Generalizations of the free-parafermionic Z(N) Baxter quantum chains:

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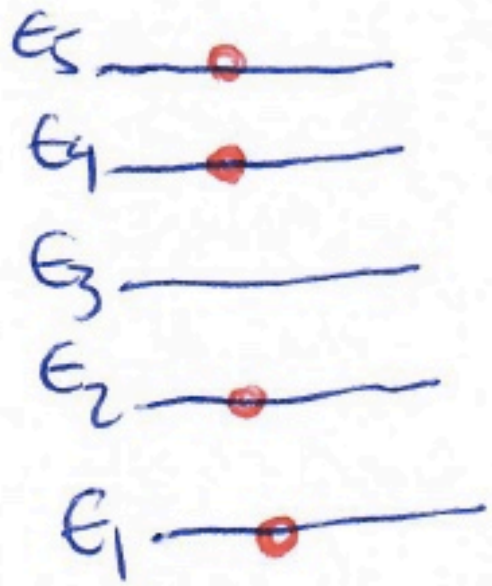
Talk: Baxter2025 Exactly Solved Models and Beyond - Celebrating the life and Achievements of Rodney James Baxter (September, 2025)

What is a free fermion and a free para fermion?

Remember: Stat Mech undergrad: "quantum gases"



$$E_{\{n_i\}} = n_1\epsilon_1 + n_2\epsilon_2 + \dots + n_{\bar{M}}\epsilon_{\bar{M}}, \quad n_i = 0,1$$

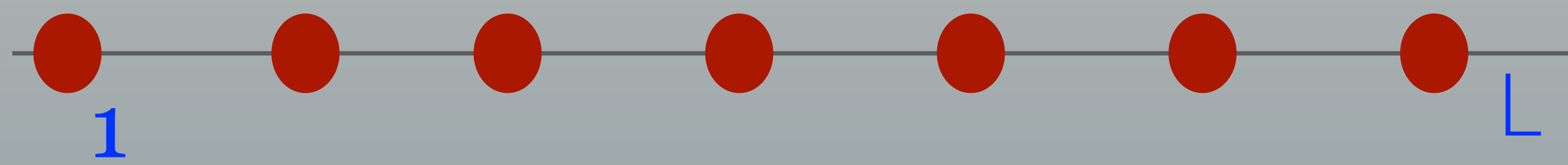
$$E_{\{n_i\}} \to E_{\{n_i\}} - 2(\epsilon_1 + \epsilon_2 + \dots + \epsilon_{\bar{M}})$$

$$E_{\{n_i\}} = -(\omega^{s_1}\epsilon_1 + \omega^{s_2}\epsilon_2 + \dots + \omega^{s_{\bar{M}}}\epsilon_{\bar{M}}),$$

$$\omega = e^{i\frac{2\pi}{2}} = -1, \quad s_i = 0,1$$

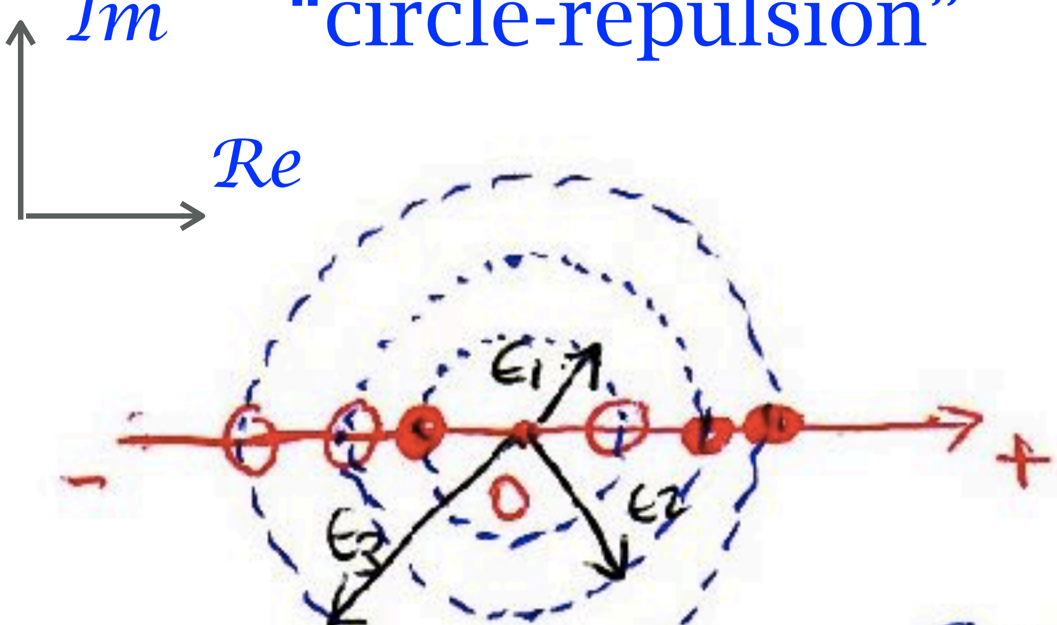
The fermions are free

$$\Xi(\beta) = (e^{-\beta\epsilon_1} + e^{\beta\epsilon_1})\cdot(e^{-\beta\epsilon_2} + e^{\beta\epsilon_2})\cdots(e^{-\beta\epsilon_{\bar{M}}} + e^{\beta\epsilon_{\bar{M}}})$$



Example: Quantum Ising model in a transverse field

"circle-repulsion"

