Guided by Baxter

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A different kind of magician

Mark Kac:

In science, as well as in other fields of human endeavor, there are two kinds of geniuses: the "ordinary" and the "magicians." An ordinary genius is a [person] that you and I would be just as good as, if we were only many times better…

It is different with the magicians. They are, to use mathematical jargon, in the orthogonal complement of where we are and the working of their minds is for all intents and purposes incomprehensible. Even after we understand what they have done, the process by which they have done it is completely dark.

Rodney's magic was a bit different from other geniuses. His genius lay in surmounting unbelievably nasty technicalities, but not by using physics intuition or abstract mathematics. He somehow saw his way through the thicket and found the underlying beauty of integrable many-body systems.

However, the beauty isn't always apparent…

As such, he left us many questions to ponder

We still haven't fully come to terms with his 1972 "solution" of the eight-vertex model and its quantum limit, the XYZ spin chain.

A solution of the Yang-Baxter equation gives you an integrable model, but in the absence of a U(1) symmetry, need more insight to get much further.

Rodney invented several tricks to find the Bethe equations for the spectrum:

- He found another model whose transfer matrix commuted with the 8v one, but which had a U(1) symmetry. With much technical genius, he then extracted the Bethe equations.
- He defined (several versions of) the mysterious Q-operator and showed that it and the transfer matrices obeyed the "T-Q" relations.

We've since learned more about Q-operators, e.g. that can they viewed as 't Hooft lines in 4d Chern-Simons gauge theory. The T-Q relations then come from the Witten effect.

Costello, Gaiotto, Yagi 2021

Still many questions remain, e.g. how do we find one in an arbitrary integrable lattice model? Rodney's derivation for 8v remains magical (and very technical).

An elegant analog arises in conformal field theory. Bazhanov, Lukyanov, Zamolodchikov 1994…

The resulting T and Q turn out to arise in (seemingly) far-removed contexts:

They are spectral determinants for ordinary differential equations.

Dorey, Tateo, 1999

In quantum impurity problems, Q is the partition function of the boundary sine-Gordon model, while T is the partition function of the anisotropic Kondo problem.

Fendley, Lesage, Saleur, 1995

T also gives the order parameter statistics of the critical Ising chain.

Lamacraft, Fendley, 2008

Surely the full significance of the T-Q relations is far from understood. Going back to Rodney's original papers is certainly a good idea.

The 8-vertex model/XYZ chain is the canonical example of an ``elliptic" integrable model, where the Boltzmann weights are written in terms of elliptic theta functions.

The lack of a U(1) symmetry makes finding any exact eigenstates extremely difficult, but Rodney found "helical" states that allow the Bethe ansatz to be implemented. The T-Q relations avoid some of the resulting technical difficulties, but introduce new mysteries of their own.

Rodney later invented the corner transfer matrix method to get further; the results for the 1-point functions are spectacular, but even more mysterious.

To bring the discussion back down to earth, let's see how far we can get with less technical methods.

The XYZ spin chain

Can learn some things by focusing on the XYZ chain instead of the full 8v model.

The XYZ Hamiltonian acts on a Hilbert space of L two-state spin systems on a chain (i.e. $(\mathbb{C}^2)^{\otimes L}$). Define operators

$$\sigma_j^r = 1 \otimes 1 \cdots 1 \otimes \sigma^r \otimes 1 \cdots \qquad r = 0, x, y, z$$

Pauli matrix acting on the j th site with $\sigma^0 = 1$

The XYZ Hamiltonian with periodic boundary conditions is

$$H_{\mathrm{XYZ}} = \sum_{j=1}^{L} \left(J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z \right)$$

Sutherland (1970) showed that H_{XYZ} commutes with the 8v transfer matrix. Rodney then showed how to write the Boltzmann weights to exploit this connection, inventing e.g. the spectral parameter in the process.

But we don't need to know anything about 8v to make some progress…

Forget Yang-Baxter, forget elliptic functions

We have shown the XYZ chain is integrable (and more) by an elementary computation.

Fendley, Gehrmann, Vernier and Verstraete;
see also Essler, Fendley and Vernier

We construct explicitly the generating function for conserved charges/higher Hamiltonians in terms of a matrix product operator (MPO).

The trick is to first write the strong zero mode operator in terms of an MPO.

