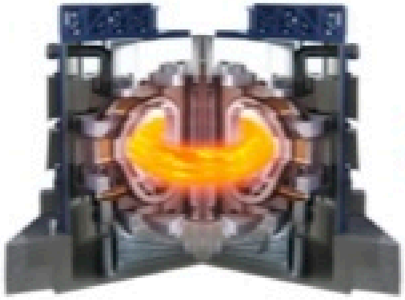

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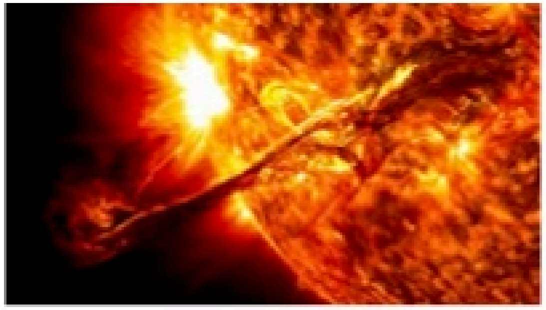

The mathematical challenges of fusion plasmas

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14 September 2023

Acknowledgement: Australian Research Council, Simons Foundation

Fusion Plasma Theory and Modelling

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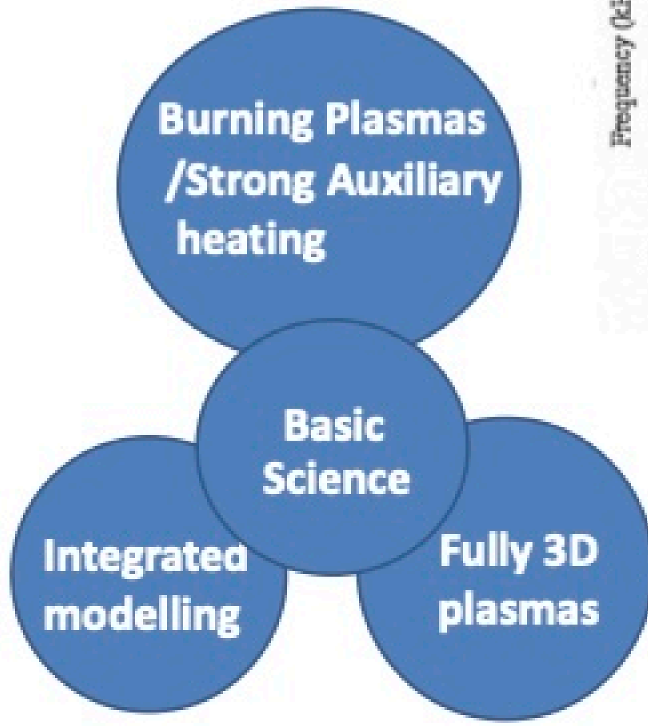
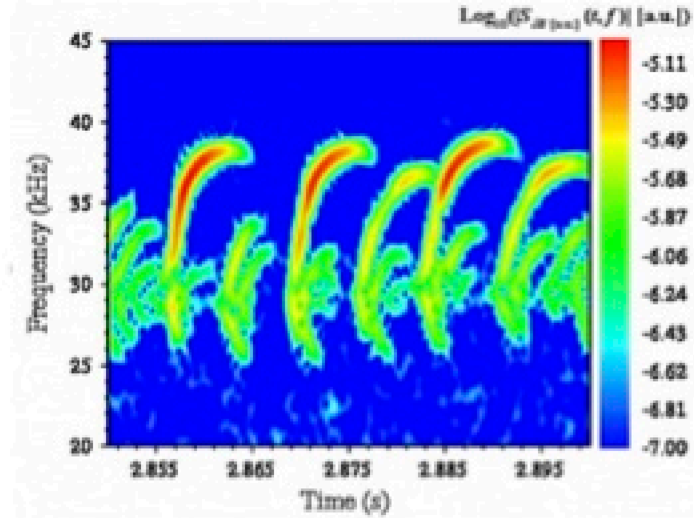
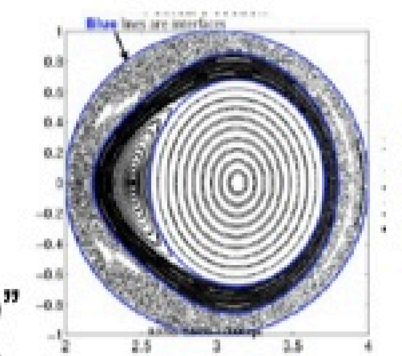
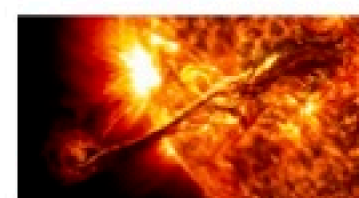
energetic ion physics



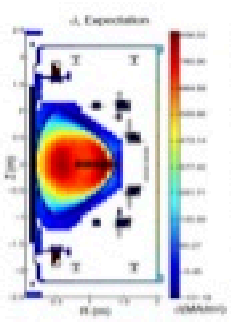




New "MRxMHD" approach to equilibrium and stability



Captures magnetic islands ….
and chaotic fields



Model-data fusion using Bayesian inference

Strong international dimension to research

Collaborators include ....

iter
china eu india japan korea russia usa

UK Atomic Energy Authority

PPPL
PRINCETON PLASMA PHYSICS LABORATORY

IPP
Max-Planck-Institut für Plasmaphysik

EPFL

KFE 한국핵융합에너지연구원
KOREA INSTITUTE OF FUSION ENERGY

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THE UNIVERSITY OF TEXAS AT AUSTIN

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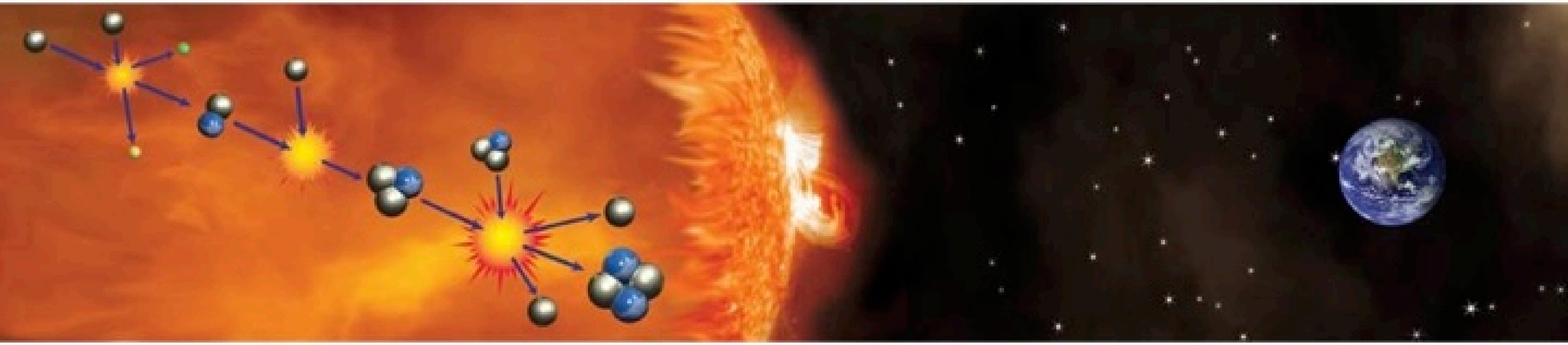
THE UNIVERSITY OF SYDNEY

Talk Plan

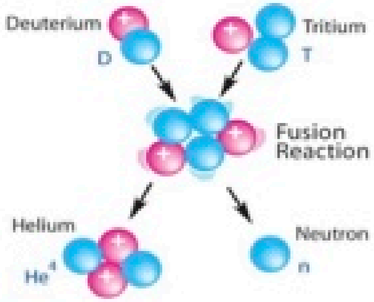
- Plasma Fusion Conditions
- Magnetic Confinement
- Tokamak Reactor – designing a power plant.
- Burning Tokamak – ITER
- Non-tokamak paths
- Mathematical Challenges
- Fusion Plasma Initiatives

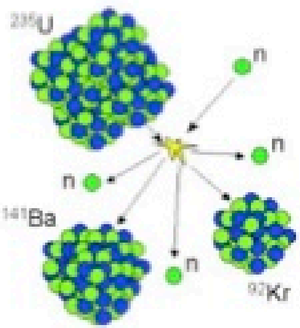
Fusion, the power of the sun and the stars, is one option

"…Prometheus steals fire from the heaven"



fusion cf fission



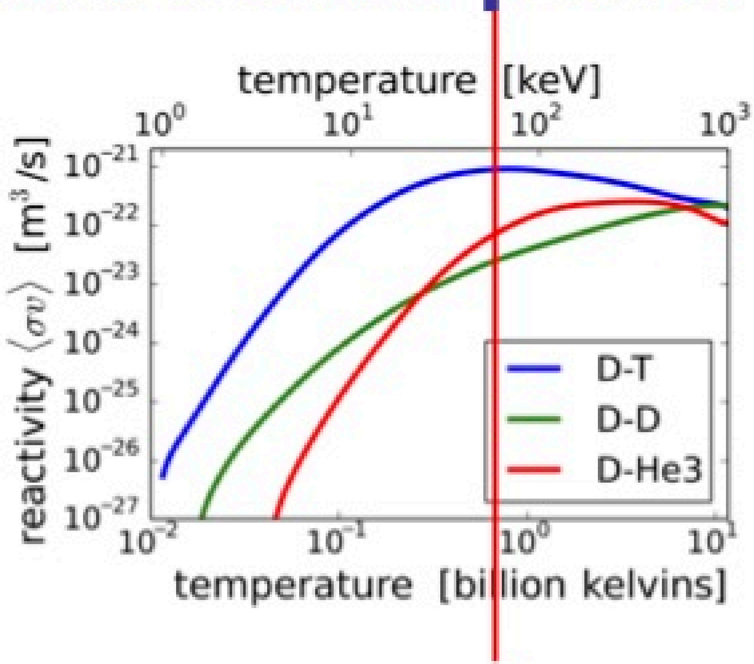


On Earth, fusion could provide:

- Large-scale energy production
- Essentially limitless fuel, available all over the world
- No greenhouse gases
- Intrinsic safety
- No long-lived radioactive waste

Conditions for *terrestrial* fusion power

- Achieve sufficiently high
  - ion temperature T_i
  - $\Rightarrow$ exceed Coulomb barrier
  - density $n_D \propto$ energy yield
  - energy confinement time τ_E



τ_E = insulation parameter: e.g. time taken for a jug of hot water to lose energy to the surroundings

≈70 keV

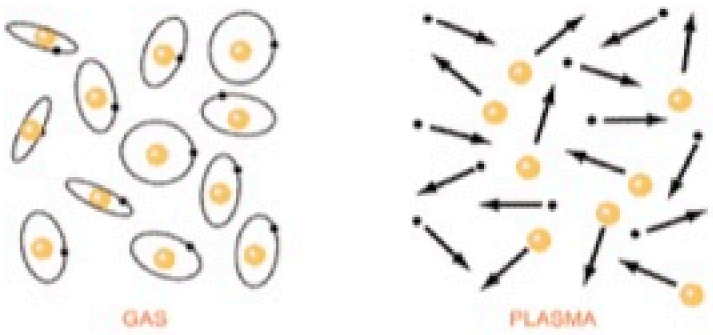
- "Lawson" ignition criteria : Fusion power > heat loss

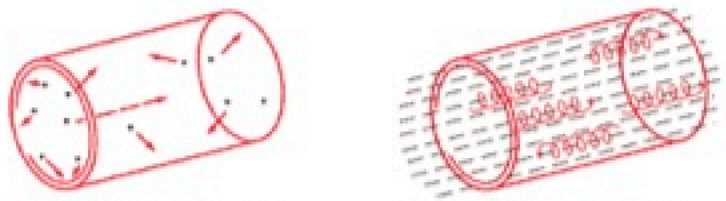
Fusion triple product → $n_D \tau_E T_i > 3 \times 10^{21}\ \mathrm{m^{-3}\ keV\ s}$

- Steady-state access requires confinement

Plasma : the 4'th state of matter

- plasma is an ionized gas





Motion of charged particles without magnetic field.