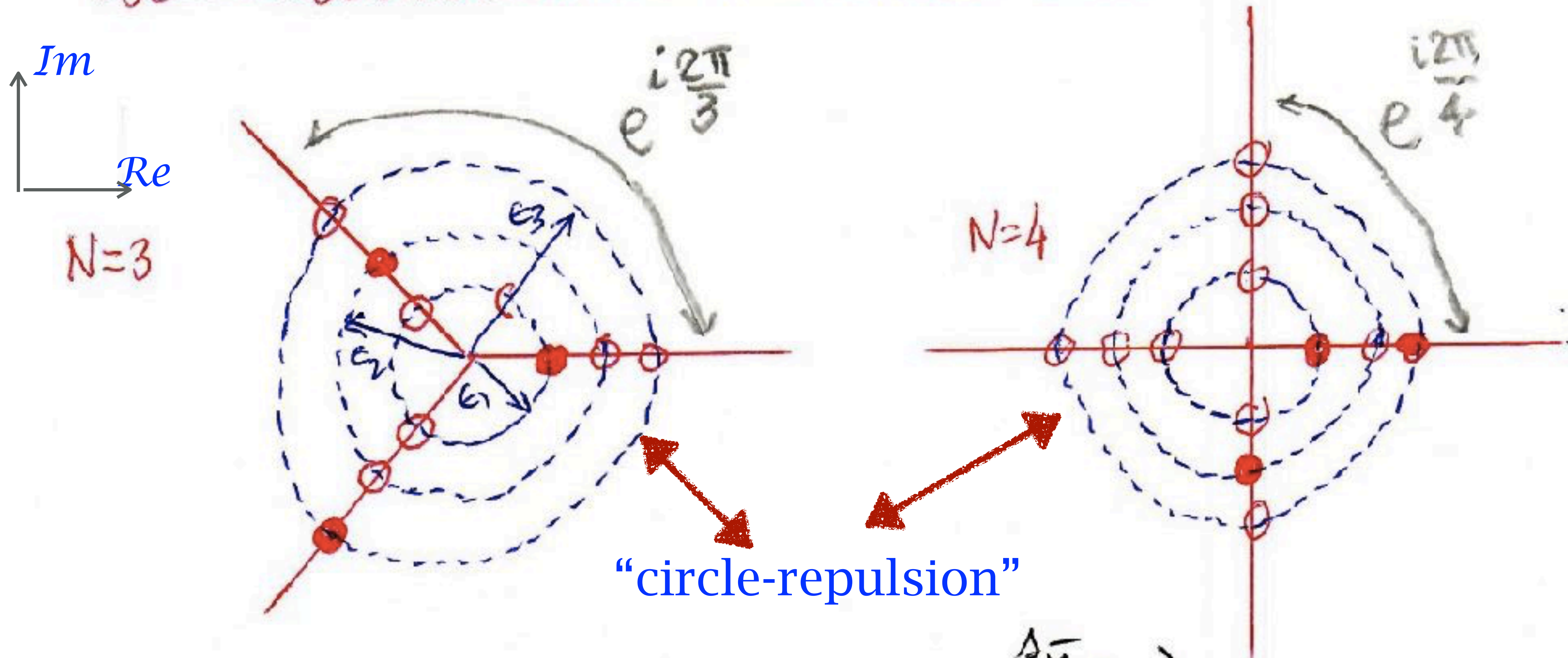


$$\mathcal{H} = -\lambda \sum_{i=1}^{L} \sigma_i^x - \sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z \qquad \text{(FBC)}$$

$$\epsilon_j = 2\cos\left(\frac{2\pi j}{2L+1}\right), \quad j = 1, 2, \dots$$

Quasi-energies $\lambda = 1$

How woud be free parafermions?



$$E_{\{s_i\}} = -(\omega^{s_1}\epsilon_1 + \omega^{s_2}\epsilon_2 + \dots + \omega^{s_{\bar{M}}}\epsilon_{\bar{M}})$$

$$s_i = 0, 1, \dots, N-1, \qquad \omega = e^{i\frac{2\pi}{N}}$$

$$\Xi_1(\beta) = \prod_{i=1}^{\bar{M}} \left(e^{\beta\epsilon_1} + e^{\beta\omega\epsilon_i} + e^{\beta\omega^2\epsilon_i} + \dots + e^{\beta\omega^{N-1}\epsilon_i} \right) \in \mathbb{R}$$

Example: Free-parafermionic Baxter chain

$$\mathcal{H}_B(\lambda) = -\lambda \sum_{i=1}^{L} X_i - \sum_{i=1}^{L-1} Z_i Z_{i+1}^{+}$$

$$XZ = \omega Z X \quad , \quad X^N = Z^N = 1 \quad , \quad Z^{+} = Z^{N-1} \leftarrow$$ $N \times N$ matrices

$$\omega = \exp(i 2\pi / N)$$

$$\epsilon_i = \epsilon_i(\lambda) \quad , \quad \epsilon_i(1) = \left[2 \cos\left(\frac{\pi i}{2L+1} \right) \right]^{2/N}$$

Ex. $N=3$

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \omega = e^{\frac{i 2\pi}{3}}$$

What is essential for fixing the eigenspectra?

Let us see in the Ising case

$$\mathcal{H}_{\mathrm{I}} = \lambda \sum_{i=1}^{L} \sigma_i^x - \sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z \qquad (\mathrm{C.P.C})$$

$$\mathcal{H}_{\mathrm{I}} = -\sum_{j=1}^{M} h_j \qquad \text{recall: } \sigma^x \sigma^z = -\sigma^z \sigma^x$$

$$h_{2i-1} = \lambda \sigma_i^x \quad , \quad h_{2i} = \sigma_i^z \sigma_{i+1}^z$$

$$h_i h_{i+1} = \omega h_{i+1} h_i \quad , \quad [h_i, h_j] = 0 \quad (|i-j|>1),$$

$$\omega = e^{i\frac{2\pi}{2}} = -1$$

For the $Z(N)$ - Baxter chain

$$\mathcal{H}_B = -\sum_{i=1}^{L} h_i$$

$$h_1 = X_1,\ h_2 = \lambda Z_1 Z_2^+,\ h_3 = X_2, \dots, h_{M-1} = \lambda Z_{M-1} Z_M^+,\ h_M = \lambda X_M$$

Algebra:

$$h_i h_{i+1} = \omega h_{i+1} h_i,\quad \omega = e^{i\frac{2\pi}{N}},\quad h_i^N = \mathbb{C}$$

$$[h_i, h_j] = 0 \text{ if } |j-i| > 1$$

Same kind of algebra

All those models are particular Cases of the more general quantum chain - with parameter $p = 1, 2, \dots$

$$\mathcal{H}^{(p)} = -\sum_{i=1}^{M} h_i$$

$\{h_i\}$ $(i = 1, 2, \dots, M)$, generators of the exchange algebra:

$$h_i h_j = \omega h_j h_i \text{ if } \alpha\, j-i \leq p$$

$$[h_i, h_j] = 0 \text{ if } |i-j| > p$$

$$\omega = \rho e^{i\alpha} \leftarrow \text{general complex c-number}$$

(in the previous cases $h_i^N = \mathbb{C}$ number)

Examples of representations $\mathcal{H}_N^{(p)}$

$N=2$ (Ising generalizations)

$$p=1 \begin{cases} \cdot\ \mathcal{H}_2^{(1)} = \text{Quantum Ising Chain} \\ \cdot\ \mathcal{H}_2^{(1)} = -\lambda_1 \sigma_1^x - \lambda_2 \sigma_1^z \sigma_2^x - \lambda_3 \sigma_2^z \sigma_3^x - \cdots - \lambda_M \sigma_{M-1}^z \sigma_M^x \end{cases}$$

$$p=2 \left\{ \cdot\ \mathcal{H}_2^{(2)} = -\lambda_1 \sigma_1^z \sigma_2^z \sigma_3^x - \lambda_2 \sigma_2^z \sigma_3^z \sigma_4^x - \cdots - \lambda_M \sigma_M^z \sigma_{M+1}^z \sigma_M^x \right.$$

