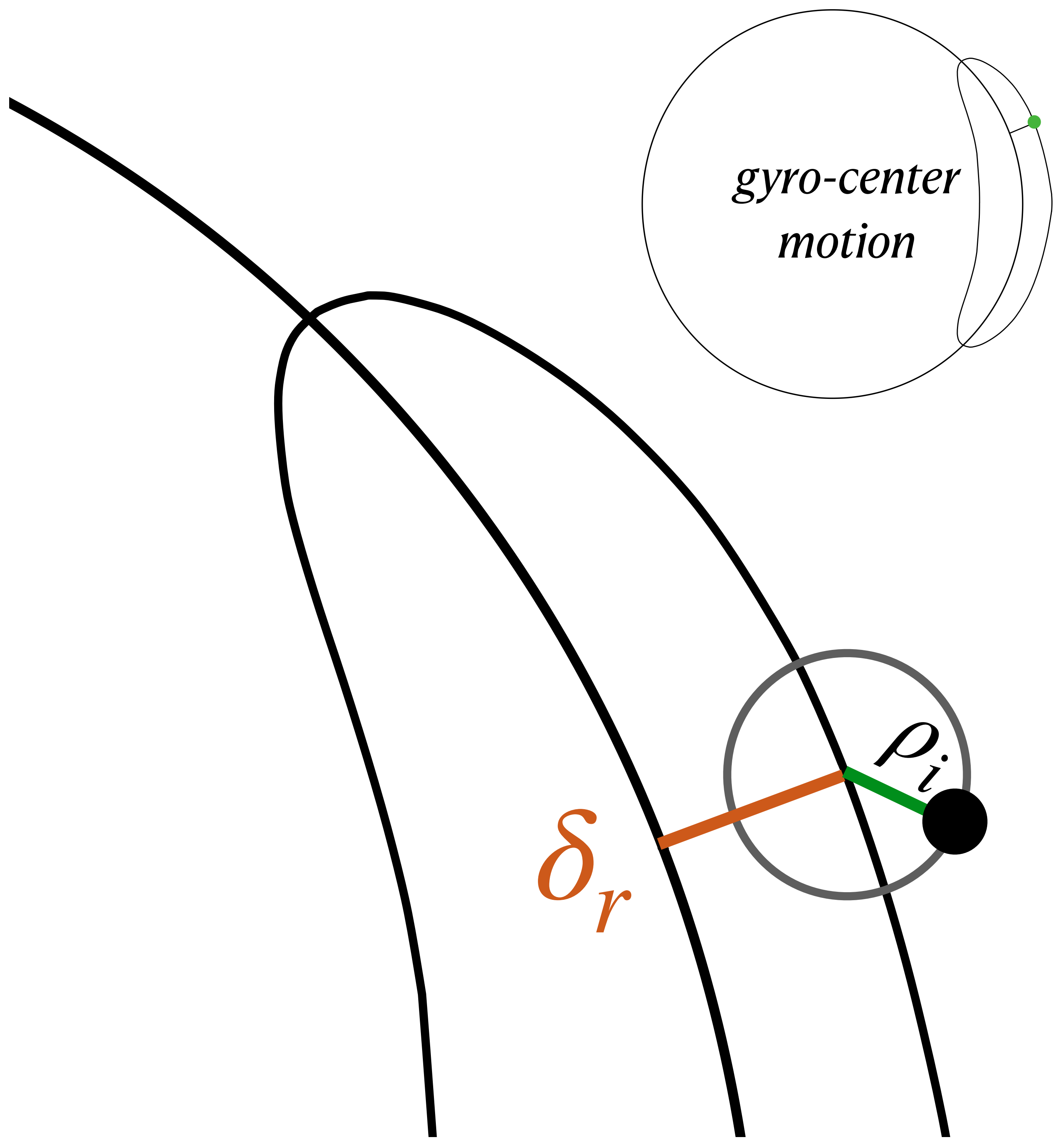


Fig: *Particle motion.*

Residual zonal flow in well-optimized stellarators

- Neglecting electron polarization ($\delta_{re} \ll \delta_{ri}, \rho_i \ll \rho_e$), we obtain

$$\phi_{\mathrm{res}} = \frac{\left\langle b_i \phi(0) \right\rangle}{\left\langle b_i \right\rangle + n_i^{-1} \left\langle \int d^3v F_{i0} k_r^2 \delta_r^2 \right\rangle}.$$

where the flux-surface average is

$$\langle \cdots \rangle = \lim_{L\to\infty} \frac{\int_{-L}^{L} (\cdots) \frac{dl}{B}}{\int_{-L}^{L} \frac{dl}{B}},$$