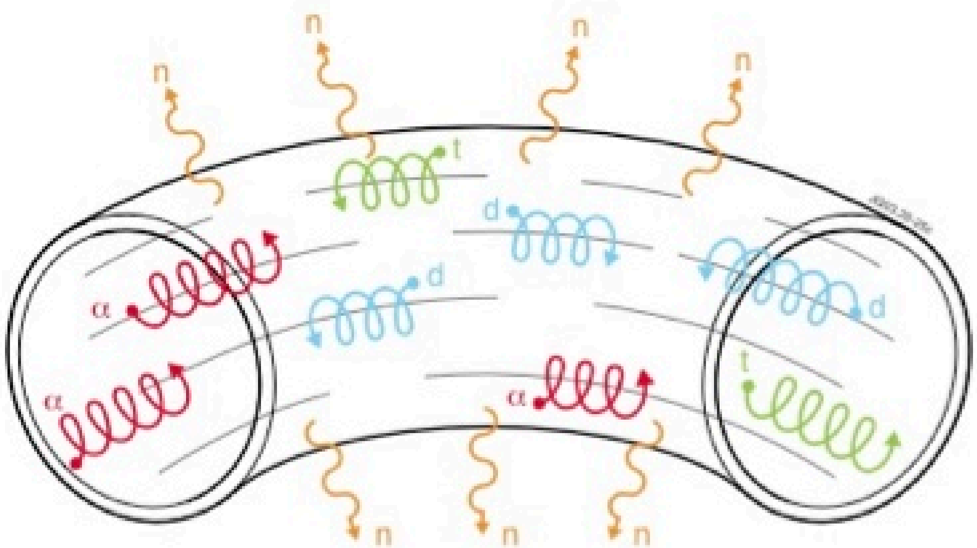
Motion of charged particles with magnetic field.

- 99.9% of the visible universe is in a plasma state

Inner region of the M100 Galaxy in the Virgo Cluster, imaged with the Hubble Space Telescope Planetary Camera at full resolution.

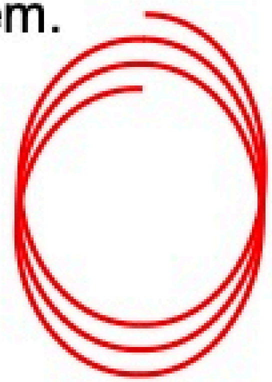
Toroidal Magnetic Confinement

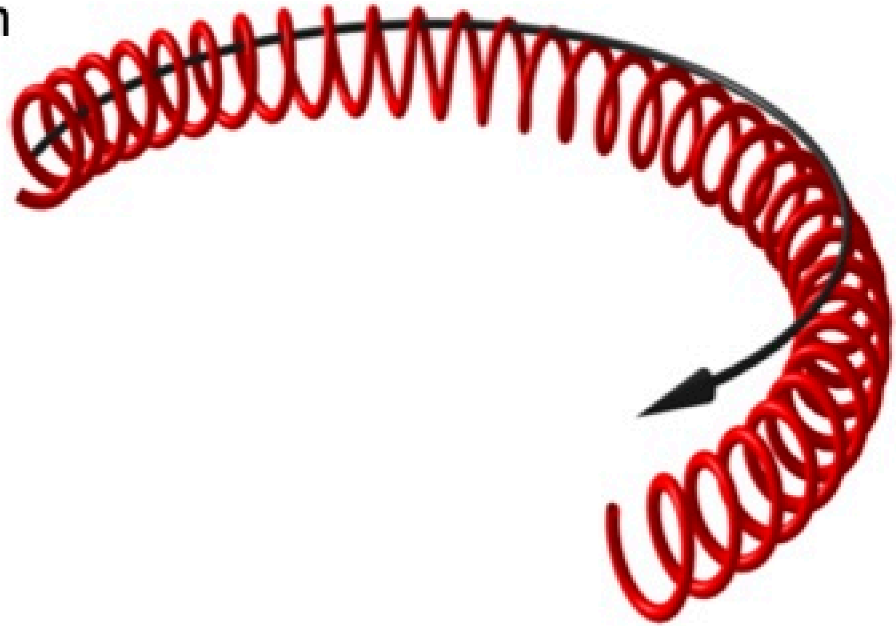
- Magnetic fields cause charged particles to spiral around field lines. Plasma particles are lost to the vessel walls only by relatively slow diffusion across the field lines



- Only charged particles (D^+, T^+, He^+…) are confined
  Neutrons escape and release energy
- Toroidal (ring shaped) device: a closed system to avoid end losses

But if field lines are bent as in an axisymmetric torus, particles drift off them.





→ To confine particles, constrain their position with a conservation law.

<u>*Noether's theorem:*</u>

For each **continuous symmetry** of a system, there is a corresponding **conserved quantity**.

Emmy Noether (1882-1935)

Axisymmetry and Noether's theorem is one way to achieve confinement

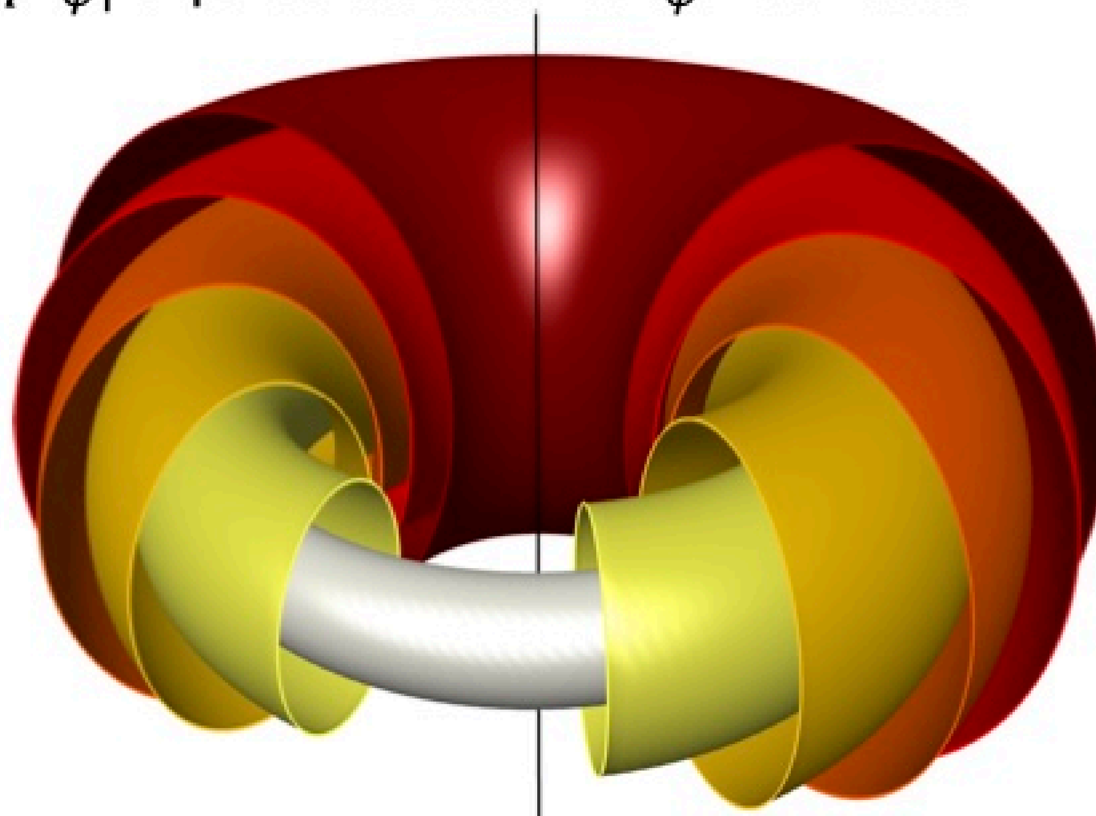
Continuous rotational symmetry ⇒ Canonical angular momentum conserved.

$$L_\phi = m v_\phi R + q A_\phi R = \text{constant}$$

Vector potential: **B**=∇×**A**

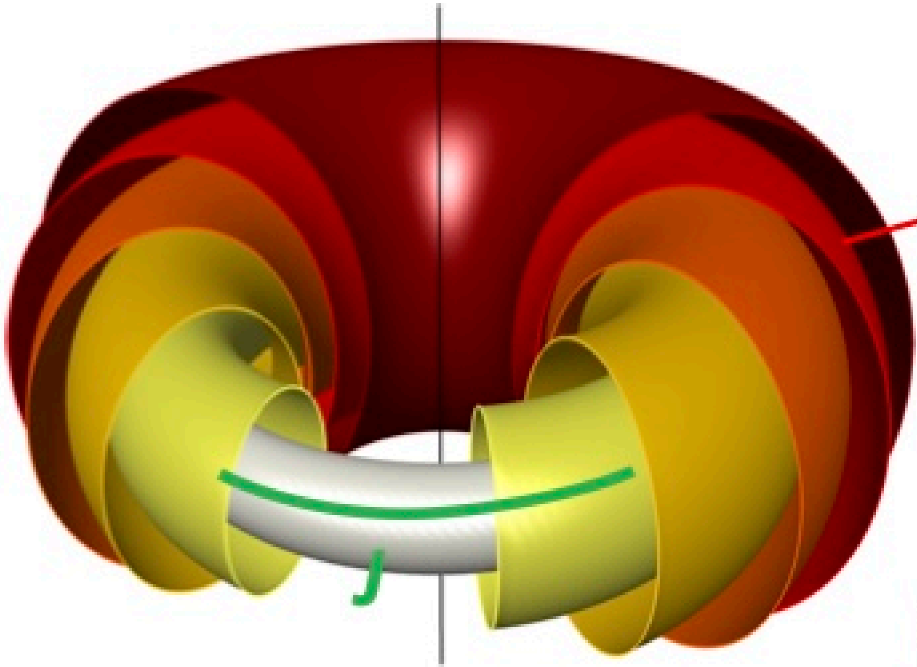
Strong B limit ⇒ $|m v_\phi| \ll |q A_\phi|$ ⇒ particles stuck to $A_\phi R$ surfaces