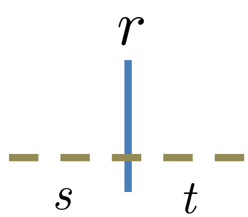
Matrix Product Operators

The MPO is built from four matrices A^r, labelled by $r \in 0, x, y, z$

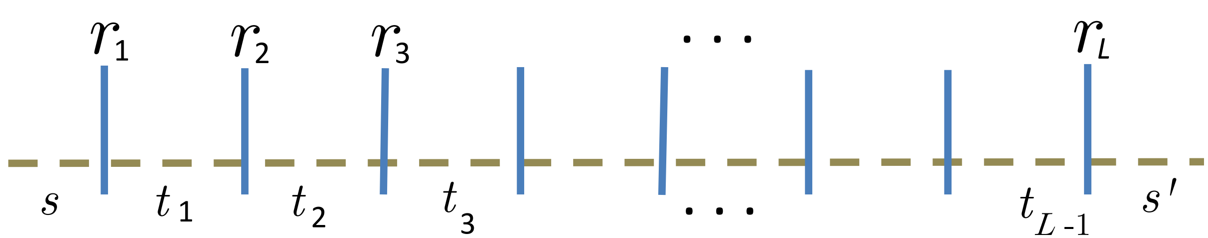
$$O_{ss'} = \sum_{\{r_j\},\{t_j\}} A^{r_1}_{st_1} A^{r_2}_{t_1 t_2} A^{r_3}_{t_2 t_3} \ldots A^{r_L}_{t_{L-1} s'} \sigma_1^{r_1} \sigma_2^{r_2} \sigma_3^{r_3} \ldots \sigma_L^{r_L}$$

Graphically:

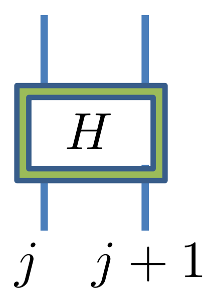
$$A^r_{st} \sigma_1^r =$$



$$O_{ss'} = \sum_{\{r_j\},\{t_j\}}$$

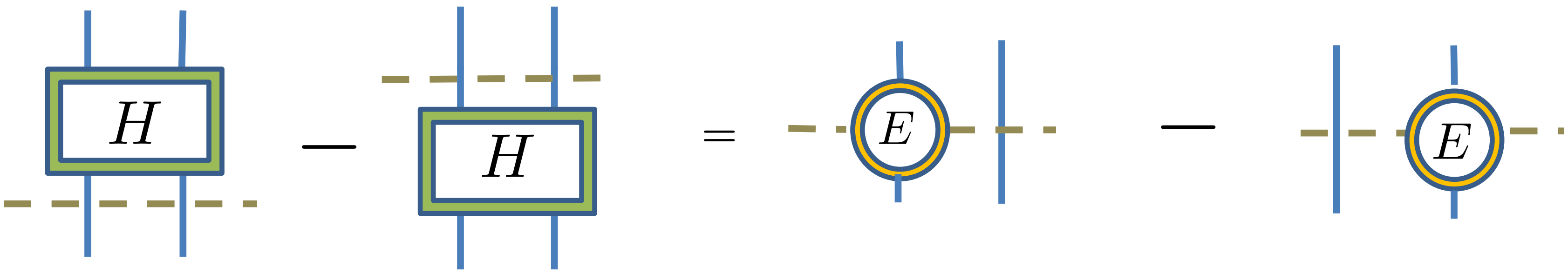


$$H_{\mathrm{XYZ}} = \sum_{j=1}^{L}$$



Conserved charges from MPOs

A conserved charge then arises if we find a solution of



for any matrices E^r. Schematically, $[H_{j,j+1}, A \otimes A] = E \otimes A - A \otimes E$

Equation is bilinear in the E^r and A^r.

For periodic b.c. the E terms can be cancelled telescopically, yielding charge O:

$$[H, O] = 0 \qquad\qquad O = \sum_{s \in 0,x,y,z} O_{ss}$$

For XYZ, we find a simple one-parameter family of solutions.

The solution and the conserved charges arise from a four-channel MPO, i.e. from four 4x4 matrices. They're very sparse: the only non-vanishing matrix elements are

$$A^0_{00} = 1, \quad A^0_{kk} = vJ_k\,, \quad A^k_{0k}A^k_{k0} = vJ_k - v^2 J_l J_m$$

$$E^k_{lm} \propto \epsilon_{klm} \qquad\qquad k,l,m = x,y,z \qquad k \neq l \neq m$$

for any value of the parameter v !