- This is a very slight generalization of Rosenbluth & Hinton's expression, allowing the initial potential to be non-zonal:

$$\Phi(0) \equiv \langle \phi(0) \rangle \neq \phi(0).$$

<u>Comparing with RH</u>

- RH expression:

$$\phi_{\mathrm{res}}^{\mathrm{RH}} = \Phi(0) \frac{\left\langle b_i \right\rangle}{\left\langle b_i \right\rangle + n_i^{-1} \left\langle \int d^3v F_{i0} k_r^2 \delta_r^2 \right\rangle}$$

- Case of **uniform initial gyro-center charge** $\left\langle b_i \phi(0) \right\rangle = b_i \phi(0)$

$$\phi_{\mathrm{res}} = \Phi(0) \frac{\left\langle b_i^{-1} \right\rangle^{-1}}{\left\langle b_i \right\rangle + n_i^{-1} \left\langle \int d^3v F_{i0} k_r^2 \delta_r^2 \right\rangle}$$

- Note the inequality between the harmonic and arithmetic means implies for this case

$$\phi_{\mathrm{res}}/\Phi(0) \leq \phi_{\mathrm{res}}^{\mathrm{RH}}/\Phi(0).$$

Residual in different stellarators

- The residual is sensitive to a phase-space average of δ_r^2:

$$D = \frac{1}{n}\left\langle \int d^3 v F_0 \delta_r^2 \right\rangle$$

- In tokamaks (and QA stellarators) the orbit width is large:
  1. Distance along the field line between bounce points is $L_\parallel \sim qR$ with $q > 1$.
  2. Low trapped particle fractions (large aspect ratio limit: $\epsilon \ll 1$), but these particles have slow transit (small $v_\parallel$).
  3. $\delta_r \sim q\rho_i/\epsilon^{1/2}$ and $D \sim f_t \delta_r^2 \sim q^2\rho_i^2/\epsilon^{1/2}$.
- The situation is similar for QH stellarators, but the connection length is smaller, $L_\parallel \sim R/|N - \iota|$, so δ_r is proportionally smaller, $D \sim |N-\iota|^{-2}\rho_i^2/\epsilon^{1/2}$.
- QI stellarators are the best, D $\lesssim \rho_i^2$.
- PSA: ZFs not to be confused with undamped equilibrium flows, which are possible in QS stellarators, but not in QI.