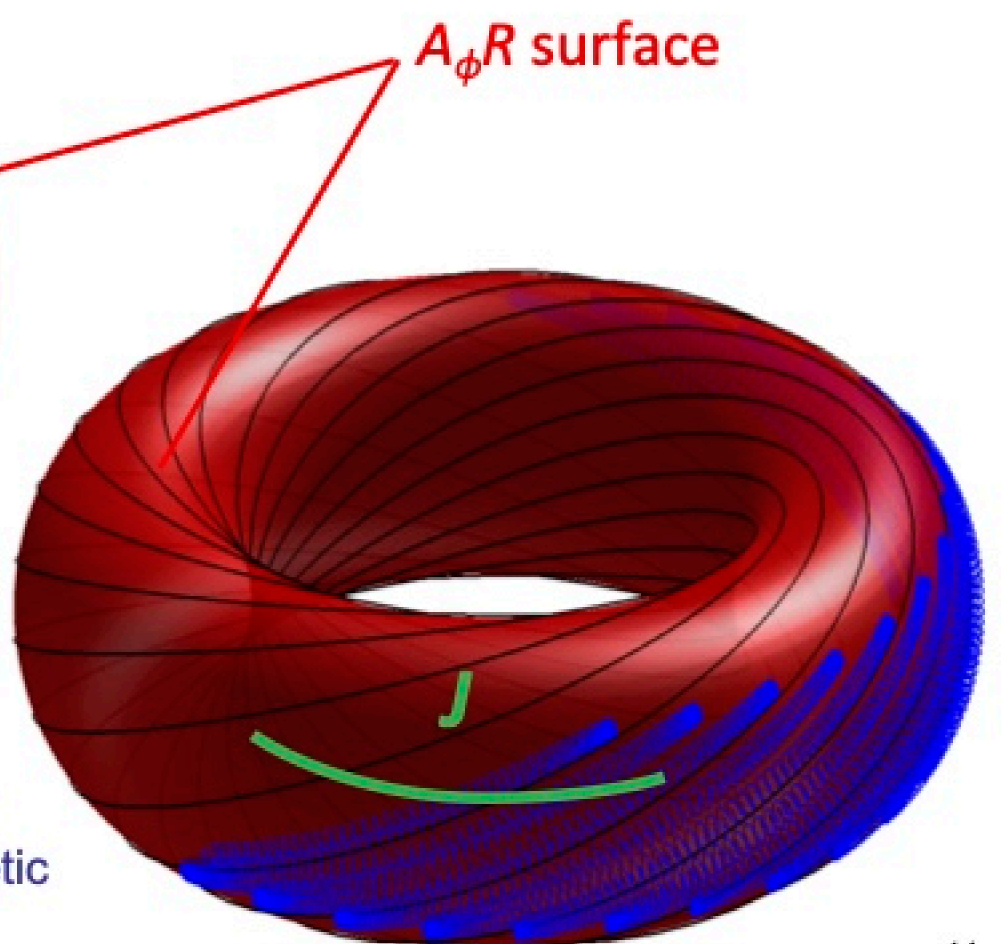
If $A_\phi R$ surfaces are bounded, then particles will be confined.



Complication: Axisymmetric confinement requires an internal current

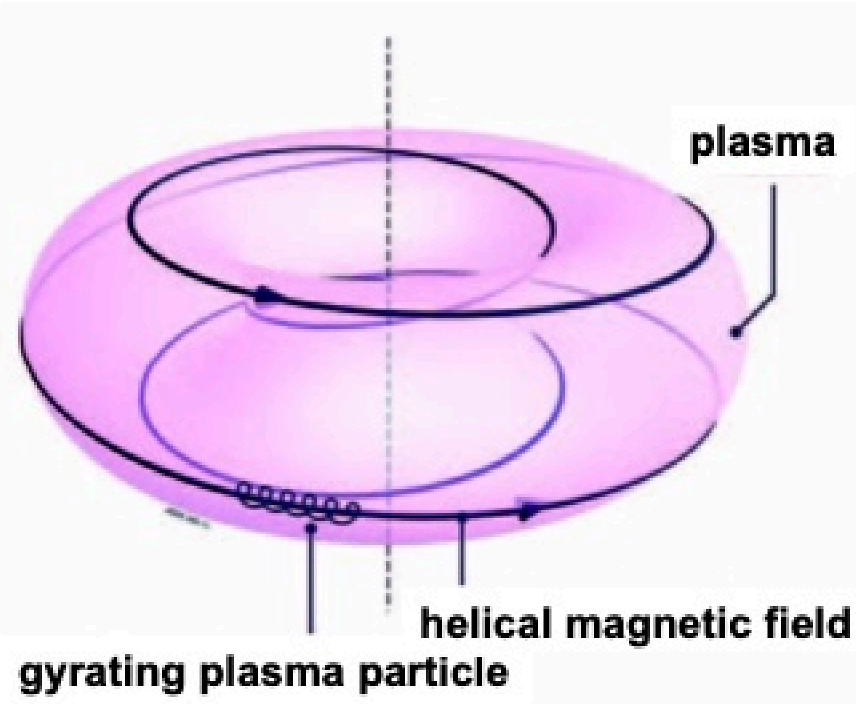
$\nabla\times(\nabla\times\mathbf{A}) = \nabla\times\mathbf{B} = \mu_0\mathbf{J} \Rightarrow$ nested $A_\phi R$ surfaces require a J_ϕ

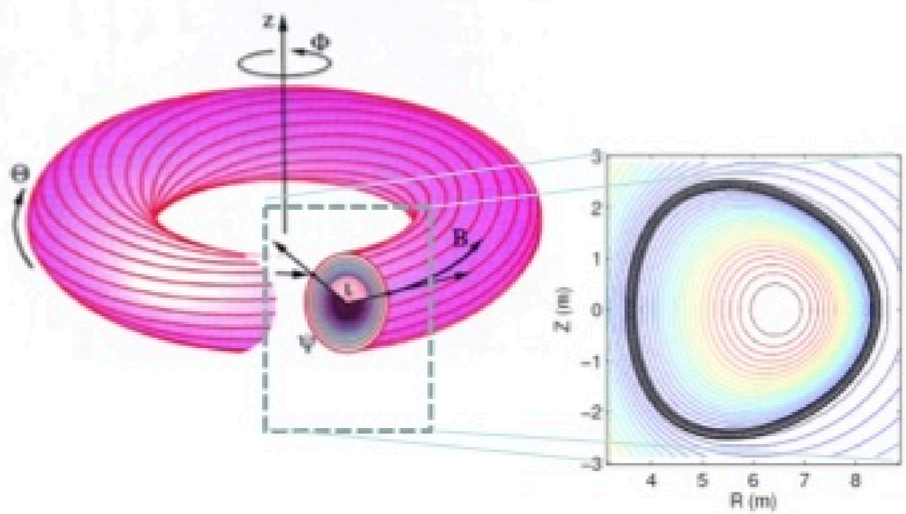




- The most successful Magnetic Confinement device is the TOKAMAK (Russian acronym for 'Toroidal Magnetic Chamber')

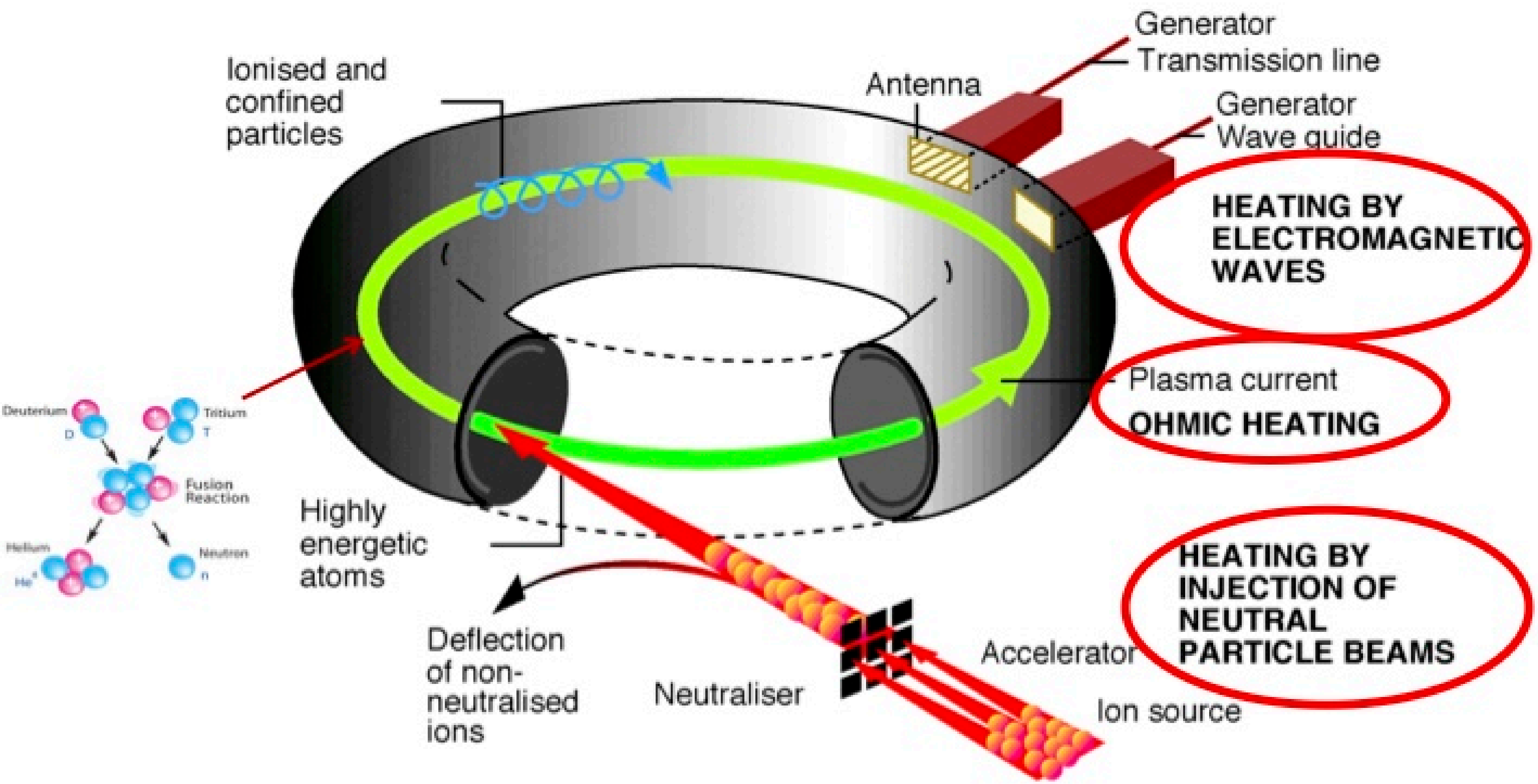
In a tokamak fields lie in flux surfaces





- If magnetic field sufficiently strong ions and electrons bound to field lines
- In a "perfect" tokamak field lines lie in flux surfaces
- Different flux surfaces are ~ thermally insulated
- Flux surfaces support pressure gradient
- Tokamaks maximise core pressure, needed to initiate fusion

How to obtain extreme temperatures?