(Fendley 3-spin model)

$$p=3 \left\{ \cdot\ \mathcal{H}_2^{(3)} = -\lambda_1 \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^x - \lambda_2 \sigma_2^z \sigma_3^z \sigma_4^z \sigma_5^x - \cdots - \lambda_M \sigma_M^z \sigma_{M+1}^z \sigma_{M+2}^z \sigma_{M+3}^x \right.$$

$$\boxed{N \text{ General}} \quad \underbrace{\sigma_i^z \to Z_i ,\ \sigma_i^x \to X_i \ : \ XZ = \omega Z X \qquad \omega = \exp(i2\pi/N)}$$

Multispin generalization of the Z(N) Baxter free-parafermionic chain

These models are particular cases of even more general free-particle models in frustrated graphs

- Elman, Chapman, Flamia (2021)
- Chapman, Flamia, Kollár (2022)
- Chapman, Elman, Mann (2023)
- Mann, Elman, Wood, Chapman (2024)
- Fendley, Pozsgay (2023)
- Pozsgay (2024, 2025)
- Fukai, Vona, Pozsgay (2025)

Generalizations of 1 dimensional models

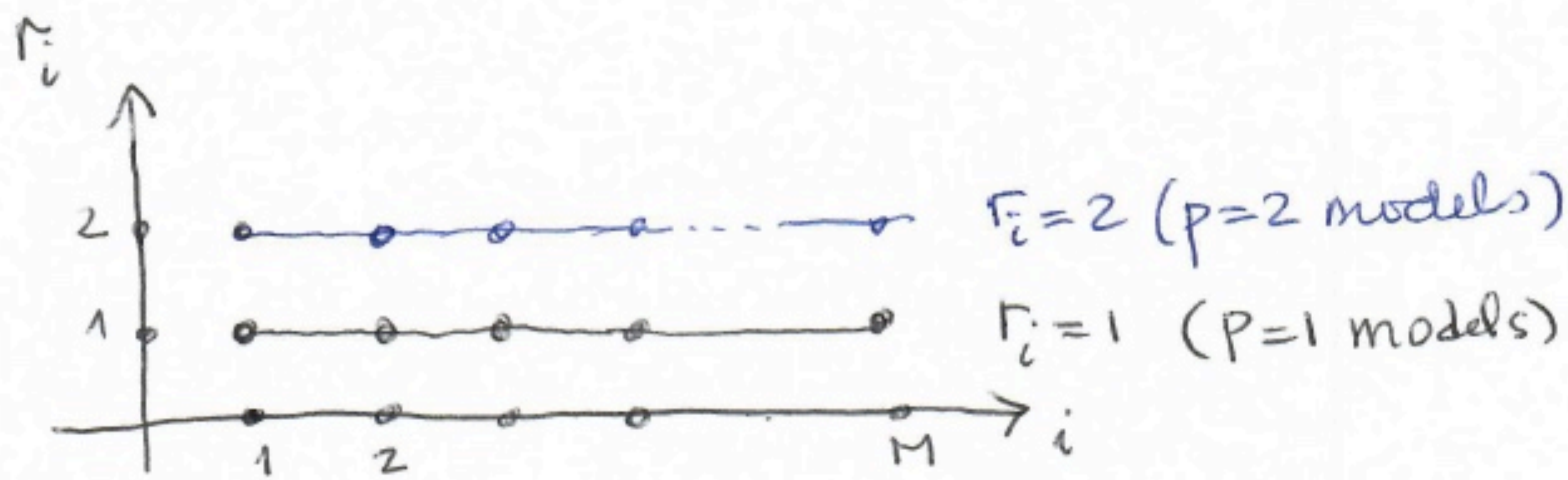
Define $\{h_i^{(r_i)}\}$ generators $r_i > 0$

$$\mathcal{H}(\lambda_1,\ldots,\lambda_M) = -\sum_{i=1}^{M} h_i^{(r_i)}$$

Condition: $\boxed{r_{i+1} \geq r_i - 1 \quad (i=1,\ldots,M)}$

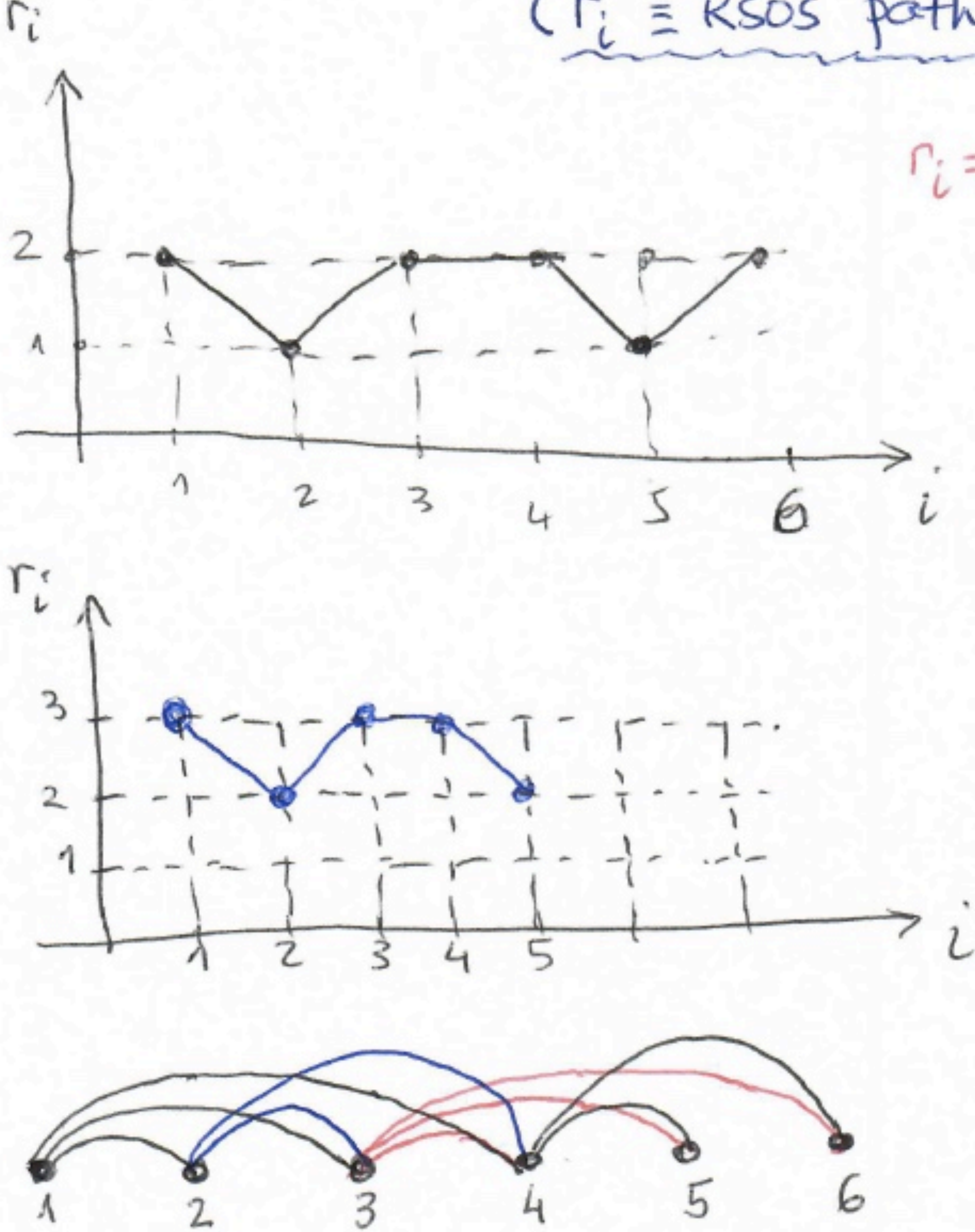
$r_1, r_2, \ldots, r_M \Rightarrow$ range of interactions (encoded on the algebra)