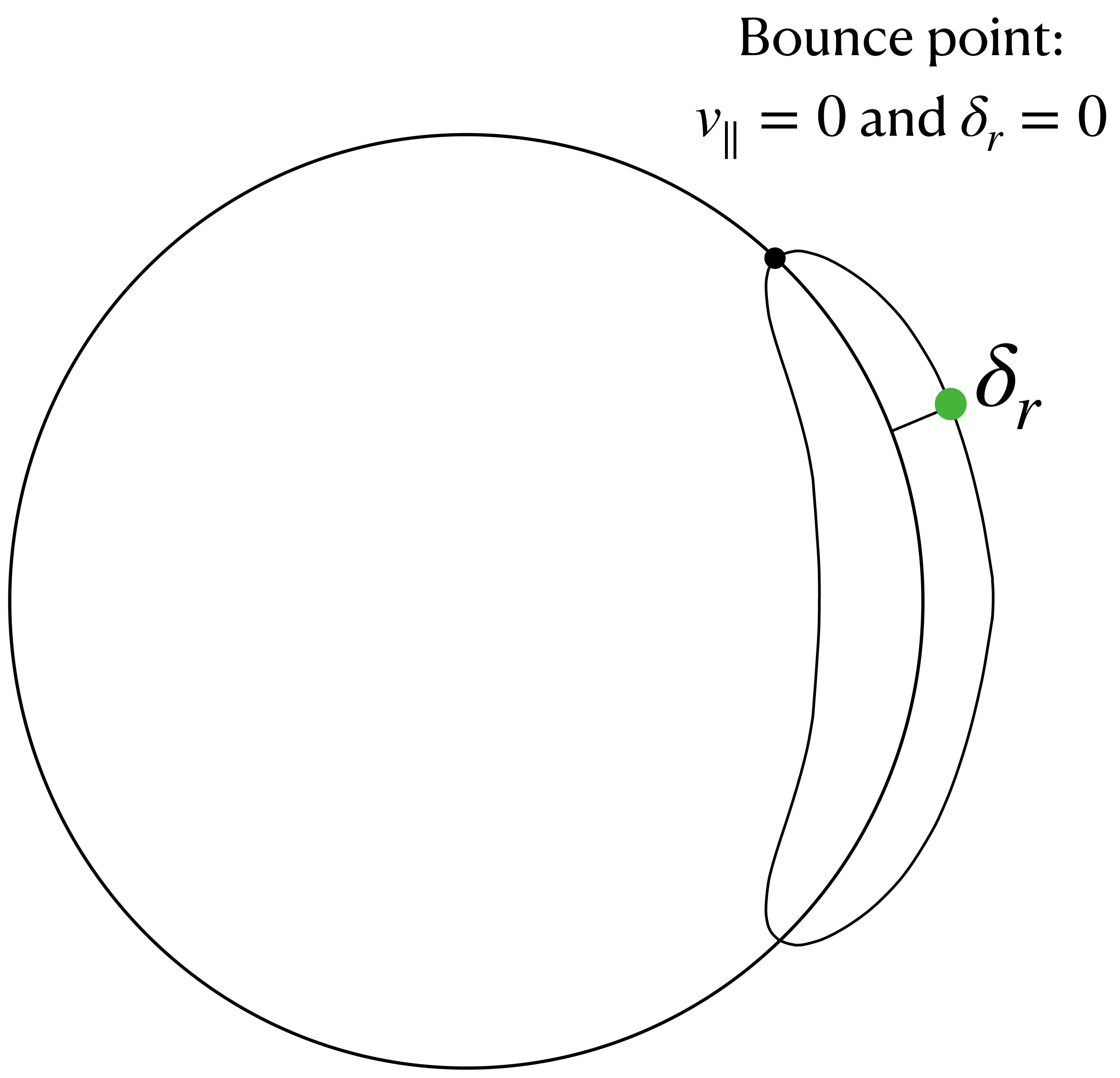


Fig: Orbit width δ_r measures the *radial excursion* from flux surfaces of particles. It is a function of phase space variables (arc length, λ, *etc.*).

Bounding the residual in QI stellarators

- Writing the flux surface average as an integral over a single field period:

$$D = \frac{1}{n}\left\langle \int d^3v F_0 \delta_r^2 \right\rangle = \frac{1}{nV'}\int_0^{2\pi} d\alpha \int_0^L \frac{dl}{B}\int_0^\infty F_0 2\pi v^2 dv \int_0^{1/B} \frac{\delta_r^2 B d\lambda}{\sqrt{1-\lambda B}}$$

- Use $dt = dl/\sqrt{1-\lambda B}$ and apply the Poincaré inequality assuming $\delta_r = 0$ at bounce points

$$D \le \frac{2}{\pi n V'}\int_0^\infty F_0 dv \int_0^{2\pi} d\alpha \int_0^{1/B_{\min}} \tau_b^2 d\lambda \int v_r^2 dt$$