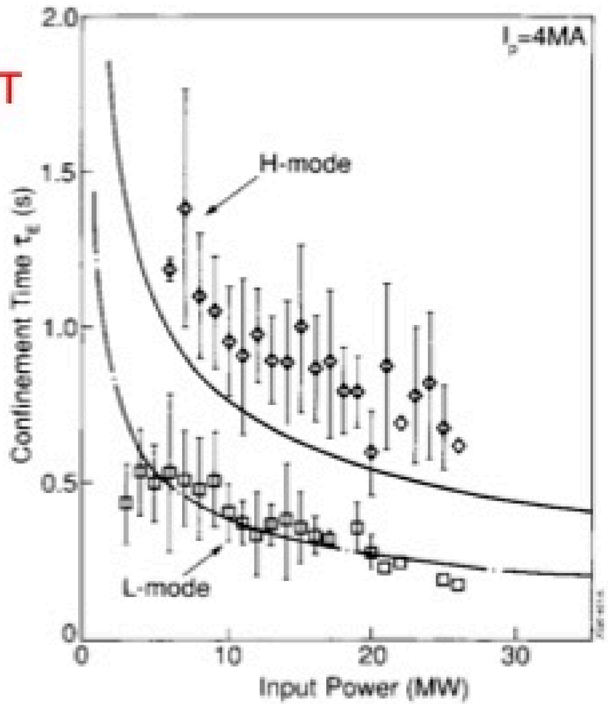


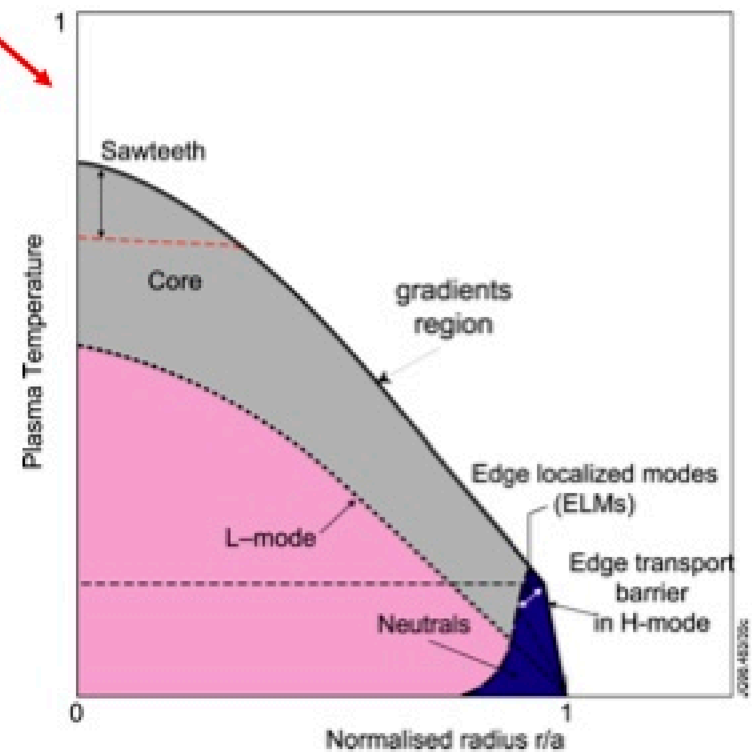
Positive ion beams: E ~ 100keV
Negative ion beams: E~ 1MeV

Toroidal Plasma Confinement: H-mode

- Plasma energy and particle transport is driven by turbulence
- It is found that the plasma confinement state (τ_E) can bifurcate:
  - two distinct plasma regimes, a low confinement (L-mode) and a high confinement (H-mode), result
  - this phenomenon has been shown to arise from changes in the plasma flow in a narrow edge region, or pedestal

JET





Plasma Confinement: τ_E scaling

- τ_E difficult to predict quantitatively: Empirical scaling used

$$\tau_E = f(H, I_p, B_0, P_H, \bar{n}, R_0, a, \kappa, A) \propto I_p^{0.93} B_0^{0.15} R_0^{1.39} a^{0.58} \kappa^{0.78} A^{0.19} P_H^{-0.69}$$

Confinement mode H-mode: H_H~1

Plasma current

toroidal magnetic field

heating power

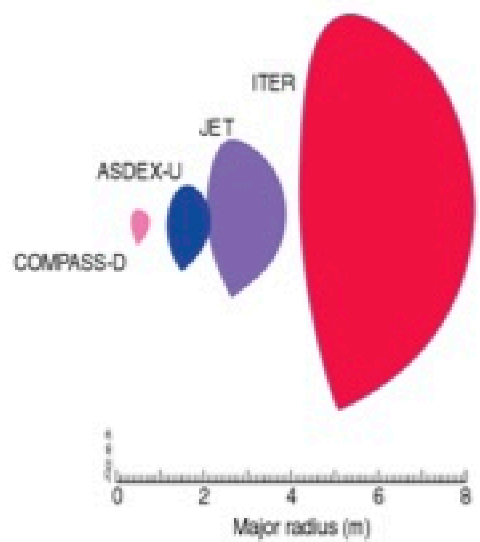
major radius

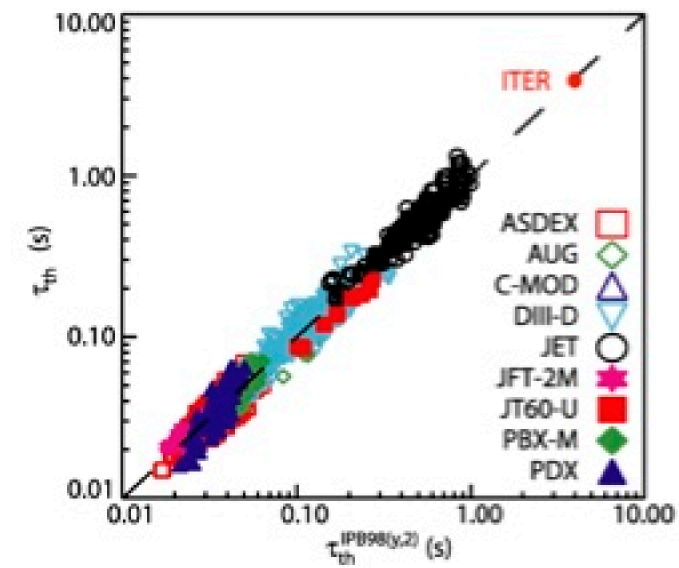
elongation = b/a

Average atomic mass

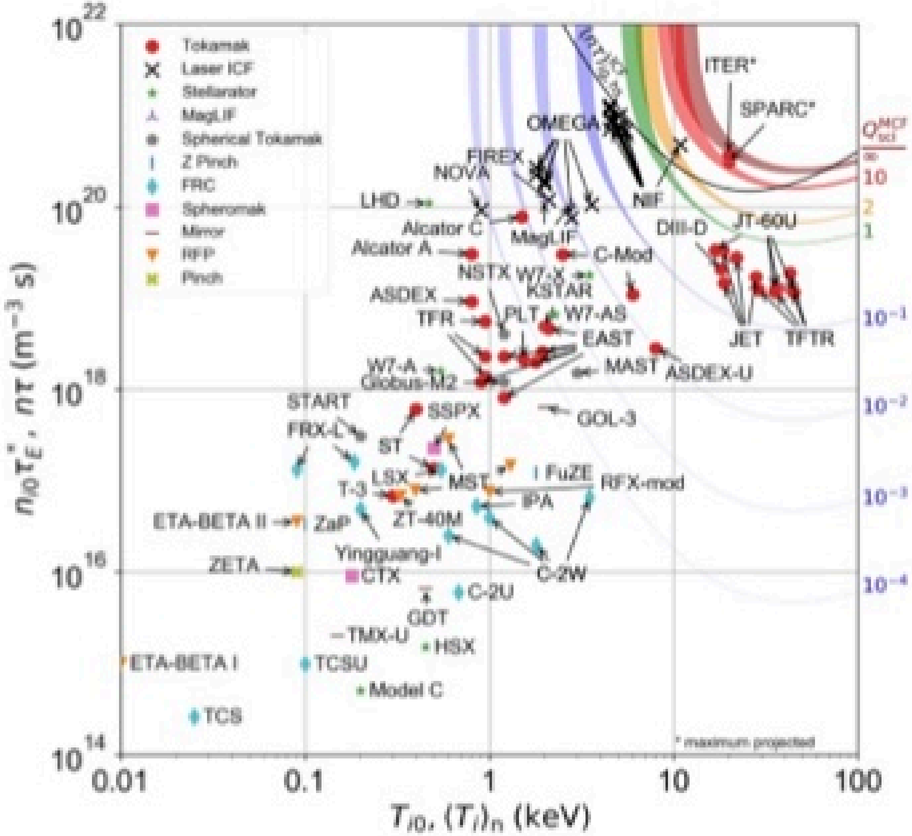
τ_E increases with machine size: bigger is better!

– H-mode turns out to be robust enough to provide the basis for the ITER design





Energy confinement : *big* is better



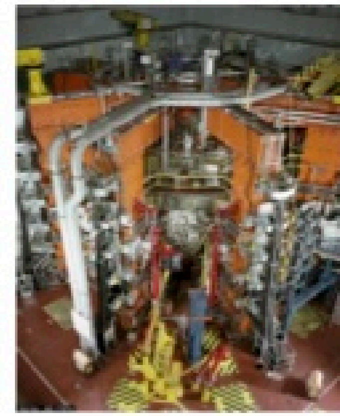
[Wurzel and Hu Physics of Plasmas 29, 062103 (2022)]

- **"Breakeven" regime :**

  $Q = P_{out} / P_{heat} = 1$

  Eg. JET

  Q=0.7, 16.1MW fusion



- **"Burning" regime : ITER**

$$D^2 + T^3 \rightarrow \underbrace{\underbrace{He^4\,(3.5\text{ MeV})}_{\geq P_{heat}} + n^1\,(14.1\text{ MeV})}_{P_{out}}$$