$$h_i^{(r_i)} h_j^{(r_j)} = \begin{cases} \omega\, h_j^{(r_j)} h_i^{(r_i)} & \text{for } (j-i) \leq r_i \\ h_j^{(r_j)} h_i^{(r_i)} & \text{for } (j-i) > r_i \end{cases}$$



($r_i \equiv$ RSOS paths)

$r_i = (2, 1, 2, 2, 1, 2 \ldots)$



The condition ("RSOS") $\boxed{r_{i+1} \geq r_i - 1}$:

Imply the range of interactions on the left of $i \Rightarrow \{l_i\}$

$$l_1 = 1 \ , \quad l_i = \underset{0<j<i}{\mathrm{Max}} \{i-j\} \ , \text{ where } r_j \geq (i-1) \ , \quad i = 2,3,\ldots$$

$$h_i^{(r_i)} \longrightarrow h_i^{(l_i, r_i)}$$

$$h_i^{(l_i,r_i)} h_j^{(l_j,r_j)} = \begin{cases} \omega^{-1} h_i^{(l_i,r_i)} h_j^{(l_j,r_j)} & \text{for } 0<(j-i)\leq l_j \\ \omega\, h_i^{(l_i,r_i)} h_j^{(l_j,r_j)} & \text{for } (j-i)\leq r_i \\ 1\, h_i^{(l_i,r_i)} h_j^{(l_j,r_j)} & \text{Otherwise} \end{cases}$$

Representations

In general a possible realization of $\mathcal{H}$ is in the "word basis"

$$|n_1, n_2, \cdots, n_M\rangle \Longleftrightarrow \left[h_1^{(\ell_1,r_1)}\right]^{n_1} \cdot \left[h_2^{(\ell_2,r_2)}\right]^{n_2} \cdots \left[h_M^{(\ell_M,r_M)}\right]^{n_M} \; ; \; n_i = 0,1,\ldots,N-1$$

If we apply

$$h_i^{(\ell_i,r_i)} |n_1,\ldots,n_M\rangle \Longleftrightarrow \underbrace{h_i^{(\ell_i,r_i)} \cdot \left\{\left[h_1^{(\ell_1,r_1)}\right]^{n_1} \cdots \left[h_i^{(\ell_i,r_i)}\right]^{n_i} \cdots \left[h_M^{(\ell_M,r_M)}\right]^{n_M}\right.}_{\prod_{j=i-\ell_i}^{i-1} \omega^{n_j} |n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_M\rangle}$$

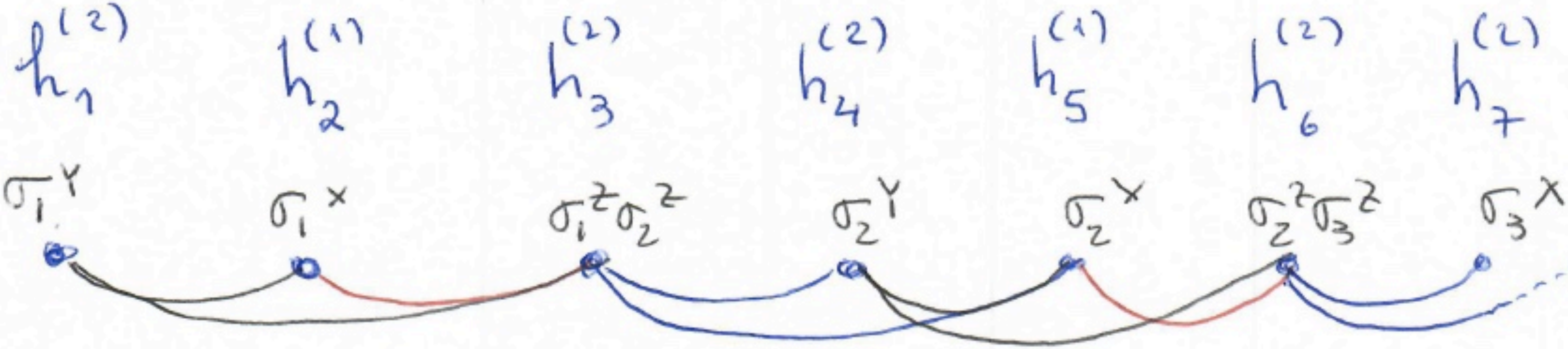
A matrix representation

$$XZ = \omega Z X \; , \; \omega = \exp\left(i\frac{2\pi}{N}\right)$$

$$\left.\begin{aligned} h_i^{(\ell_i,r_i)} &= \lambda_i \Big(\prod_{j=i-\ell_i}^{i-1} Z_j\Big) X_i \\ \mathcal{H}^{(N,r_1,\ldots,r_M)} &= -\sum_{i=1}^{M} \lambda_i \Big(\prod_{j=i-\ell_i}^{i-j} Z_j\Big) X_i \end{aligned}\right\} \text{General range Parafermionic models}$$

Example:

$\boxed{N=2}$ $\{r_i\} = 2, 1, 2, 2, 1, 2, 2, 1, \ldots$



$$\mathcal{H} = \underbrace{-J\sum_i \sigma_i^Z\sigma_{i+1}^Z - h_x\sum_i \sigma_i^X}_{\text{"standard Ising"}} \underbrace{- h_y\sum_i \sigma_i^Y}_{\text{additional}}$$

$\boxed{\text{General } N>2}$

$$\mathcal{H} = \underbrace{-J\sum_i Z_i^+ Z_{i+1} - h_x\sum_i X_i}_{\text{"Standard Baxter free-parafermion"}} \underbrace{- h_y\sum_i Z_i X_i}_{\text{additional}}$$

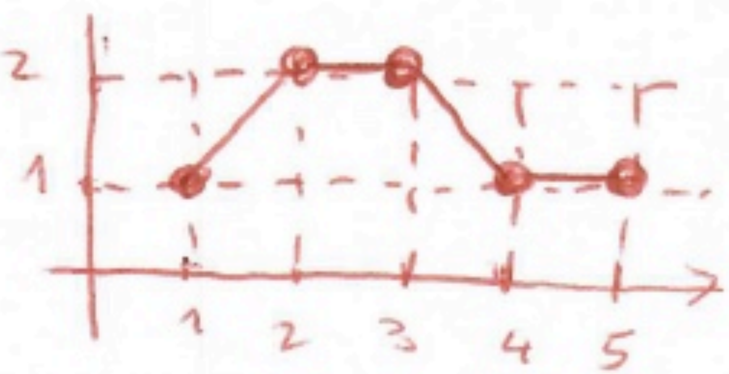
Integrability for general $Z(N)$ chains: Where it comes ?