This MPO is much simpler than the 8v transfer matrix, only involving operators of the form $\cdots\sigma^k \otimes \cdots \otimes \sigma^k \otimes \cdots \otimes \sigma^l \otimes \cdots \otimes \sigma^l \otimes \cdots \otimes \sigma^m \otimes \cdots \otimes \sigma^m \cdots$

the dots include only identity operators

Moreover, it involves only $J_x,\ J_y,\ J_z$ directly: no elliptic theta functions!

This MPO generates a sequence of conserved charges whose number grows with L.

For periodic b.c., looks like it's half of those you get from the YBE.

We now can show easily that the open XYZ chain remains integrable for arbitrary boundary magnetic fields at the edges, i.e. add to Hamiltonian

$$H_{\text{left}} = h^x \sigma_L^x + h^y \sigma_L^y + h^z \sigma_L^z \qquad H_{\text{right}} = \tilde{h}^x \sigma_L^x + \tilde{h}^y \sigma_L^y + \tilde{h}^z \sigma_L^z$$

The coefficient of each operator in the periodic MPO is $\mathrm{tr}\ A^{r_1} A^{r_2} A^{r_3} \ldots A^{r_L}$

For the open chain, we instead take them to be $B^{r_1} A^{r_2} A^{r_3} \ldots \widetilde{B}^{r_L}$,
where $B^{r_1},\ \widetilde{B}^{r_L}$ are 4-component vectors instead of 4x4 matrices.

Conserved charge comes from solving boundary relations of the form

$$\left[H_{1,2} + H_{\text{left}},\ B^{r_1} A^{r_2}\right] = -B^{r_1} E^{r_2}$$

$$\left[H_{L-1,L} + H_{\text{right}},\ A^{r_{L-1}} \widetilde{B}^{r_L}\right] = -E^{r_{L-1}} \widetilde{B}^{r_L}$$

where $A,\ E$ must be the same as the bulk ones.

These relations are linear in the B. Moreover, the A are nicer than the Boltzmann weights of the 8v model. It's much simpler than solving boundary Yang-Baxter (i.e. Sklyanin) to find the conserved charges and ensuing integrability.

Can also generalise to more complicated b.c. simply by gluing nearby spins.

A preview of things to come

- This MPO can be written in terms of (sums of) products of 8v transfer matrices.

- This MPO sheds some light on the helical states, the exact eigenstates Baxter constructed. Maybe the XYZ Bethe ansatz can be simplified as well ?!?

- The higher-spin generalization can be written in an extremely interesting and very simple algebraic form, again without elliptic functions. work in progress with Eric Vernier

- Very likely the latter will generalize nicely away from $\mathrm{sl}(2)$.

- Not obvious, however, what this MPO has to do with the usual Q-operators.

Going beyond traditional YBE-type methods

I hope I've been convincing that there remain many interesting things to be learned about the eight-vertex model and the XYZ spin chain. There's at least a hint here that there exist non-traditional (and perhaps simpler) methods.

Rodney's work on the integrable N-state chiral Potts model certainly points this way.

-- The magnetization in the ordered phase is unbelievably simple:

$$\langle \sigma \rangle = (1-s)^{\frac{N-1}{2N^2}}$$

with s parametrizing the distance from criticality. Rodney found a determinantal proof of this formula. Is there a parafermionic generalization of Szego's theorem used for the $N=2$ Ising case?

Baxter 2010

-- Rodney found a non-Hermitian "free parafermion" Hamiltonian that can be solved (with open boundary conditions) by a very non-trivial generalisation of free-fermion methods. As with XYZ, the proof that the model is integrable is now elementary, but everything else is still rather intricate.

Baxter 1989; Fendley 2013; Baxter 2013; Au-Yang+Perk 2014-6

Many other questions Rodney left us

- What's with the corner transfer matrix? Where does it come from?

- What's with 3d integrable models? Rodney's papers here are well worth pondering.

- What's the algebraic structure behind elliptic integrable models?
  O elliptic quantum group, where art thou??

- Still not easy to solve integrable models absent U(1) symmetry, c.f. free fermions in disguise with periodic b.c. Is there a systematic way to generalize Rodney's 8v work and find reference states for the Bethe ansatz?

Many thanks to Murray and Vladimir for organizing such a lovely tribute to the amazing work of a lovely man.