Q>5 ⇒ITER

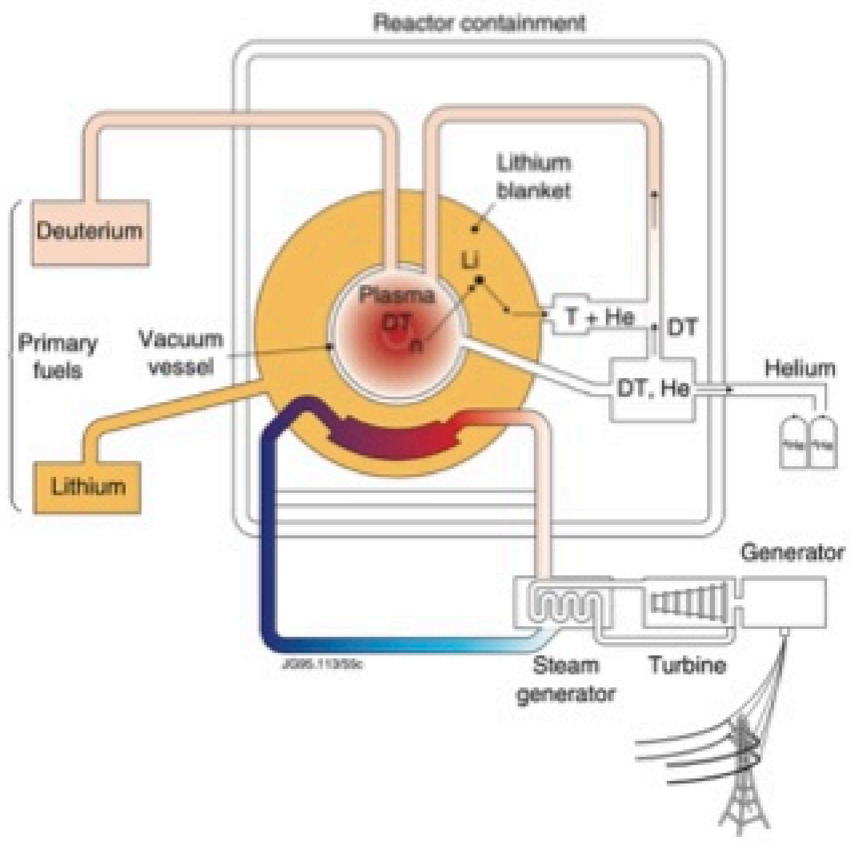
- **"Ignition" regime, $Q \rightarrow \infty$**

Tokamak Reactor

Designing a tokamak fusion reactor—How does plasma physics fit in?

Cite as: Phys. Plasmas 22, 070901 (2015); https://doi.org/10.1063/1.4923266
Submitted: 22 January 2015 • Accepted: 01 May 2015 • Published Online: 01 July 2015

J. P. Freidberg, F. J. Mangiarotti and J. Minervini



Engineering & Nuclear Constraints Informs Physics Requirements

TABLE II. Basic engineering and nuclear physics constraints.

Quantity	Symbol	Limiting value
Electric power output	P_E	1000 MW
Maximum neutron wall loading	P_W	4 MW/m^2
Maximum magnetic field at the coil	$B_{\max}$	13 T
Maximum mechanical stress on the magnet	$\sigma_{\max}$	600 MPa
Maximum superconducting coil current density averaged over the winding pack	$J_{\max}$	20 MA/m^2
Thermal conversion efficiency	η_T	0.4
Maximum RF recirculating power fraction	f_{RP}	0.1
Wall to absorbed RF power conversion efficiency	η_{RF}	0.4
Temperature at $[\langle\sigma v\rangle/T^2]_{\max}$	$\bar{T}$	14 keV
Fast neutron slowing down cross section in *Li*-7	σ_S	2 barns
Slow neutron breeding cross section in *Li*-6	σ_B	950 barns at 0.025 eV

TABLE I. Reactor parameters to be determined.

Quantity	Symbol
Minor radius of the plasma	a
Major radius of the plasma	R_0
Elongation	κ
Thickness of the blanket region	b
Thickness of the TF magnets	c
Plasma temperature	T
Plasma density	n
Plasma pressure	p
Energy confinement time	τ_E
Magnetic field at $R = R_0$	B_0
Normalized plasma pressure	β
Plasma current	I
Normalized inverse current	q_*
Bootstrap fraction	f_B

Reactor Geometry

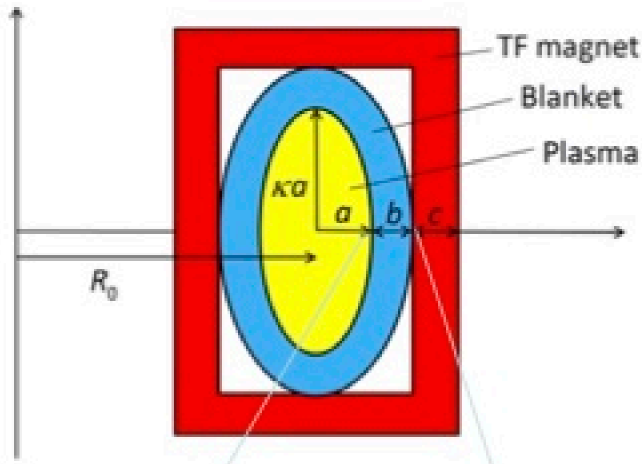
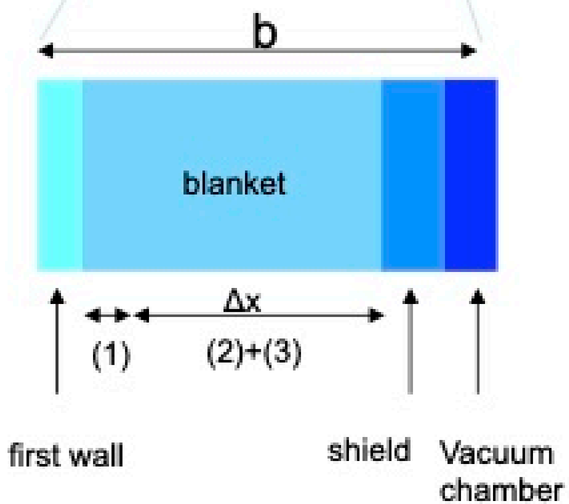


FIG. 1. Simple model for the reactor geometry.



A. Plasma Elongation

- Increasing κ reduces cost and increases β
- Vertically unstable at large kappa

Compromise κ = 1.7

B. Blanket

- Blanket comprises
  (1) a narrow region for neutron multiplication,
  (2) a moderating region,
  (3) a breeding region to produce T.
- Assume blanket is composed of natural Li
- Δx is determined by distance required for 14MeV neutron to slow down and breed T from Li-6

Yields $\Delta x \approx 0.9$m, and b $\approx$ 1.2m

Reactor Geometry

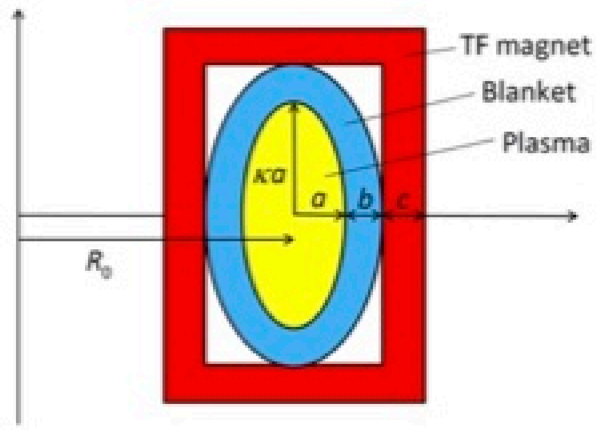


FIG. 1. Simple model for the reactor geometry.

C. Wall Loading Limit

$$P_n = P_W S$$

Total neutron power (MW)

Neutron wall loading limit (MW/m^2)

Surface area of plasma : $S \approx 4\pi^2 R_0 a[(1+\kappa^2)/2]^{1/2}$

- $P_n = \frac{En}{Ef} P_f$
- Assume steam cycle with efficiency $\eta_T = 0.4$ and solve for R_0 :

$$R_0 = \left[\frac{1}{4\pi^2}\frac{E_n}{E_F}\frac{P_E}{\eta_T P_W}\left(\frac{2}{1+\kappa^2}\right)^{1/2}\right]\left(\frac{1}{a}\right) = \frac{7.16}{a}\ \text{m}.$$

Conventional aspect ratio R_0/a ~4 ⇒ R_0=5.34m.

Reactor Geometry

D. Central field strength B.

$$\tau_E = f\left(H, I_p, B_0, P_H, \bar{n}, R_0, a, \kappa, A\right) \propto I_p^{0.93} B_0^{0.15} R_0^{1.39} a^{0.58} \kappa^{0.78} A^{0.19} P_H^{-0.69}$$

- Maximise B. Maximum field that Niobium-Tin superconducting magnets can support is 13T.
- Field at centre of machine is then B_0=6.8T

E. Coil thickness c=c(a,B_0)

$$c = c_M + c_J$$

Thickness of superconducting winding pack.

Thickness of structural material needed to support magnet stresses

- Combine force analysis on coils (JxB) and winding pack ⇒ c~1m

Plasma Physics Requirements

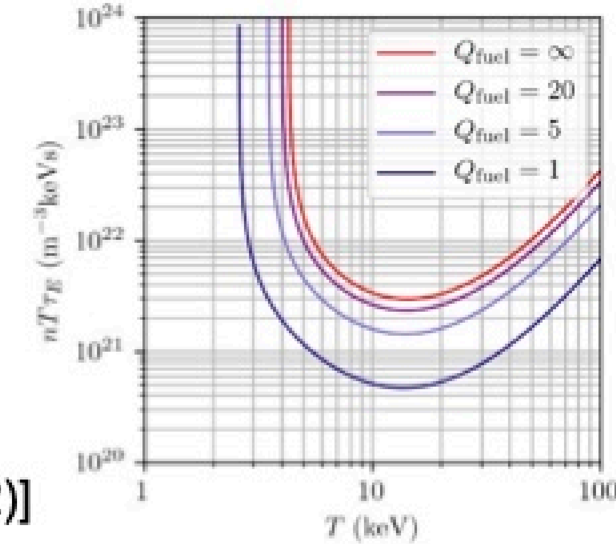
F. Average Plasma Temperature: Ignition condition

- Cross section peaks at $\langle \sigma v \rangle$ = 70keV
- Triple product vs T required to achieve

$$Q_{fuel} = E_f / E_{abs}$$

$\Rightarrow \bar{T}$~14keV

[Wurzel and Hu Physics of Plasmas 29, 062103 (2022)]



G. Average Plasma Pressure

Require P_{fus} produces all P_E at thermal efficiency η_T : $\frac{\eta_T E_F}{16} \int p^2 \frac{\langle \sigma v \rangle}{T^2} d\mathbf{r} = P_E.$

Assuming simple polynomial profiles for pressure, temperature and density (**no H-mode profile!**) gives:

$\bar{p} = \frac{8.76}{a^{1/2}}\ atm. \approx 7.6$ atm. Hence, $\beta = \frac{2\mu_0 \bar{p}}{B_0^2} = \frac{1.31}{a^{1/2}(1-\varepsilon_B)^2}\ \% \quad = 4.13\%$

Plasma Physics Requirements

H. Average density

P ~ 2 nT $\Rightarrow \bar{n} \approx \bar{p}/(2\bar{T})$= 1.43 x 10^{20} m^{-3}

- Number density of air is 2.5 x 10^{25} m^{-3}
- Number density of water is 3.3 x 10^{28} m^{-3}
- Number density of diamond 1.76 x 10^{29} m^{-3}
- Number density of core of Sun 9.5 x 10^{31} m^{-3}