Commuting charges $\{\mathcal{H}_M^{(\ell)}\}$:

$$\mathcal{H}_M^{(1)} = -\mathcal{H} = h_1^{(r_1)} + h_2^{(r_2)} + \cdots + h_M^{(r_M)}$$

$$\mathcal{H}_M^{(2)} = \sum_{\substack{j_1, j_2 = 1 \\ (|j_2 - j_1| \geq r_{j_1})}}^{M} h_{j_1}^{(r_{j_1})} h_{j_2}^{(r_{j_2})}$$

Example: $r_1 = 1,\ r_2 = 2,\ r_3 = 2,\ r_4 = 1,\ r_5 = 1$



$$\mathcal{H}_5^{(1,2,2,1,1)} = h_1^{(1)} h_3^{(2)} + h_1^{(1)} h_4^{(1)} + h_1^{(1)} h_5^{(1)} + h_2^{(2)} h_5^{(1)}$$

general ℓ-charge

$$\mathcal{H}_M^{(\ell)} = \sum_{\substack{j_1, j_2, \ldots, j_\ell = 1 \\ |j_{i+1} - j_i| \geq r_i}}^{M} h_{j_1}^{(r_{j_1})} h_{j_2}^{(r_{j_2})} \cdots h_{j_\ell}^{(r_{j_\ell})} \quad , \quad \ell = 1, 2, \ldots$$

The number of charges $\mathcal{H}^{(\ell)} \equiv \overline{M}$ (depend on the model)

Theorem $\left[\mathcal{H}_M^{(\ell)}, \mathcal{H}_M^{(\ell')}\right] = 0 \quad \forall \ell, \ell'$ (in involution)

in general $\left\{\mathcal{H}_M^{(\ell)}\right\} \xrightarrow[M\to\infty]{}$ ∞ set of conserved charges

$\Downarrow$

The models are exact integrable

Are free-particle models ?

Answer: Yes due to the Inverse relation

Generating function $G_M(u) = \sum_{\ell=0}^{\overline{M}} (-u)^\ell \mathcal{H}_M^{(\ell)}$

Satisfy the inverse relation:

$Z(2)$: $G_M(u)\, G_M(-u) = P_M(u^2) \leftarrow$ polynomial in u^2 !!!

$Z(N)$: $G_M(u)\, G_M(\omega u) \cdots G_M(\omega^{N-1} u) = P_M(u^N)$!!!

The conserved charges satisfy the recurrence:

$$\mathcal{H}_M^{(\ell)} = \mathcal{H}_{M-1}^{(\ell)} + h_M^{(r_M)}\, \mathcal{H}_{M-(\ell_M+1)}^{(\ell-1)} \quad , \left(h_M^{(r_M)} \equiv h_M^{(\ell_M, r_M)} \right)$$

$$\mathcal{H}_M^{(1)} = -\mathcal{H}$$

$$\Downarrow$$

Polynomials (recursion)

$$\boxed{\begin{aligned} &\underline{P}_M(z) = P_{M-1}(z) - \lambda_M^N\, z\, \underline{P}_{M-(\ell_M+1)} \\ &P_M^{(0)}(z) = 1 \,, \text{ for } M \le 0 \end{aligned}}$$

The inversion relation $\Longleftrightarrow$ Eigenenergies of $\mathcal{H}$ is given in terms of the zeros of the polynomial

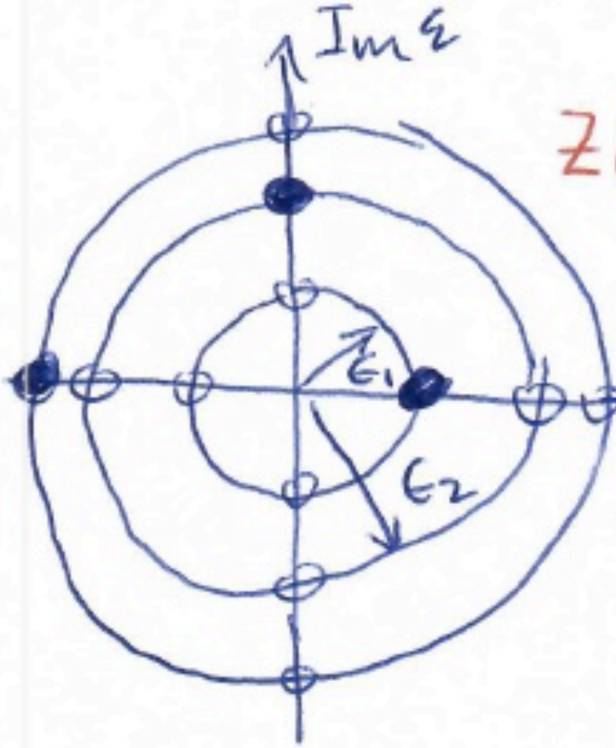
$$P_M(z_i)=0 \quad , \; i=1,2,\cdots,\bar{M}$$

Free-particle

Energies:

