q-de Rham cohomology

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Number Theory Down Under, ANU
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The plan

- de Rham cohomology
- *q*-de Rham cohomology
- Prismatic cohomology
- Sketch of proof

de Rham cohomology

de Rham cohomology

Differential k-forms on $\mathbf{A}^n_{\mathbf{Z}}$:

$$\Omega^k_{\mathbf{Z}[x_1,\dots,x_n]} = \bigoplus_{i_1<\cdots<i_k} \mathbf{Z}[x_1,\dots,x_n]dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

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Exterior derivative $\nabla : \Omega^k_{\mathbf{Z}[x_1,\ldots,x_n]} \longrightarrow \Omega^{k+1}_{\mathbf{Z}[x_1,\ldots,x_n]}$

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$$H^i_{\mathrm{dR}}(\mathbf{Z}[x_1,\ldots,x_n]) = H^i(\Omega^\bullet_{\mathbf{Z}[x_1,\ldots,x_n]})$$

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$$\begin{aligned} H^0_{\mathrm{dR}}(\mathbf{Z}[x,y]) &= \mathbf{Z} \\ H^1_{\mathrm{dR}}(\mathbf{Z}[x,y]) &= \bigoplus_{i,j=0}^{\infty} \frac{\mathbf{Z}}{d_{ij}\mathbf{Z}} \left(\frac{i+1}{d_{ij}} x^i y^{j+1} dx + \frac{j+1}{d_{ij}} x^{i+1} y^j \right) \\ H^2_{\mathrm{dR}}(\mathbf{Z}[x,y]) &= \bigoplus_{i,j=0}^{\infty} \frac{\mathbf{Z}}{l_{ij}\mathbf{Z}} x^i y^j dx \wedge dy \end{aligned}$$

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Theorem (Grothendieck)

There is a natural quasi-isomorphism

$$\Omega^\bullet_{\mathbf{Z}[x_1,\ldots,x_n]} \xrightarrow{\sim} C_{\mathrm{dR}}(\mathbf{Z}[x_1, \ldots, x_n])^\bullet$$

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Grothendieck's Theorem is just integration

$$\mathbf{Z}[x]dx \ni f(x)dx \longmapsto \int_y^x f(z)dz \in \mathbf{Z}[x,y]\langle\langle x-y\rangle\rangle$$

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q-derivative of $f(x) \in \mathbf{Z}[x]$

$$\nabla_{x,q}(f) = \frac{f(qx) - f(x)}{qx - x} \quad , \quad \nabla_{x,q}(x^n) = [n]_q x^{n-1}$$

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- *replace ∇_q with $\xi\nabla_q$ for a certain $\xi \in S_q$*

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- $W(R) \subset \prod_{\mathbf{N}} R$ largest subring on which $\varphi_p((r_n)_n) = (r_{pn})_n$ is lifts the Frobenius

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$$\text{Hom}(\mathbf{Z}\{\xi\}, R) \xrightarrow{\sim} W(R) : \theta \mapsto (\theta(\varphi_n(\xi)))_n$$

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Remark

Integral prismatic cohomology really lives on a stack $\Sigma_{\mathbf{Z}}$ which admits a cover $\mathrm{Spf}(\mathbf{Z}\{\xi\}_{\text{dist}}) \longrightarrow \Sigma_{\mathbf{Z}}$

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There is a natural quasi-isomorphism

$$C_\Delta(\mathbf{Z}[x_1, \ldots, x_n])^\bullet \otimes_{\mathbf{Z}\{\xi\}_{\text{dist}}} S_q \xrightarrow{\sim} (q\Omega^\bullet_{\mathbf{Z}[x_1,\ldots,x_n]} \otimes_{\mathbf{Z}[[q-1]]} S_q, \xi\nabla_q)$$

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Thank you!