I. Energy Confinement time

Require that in steady state thermal conduction losses are balanced by alpha particle heating (**ignited plasma**)

$$P_\alpha = \frac{3}{2\tau_E}\int p\,d\mathbf{r} = \frac{3}{2}\frac{V_P\bar{p}}{\tau_E},$$

Solving gives

$$\tau_E = 3\pi^2 R_0 a^2 \kappa \left(\frac{E_F}{E_\alpha}\frac{\eta_T}{P_E}\right)\bar{p} = 0.81\, a^{1/2}\ \text{s.}$$

τ_E (ignited)=0.94s

Plasma Physics Requirements

J. Plasma Current

For ELMy H-modes, empirical scaling

$$\tau_E = 0.145H \frac{I^{0.93} R^{1.39} a^{0.58} \kappa^{0.78} \bar{n}^{0.41} B_0^{0.15} A^{0.19}}{P_\alpha^{0.69}} \text{ s}$$

Assume H = 1. Solve τ_E (empirical) = τ_E (ignited)

$$I = \frac{7.98}{H^{1.08}} \frac{\tau_E^{1.08}}{R_0^{1.49} a^{0.62} \kappa^{0.84} \bar{n}^{0.44} B_0^{0.16} A^{0.20}} \left[\left(\frac{E_\alpha}{E_F} \right) \frac{P_E}{\eta_T} \right]^{0.74}$$

$$= 12.1 \frac{a^{1.63}}{B_0^{0.16}} \text{ MA.}$$

For reference values of a and B, I=14.3MA

Yields q*

$$q_* = \frac{2\pi a^2 B_0}{\mu_0 R_0 I} \left(\frac{1+\kappa^2}{2} \right) = 0.112\, a^{1.37} B_0^{1.16},$$

$q_* = 1.56$

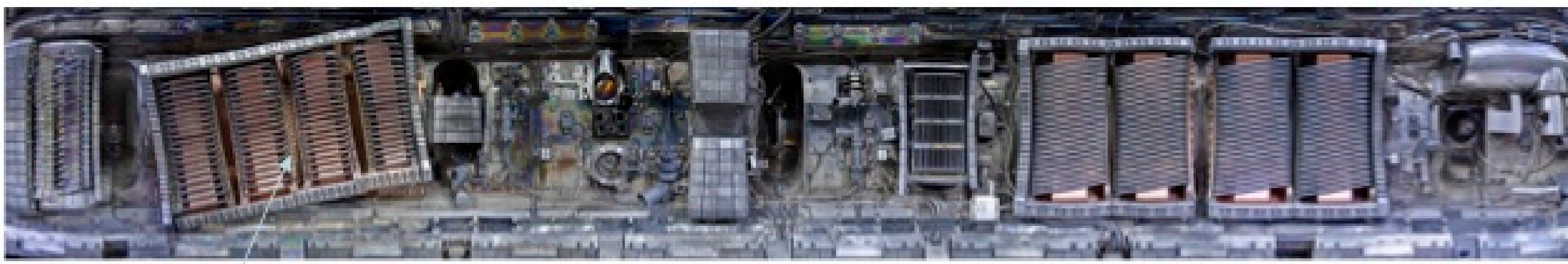
Plasma Physics Requirements

K. Bootstrap fraction f_{BS}

Steady-state tokamak requires non-inductive drive: $I_p = I_{CD} + I_{BS}$ $\boxed{I_{OHMIC} = 0}$

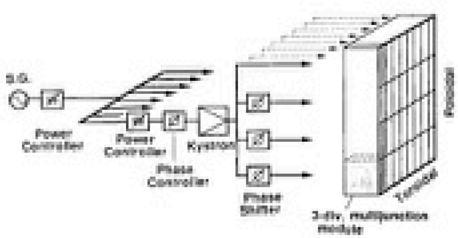
I_{BS} Bootstrap: arises due to the effect of trapped particles and density gradient.

I_{CD} Driven by radio frequency, either lower hybrid, ion cyclotron or electron cyclotron.



Lower Hybrid launcher

ACLACTOR CMOD outer wall



Required recirculating power < 0.1.

Assume coupling RF power is 50% efficient from wall to klystron and 80% from klystron to plasma $\Rightarrow I_{CD} = 2.2$MA.

I_{BS}=14.3MA. Hence I_{BS}=12.1MA and f_{BS}= 0.84.

Can a tokamak plasma meet these requirements?

- Density $n_D(0)$ Limit : Stability to disruption from current-driven modes (Greenwald limit) mostly at plasma edge

$$\bar{n} < \bar{n}_G \equiv \frac{I(MA)}{\pi a^2} \qquad \Rightarrow 1.43 < 2.56 \qquad ✓$$

- Beta Limit : Stability to disruption from pressure-gradient-driven modes (Troyon limit)

$$\beta < \beta_T \equiv \beta_N \frac{I}{aB_0} \qquad \Rightarrow 4.13 < 4.39 \qquad ✓$$

- Kink safety limit: Stability to disruption from current-gradient-driven modes in plasma core

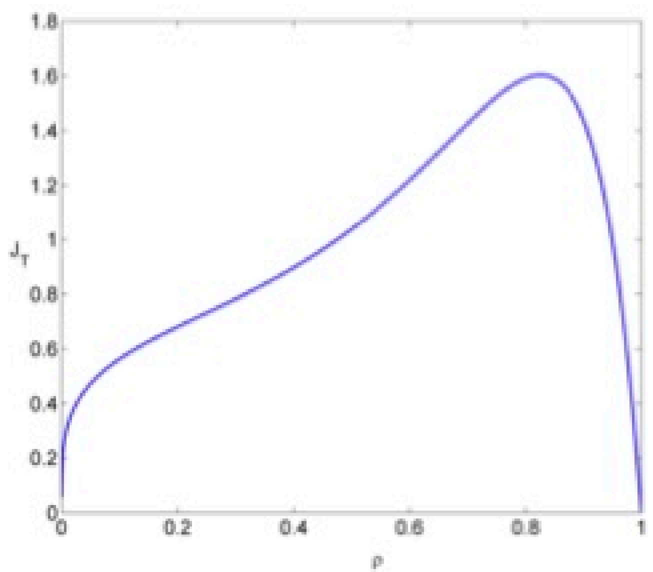
$$q_* > q_K \approx 2 \qquad \Rightarrow 1.56 > 2 \qquad \times$$

Can a tokamak plasma meet these requirements?

- Bootstrap fraction: Theory can compute the local "neoclassical bootstrap current"

$$J_B(\rho) = -2.44\left(\frac{r}{R_0}\right)^{1/2}\left(\frac{p}{B_\theta}\right)\left(\frac{1}{n}\frac{\partial n}{\partial r} + 0.055\frac{1}{T}\frac{\partial T}{\partial r}\right),$$

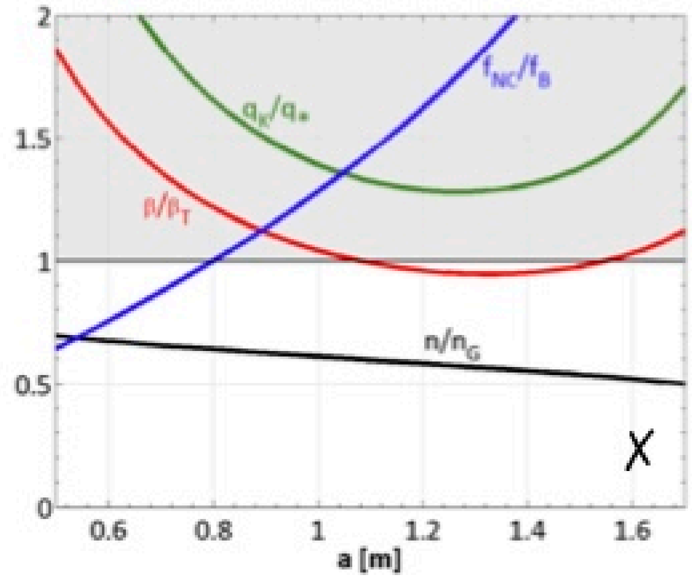
B_θ profile a function of J_T (or q) profile.



$$f_{NC} = \frac{I_B}{I} = \frac{\int J_B(\rho)d\tau}{I} > f_{BS}$$

$\Rightarrow$ 0.44 >0.84 ✕

Can we find an operating regime?

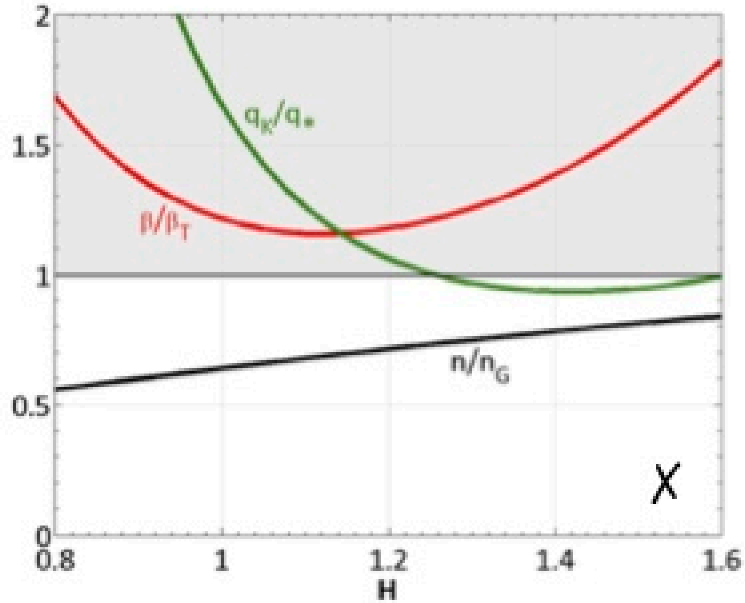


Allow a to vary

Operating region is unshaded region

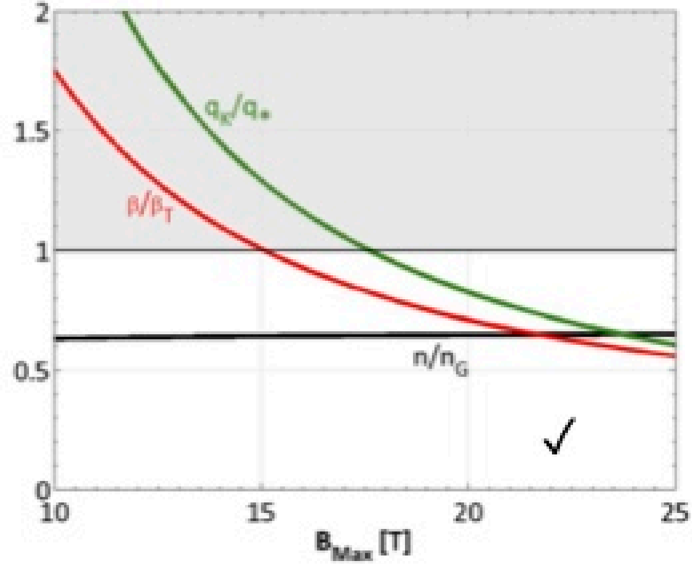
Successful operation requires an operating point (vertical line) with all conditions satisfied, i.e.