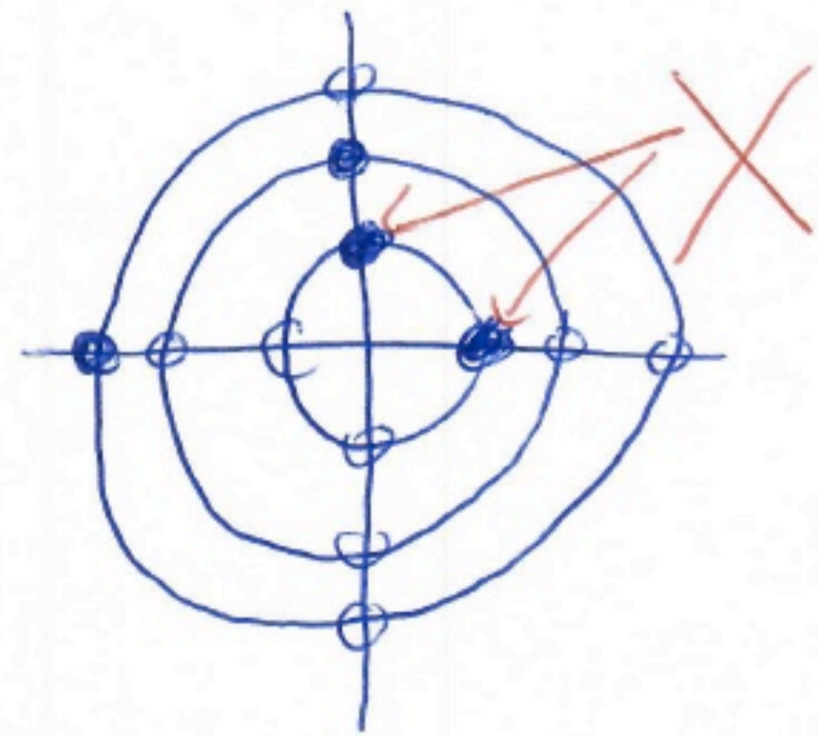
$$E(s_1,s_2,\cdots,s_{\bar{M}}) = -\sum_{i=1}^{\bar{M}} \omega^{s_i}\epsilon_i \quad ; \quad \epsilon_i = z_i^{-1/N}$$

For each ϵ_i we have to choose one and only one $s_i=0,1,\cdots,N-1$



Z(4)

"Circle exclusion"

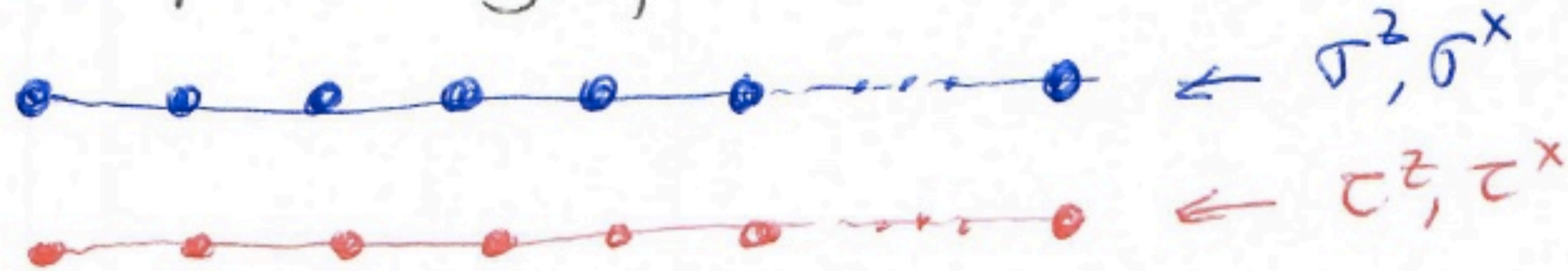


Is it possible to obtain quantum chains with the same quasi-energies ϵ_i but with "no circle repulsion"?

"Circle-repulsion" × "no" circle-repulsion"

Known Correspondence

Decoupled Ising quantum chains



$$\mathcal{H}^{\mathrm{Ising}^2} = -\sum_i \left(\sigma_i^z \sigma_{i+1}^z + \sigma_i^x\right) - \sum_i \left(\tau_i^z \tau_{i+1}^z + \tau_i^x\right)$$

XY quantum chain (*XX* quantum chains)

$$\mathcal{H}^{XY} = -\sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right)$$

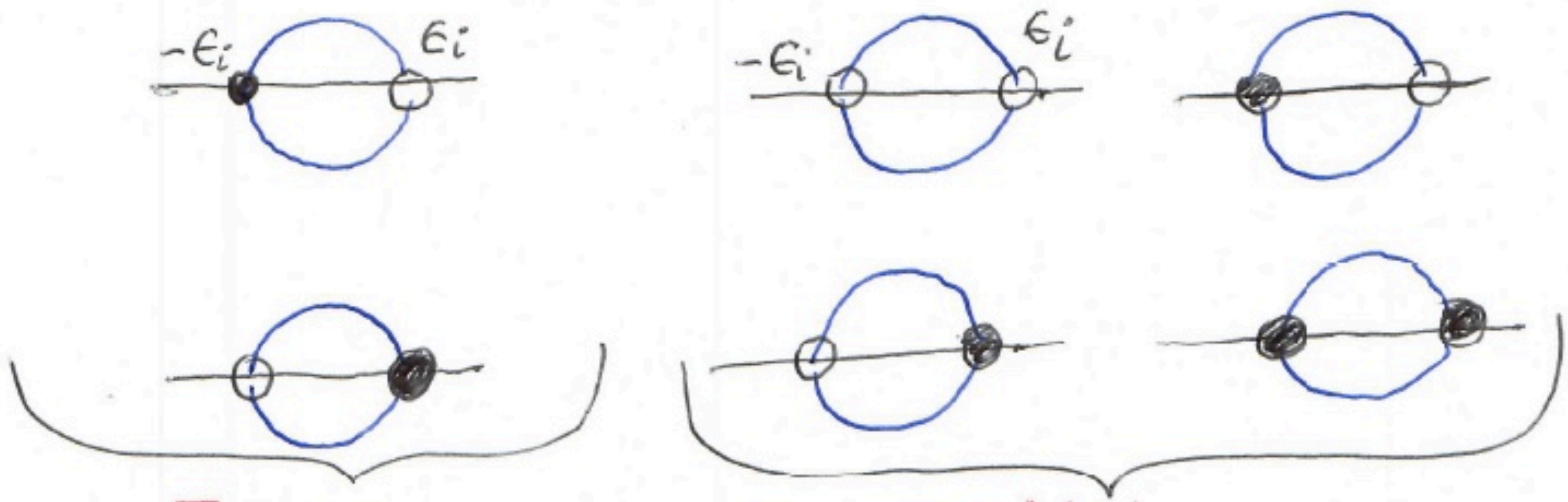
or

$$\mathcal{H}^{XY} = -\sum \left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+\right), \quad \sigma^{\pm} = \sigma^x \pm i\sigma^y$$

"Circle repulsion" x no "circle repulsion"

Ex. Quantum Ising $Z(2)$ x XY-quantum chain ($U(1)$)

For each quasi-energy ϵ_i



Ising XY

The spectrum of the XY model is the same as the one of two decoupled Ising chains.