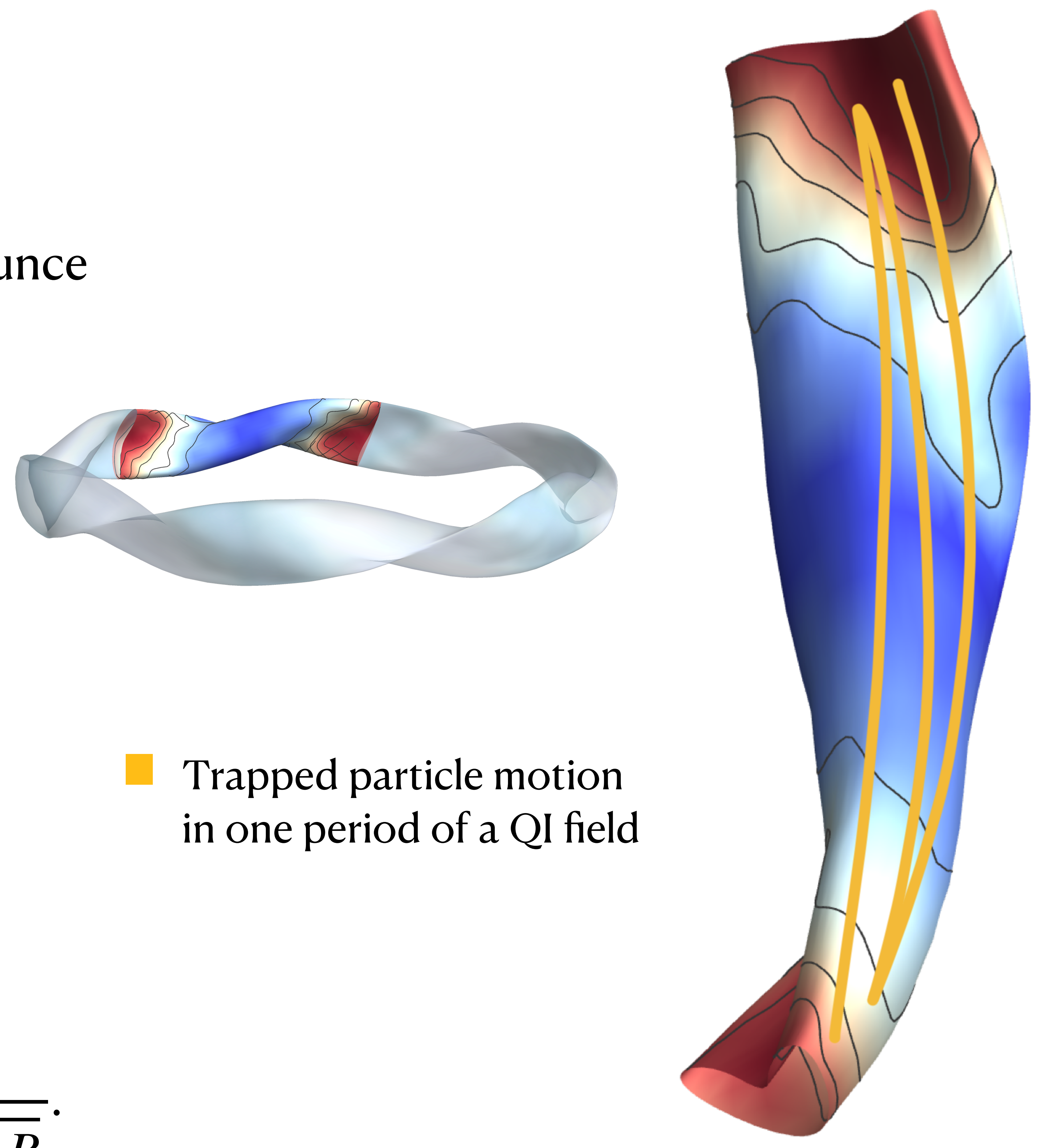
$$\tau_b(\lambda) = \int_{\lambda B(l)<1} \frac{dl}{\sqrt{1-\lambda B(l)}}$$

- For QI, toroidal current is small, $\mathbf{B} = G\nabla\varphi + K\nabla\psi$, so v_r can be written

$$v_r = \frac{v_\parallel^2 + v_\perp^2/2}{\Omega}(\mathbf{b}\times\nabla\ln B)\cdot\nabla r = -\frac{v^2 r'(\psi)}{\Omega}\left(1-\frac{\lambda B}{2}\right)\frac{\partial B}{\partial \theta}.$$

- We find the more rigorous bound

$$D \le \frac{3mT}{2\pi^2 e^2 V'}\left(\frac{dr}{d\psi}\right)^2 \int_0^{2\pi} d\alpha \int_0^L \left(\frac{\partial \ln B}{\partial \theta}\right)^2 dl \int_0^{1/B} \tau_b^2 \left(1-\frac{\lambda B}{2}\right)^2 \frac{d\lambda}{\sqrt{1-\lambda B}}.$$



Trapped particle motion in one period of a QI field

Bounding the residual in QI stellarators

- Estimate from the bound

$$D \leq \underbrace{\frac{3mT}{2\pi^2 e^2 V'}\left(\frac{dr}{d\psi}\right)^2}_{\sim B\rho_i^2/(r^2 L)} \underbrace{\int_0^{2\pi} d\alpha \int_0^L \left(\frac{\partial \ln B}{\partial \theta}\right)^2 dl}_{\sim L(\partial_\theta \ln B)^2 \sim L(r\kappa)^2 \sim r^2/L} \underbrace{\int_0^{1/B} \tau_b^2 \left(1 - \frac{\lambda B}{2}\right)^2 \frac{d\lambda}{\sqrt{1-\lambda B}}}_{\sim L^2/B}$$

- We get roughly

$$D \lesssim \rho_i^2.$$

Final Thought: Zonal flow Optimization

- Do stellarator geometries exist that have large stable zonal flows?
- How to go about finding them.
  1. Choose the right stellarator (QI).
  2. Achieve good collisionless particle confinement.
  3. Try to optimize for small orbit widths (large residual, low damping rate of GAMs) — rigorous bounds derived here, and more detailed targets being developed.
  4. Optimizing for large Dimits shift?