$$\frac{\bar{n}}{\bar{n}_G} < 1, \qquad \frac{\beta}{\beta_T} < 1, \qquad \frac{q_K}{q_*} < 1, \qquad \frac{f_{NC}}{f_{BS}} < 1$$



Choose a such that $f_B = f_{NC}$
Vary H

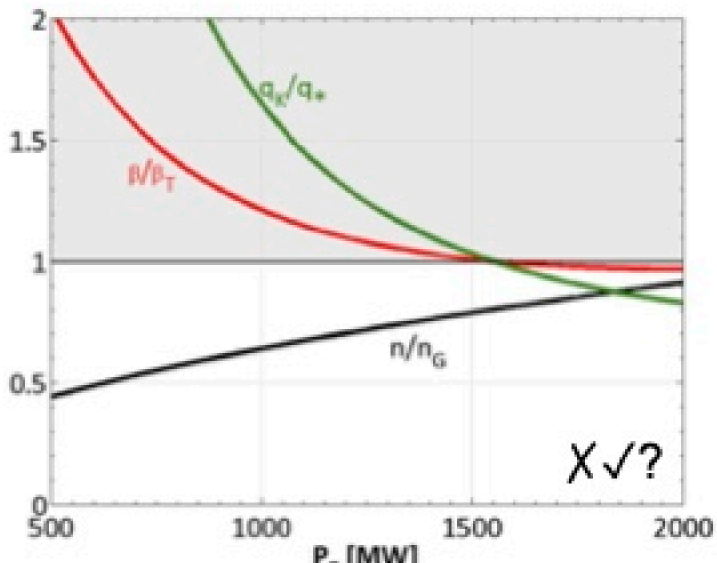
Can we find an operating regime?



Choose *a* such that $f_B = f_{NC}$
Increase B_{max}

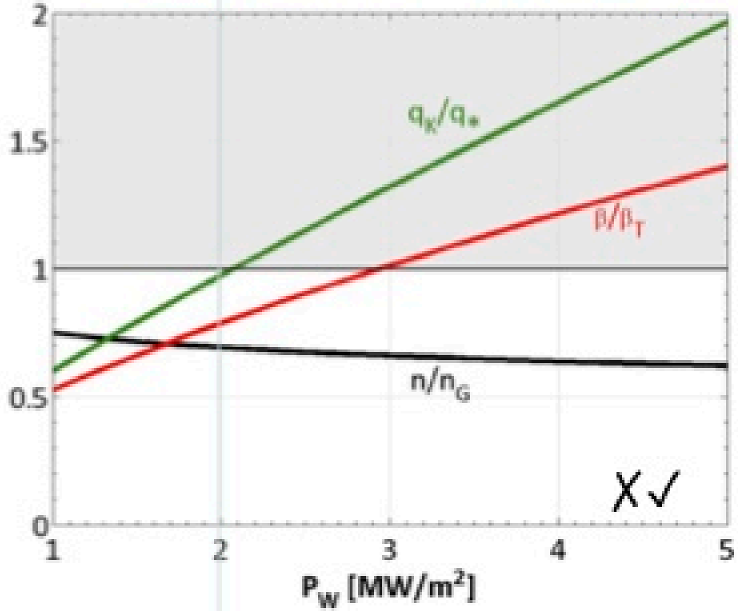
Aim of SPARC (MIT): Can 20T field coils be built and demonstrated in a tokamak?

Attracted more than USD $2bn in investment



Choose *a* such that $f_B = f_{NC}$
Increase electric power PE

Can we find an operating regime?



Choose a such that $f_B = f_{NC}$

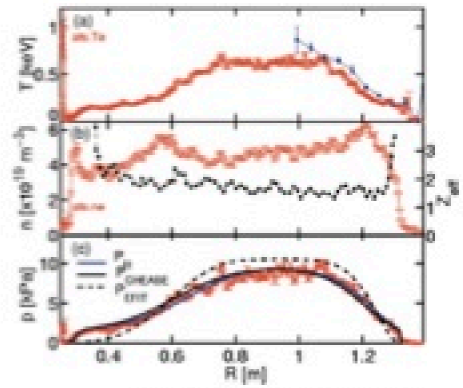
We have assumed a wall loading constraint of 4MW

Relax wall loading constraint

Machine has 10.2m major radius !

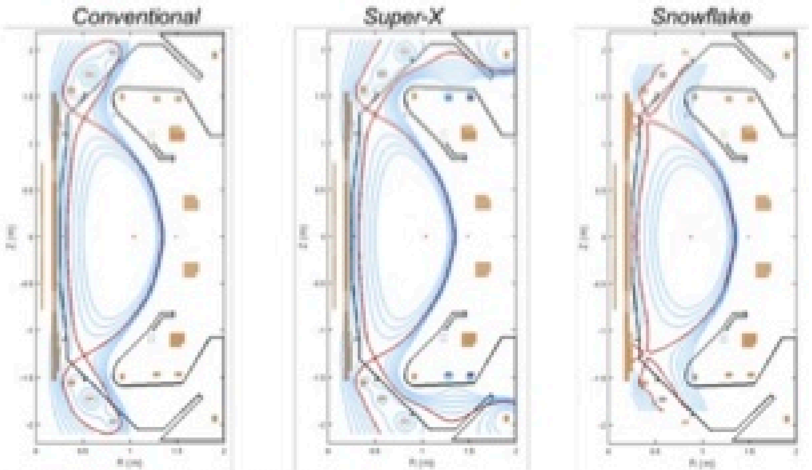
What's missing (from a tokamak study)?

- Mostly a 0D analysis
  - Plasma profiles assumed and integrated.
  - Tokamak plasmas can support "H mode"
  - Confinement and stability a function of current and pressure profiles

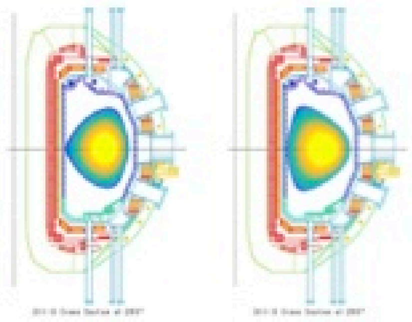


MAST plasmas

- Plasma shaping
  - Cross section is not elliptical
  - Diverters
  - Negative triangularity?



MAST-U plasmas



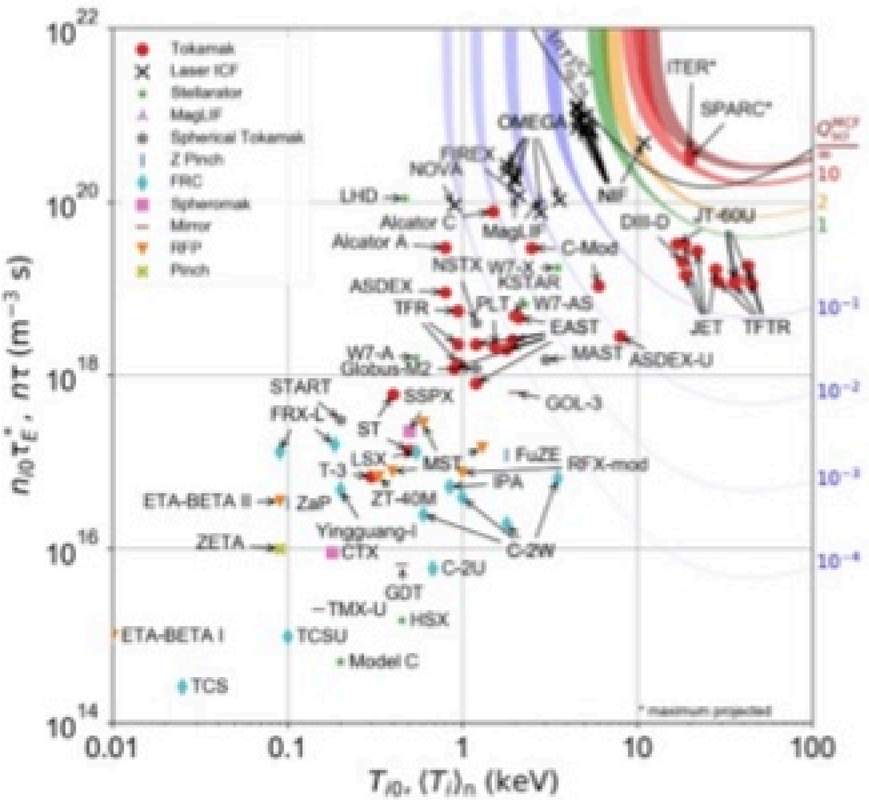
DIII-D plasmas

- Aspect ratio : Spherical or Compact Tokamak. :
  H. Bruhns *et al* 1987 *Nucl. Fusion* **27** 2178, Submitted 21 July 1987
  G.A. Collins et al 1988 Nucl. Fusion 28 255. Submitted 20 July 1987

The Next Step - ITER

Parameter	**Power Plant 101**	*ITER*
Major radius, R_0 (m)	5.34	6.2
Minor radius, a (m)	1.34	2.0
Elongation, κ	1.7	1.7
Toroidal field at R_0, B_T (T)	6.83	5.3
Plasma current, I_p (MA)	14.3	15
Edge safety factor, q_{95}	1.68	3.0
Confinement enhancement, $H_{H98}(y,2)$	1.0	1.0
Normalised beta, β_N	2.6	1.8
Average electron density, $\langle n_e \rangle$ ($10^{19}m^{-3}$)	14.3	10.1
Fraction of Greenwald limit, $\langle n_e \rangle / n_{GW}$	0.56	0.85
Average ion temperature, $\langle T_i \rangle$ (keV)	14.0	8.0
Average electron temperature, $\langle T_e \rangle$ (keV)	14.0	8.8
Neutral beam power, P_{NB} (MW)	?	33
RF power, P_{RF} (MW)	100	7
Fusion power, P_{fusion} (MW)	2500	500
Fusion gain, $Q=P_{fusion}/(P_{NB}+P_{RF})$	25	10
Non inductive current fraction, I_{NI}/I_p (%)	84	28
Burn time (s)	∞	400

Why ITER ?