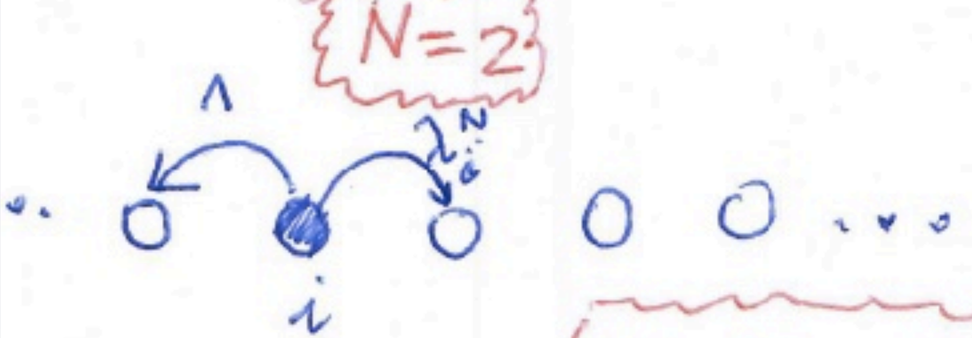
(XXZ quantum chain x Ashkin-Teller chain)

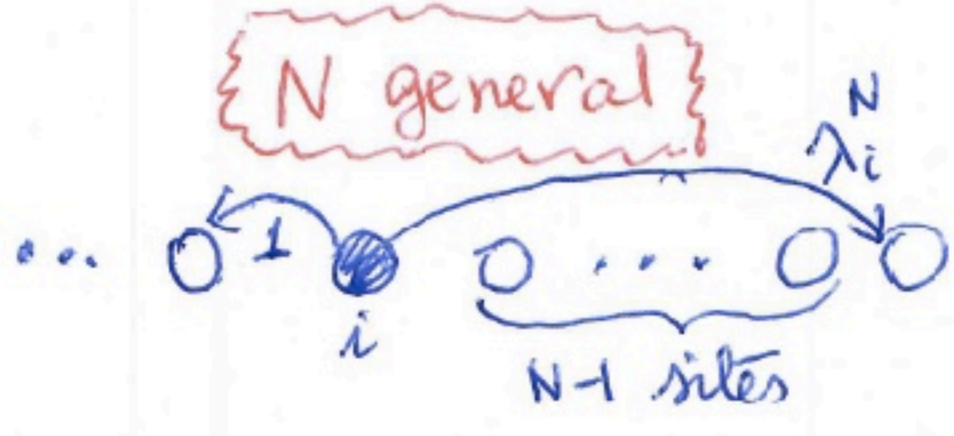
Constructing models with no-"circle-repulsion"
(same pseudo-energies as the Z(N) free-parafermionic quantum chains

XY models with N-multispin interactions.

$$\mathcal{H}_M^{(N,XY)}(\{\lambda_i\}) = \sum_{i=1}^{M+N-2} \sigma_i^+ \sigma_{i+1}^- + \sum_{i=1}^{M} \underbrace{\lambda_i^N}_{\text{coupling constants}} \sigma_i^- \left(\prod_{j=i+1}^{i+N-2} \sigma_j^z \right) \sigma_{i+N-1}^+$$

$\sigma^{\pm} = \sigma^x \pm i\sigma^y$ ($\sigma^x, \sigma^y \to$ Pauli matrices)





U(1) symmetry

$$\left[\sum_{\ell} \sigma_\ell^z, \mathcal{H} \right] = 0 \,!$$

$\boxed{\text{Introduce fermions}^{(c_i, c_i^+)} :}$ Jordan-Wigner transformation : $\sigma_i^\pm \longrightarrow c_i^\pm$

$$\mathcal{H} = -\sum_{i,j=1}^{M+N-1} c_i^+ A_{ij} c_j \Leftarrow \text{bilinear in fermions}$$

$$A_{ij} = \delta_{j,i+1} + \lambda_j^N \delta_{j,i+1-N} \Leftarrow \text{(generalized tri-diagonal)}$$

Diagonalization ("Bogoliubov")

Eigenspectrum
$$E_{s_1,\dots,s_{M+N-1}} = -\sum_{k=1}^{M+N-1} s_k \Lambda_k \;, \quad s_k = 0, 1$$

$\Lambda_k \rightarrow$ Eigenvalues of the connectivity matrix $\mathbb{A}$

Define: $\underline{P}_M^{(N-1)}(u) = \mathrm{Det}\,(\mathbb{1} - \mathbb{A}u)$,

$$\Lambda_k \Rightarrow P_M^{(N-1)}(u_k) = 0 \;, \quad \Lambda_k = \frac{1}{u_k}$$

(we need the zeroes of $\underline{P}_M^{(N-1)}(z_i) = 0$) !!

Laplace cofactor's rule of determinants

$$\underline{P}^{(N-1)}_M(z) = P^{(N-1)}_{M-1}(z) - z\lambda_M^N \, \underline{P}^{(N-1)}_{M-N}(z) \, , \quad P^{(N-1)}_M(z) = 1,\ M \le 0$$

to compare with the $Z(N)$-p-polynomials.

$$\underline{P}^{(p)}_M(z) = \underline{P}^{(p)}_{M-1}(z) - z\lambda_M^N \, \underline{P}^{(p)}_{M-(p+1)}(z) \, , \quad P^{(p)}_M(z) = 1 ;\ M \le 0$$

If $\boxed{N = p+1}$ $\leftarrow$ Same pseudo-energies $\epsilon_i = \frac{1}{z_i^N}$

$P(z_i) = 0$

$N=2$, $p=1$ (Ising) $\longleftrightarrow$ $N=2$ XY model

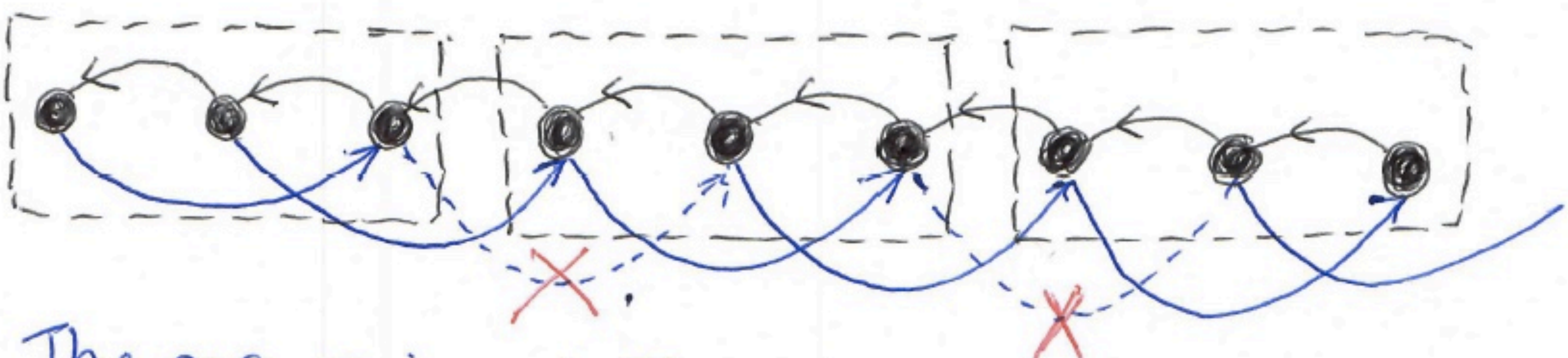
$N=3$, $p=2$ (parafermionic version of the Fendley's model) $\longleftrightarrow$ $N=3$ XY model

$\vdots$

How to generate the others $Z(N)$ - free parafermionic Baxter models ($Z(N)$ with $p=1$) ?

Split the M coupling constants $\{\lambda_i\}$ in cells of size N, as take as non-zero only the first two in each cell.

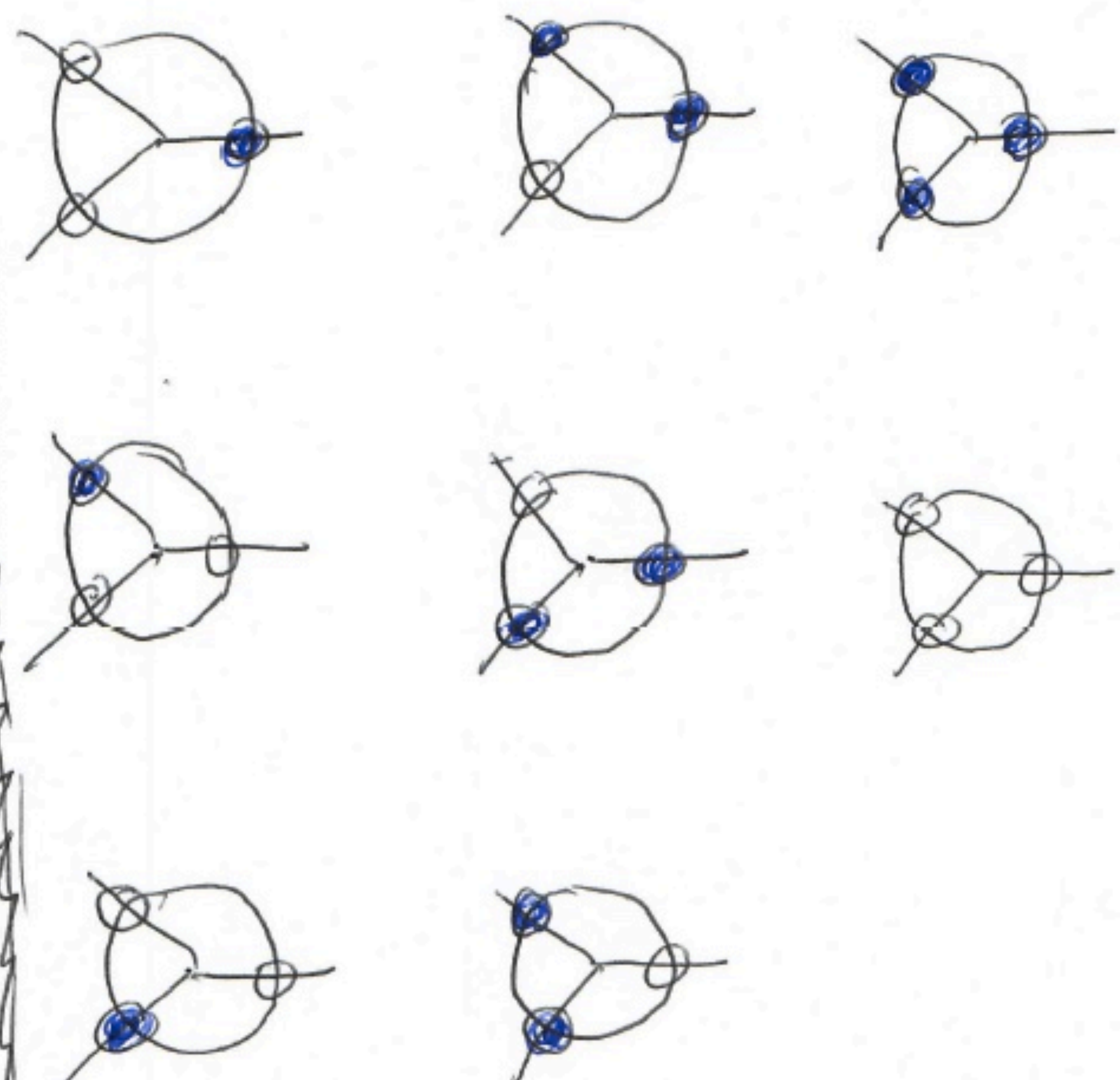
Ex. $N=3$



The recursion relations become the same as the $Z(N)$ - free parafermionic Baxter chains ($p=1$).

Example: spectral comparison for each pseudo-energy ϵ_i

$Z(3)$ - Baxter
$(P=1)$



$N=3$ multispin XY

Critical Properties

remark: # of non-zero quasi-energies ϵ_i : $\bar{M} < M$

Hilbert space dimension M^N

The spectra has a global degenerascy

$$\left(\frac{M}{\bar{M}}\right)^N \quad \text{(extensively large !!)}$$

The models have multicritical points

$\boxed{\bullet \text{ The homogeneous models } r_1 = r_2 = \cdots = r_M = p \quad (p = 1, 2, \dots)}$

		$N=2$	$N=3$	$N=4$	$N=5$
$p=1$	z	1	$2/3$	$2/4$	$2/5$
	α	0	$1/3$	$2/4$	$3/5$
$p=2$	z	$3/2$	1	$3/4$	$2/5$
	α	0	0	$\cdot 1/4$	$4/5$
$p=3$	z	2	$4/3$	1	$4/5$
	α	0	0	0	$1/5$

$$z = \frac{P+1}{N} \quad , \quad \alpha = \mathrm{Max}\left\{0,\ 1-(P+1)/N\right\}$$

New universatity clan of critical behavior

- Models where $r_1 = r_2 = \dots r_M = p$, with $\boxed{pM = N}$ $\Downarrow$ $z = 1$

  We have conformal invariance (although the models are non-Hermitean), with central charge $\boxed{c=1}$.

- The related N-multispin XX models, when in periodic lattices, show quite distinct physical behavior (as compared to the free boundary cases)

- The inhomogeneous interacting range models $(r_{i+1} > (r_i - 1))$

  - Multi-critical points where critical exponents depend on $\{r_1, r_2, \cdots, r_M\}$.

FINAL OBSERVATION

The simple curiosity of a single brilliant man:

$$\boxed{H_{\text{baxter}} = -\lambda \sum_i X_i - \sum_i Z_i Z_{i+1}^{+}}$$

Drives the storms
in several minds

Thank you Rodney