1) **Achieve a deuterium-tritium plasma in which the fusion conditions are sustained mostly by internal fusion heating. Achieve a burning plasma.**

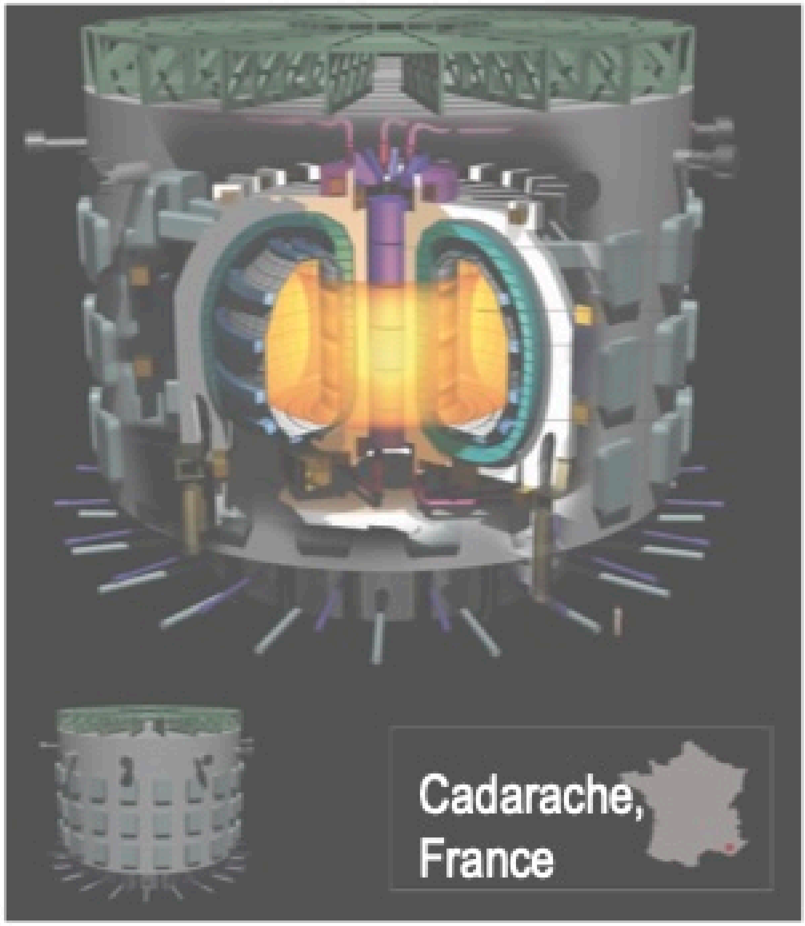
2) **Generate 500 MW of fusion power for long pulses**

3) **Contribute to the demonstration of the integrated operation of technologies for a fusion power plant**

4) **Test tritium breeding**

5) **Demonstrate the safety characteristics of a fusion device**

ITER



- Fusion power = 500MW
- Power Gain (Q) > 10
- Temperature ~ 100 million °C
- Growing Consortium

- Collaboration agreements with
  - ➢International Atomic Energy Agency
  - ➢Principality of Monaco 16/01/2008
  - ➢CERN – 10/03/2008
  - ➢**Australia 30/09/2016**
  - ➢ Khazakstan 11/06/2017

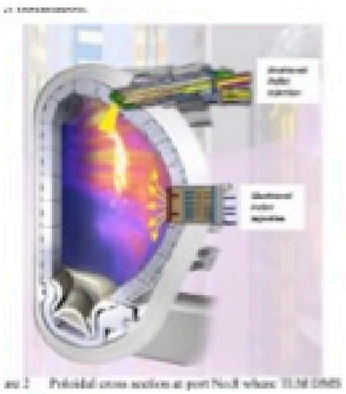
Construction +10 year operation cost ~$40 billion ?

ITER facilities more than 77% completed

17 March 2023

(Some) Tokamak Science Challenges

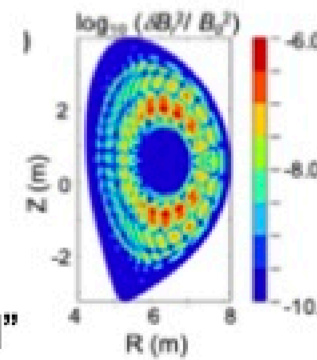
Disruption Avoidance / Mitigation



- Some plasma configurations unstable to low (m,n) modes
- Entire stored energy released on resistive diffusive time scale
- Unmitigated major disruptions at high stored energies can cause masses of the order of kilograms to be melted.

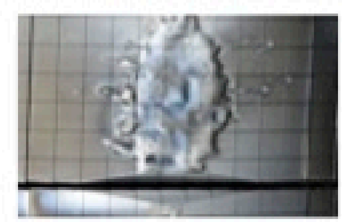
[M. Lehnen et al. / Journal of Nuclear Materials 463 (2015) 39–48]

Energetic particle confinement:



Fusion alphas: As 3.5MeV alphas slow they can drive wave modes of the plasma, inhibiting ignition and damaging the wall

Runaway electrons. Mostly current current by electron flow “thermal” electrons. If Electric field exceeds Dreicer field a supra-thermal population can exist.



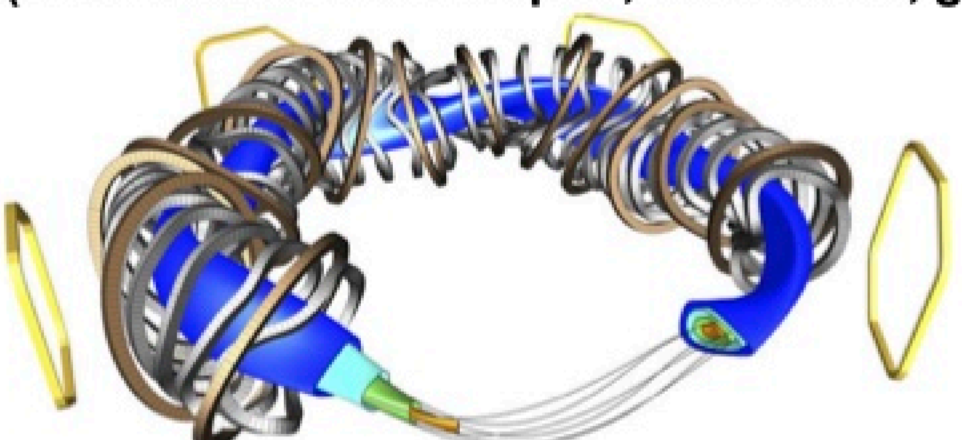
Materials (common to all D-T fusion)

Life-fluence of neutrons:
current experiments ~10^{-9} dpa, ITER ~ 1 dpa, DEMO ~100dpa

Stellarator Concept

- Tokamak: field line helicity created by internal plasma current
- Stellarator: field line helicity created by twist of field coils
  - ✓ Eliminates disruptive current-driven instabilities,
  - × field coils are an engineering challenge,
  - × flux surfaces aren't guaranteed.

e.g. W7-X : a first-generation computationally optimised stellarator (low neoclassical transport, low current, good stability, and "good" flux surfaces)



- € 1 billion experiment. Construction started in ~2000.
- Aim: evaluate fusion reactor using stellarator technology.
- Opened by Chancellor Merkel in February 2016

Hidden Symmetries and Fusion Energy

Simons Foundation Collaboration on Hidden Symmetries and Fusion Energy, **2018 - 2025** https://hiddensymmetries.princeton.edu/

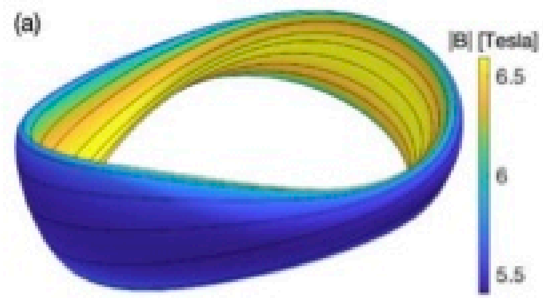
Objective: To create and exploit an effective mathematical and computational framework for the design of stellarators with hidden symmetries. *(good flux surfaces, good particle confinement, high bootstrap fraction)*

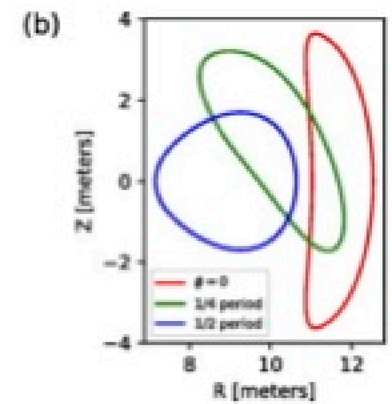
Deliverables: Optimum design principles of a stellarator, a modern optimization code (SIMSOPT, SIMonS OPTimization code) that can exploit the full power of petascale and exascale computers, and designs of next-generation stellarator experiments

https://github.com/hiddenSymmetries/simsopt

Physics of Plasmas 29, 082501 (2022)

Optimized quasi-axisymmetric configuration with energetic particle confinement, self-consistent bootstrap





TYPE ONE ENERGY THEA ENERGY Proxima Fusion

Characteristics of fusion physics

- Multi-scale:
  - Spatial: electron gyro-radius 5×10^{-6} m $\rightarrow$ 10m (device scale)
  - Temporal: electron gyration 4×10^{-10}s $\rightarrow \tau_E \sim$ 3 seconds
  - Generates many expansion parameters for asymptotic analysis
- Multi-dimensional:
  - 0-1D: energy & particle confinement scaling
  - 2D: Tokamak equilibrium model
  - 3D: Tokamak stability, stellarator fields
  - 4D: Transport, diffusion
  - 6D: Phase space (3D spatial, 2D velocity, time)
- Strong nonlinearity

Some Mathematics Challenges

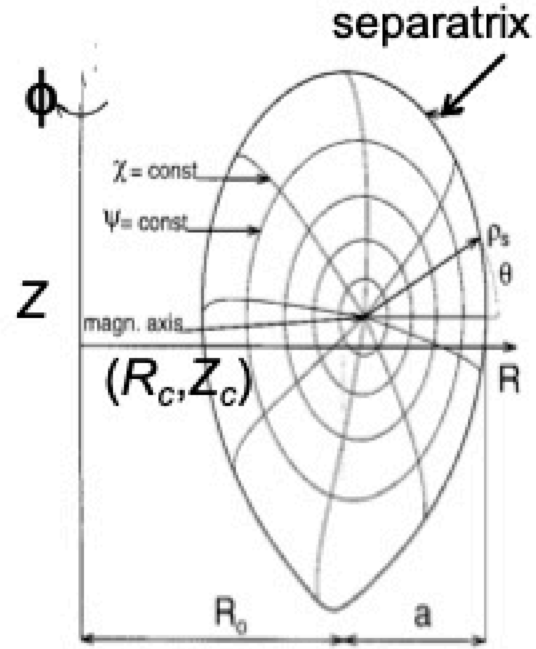
- Force Balance
  - Solutions of $J \times B = \nabla P$ (+ anisotropy and flow) in 2D [Hole, Qu]
  - Existence of $JxB = \nabla P$ in 3D: Grad Conjecture
  - Multiple Region relaxed MHD, SPEC [Dewar, Hudson, Hole]
  - Discontinuous Galerkin MRxMHD [R. Tang]
  - Topological Data Analysis of field lines [Bohr, Robins]
- Stability
  - Stability of tokamaks with flow / anisotropy [Doak]
  - Impact of islands and chaos on MHD modes [Thomas]
  - Stellarator Stability [*Kumar*]
  - Wave-particle and nonlinear evolution [*Hezaveh*]
- Time-dependent problems
  - Structure preserving discretisations [Jeyakumar]
  - Anisotropic diffusion equation [Muir]
- Computational Approaches
  - Data Science and AI [Gretton]

What is *B* for a stationary tokamak plasma?

(1) $\mathbf{J}\times\mathbf{B}=\nabla p \Rightarrow \begin{cases} \mathbf{B}\cdot\nabla p = 0 \Rightarrow & \text{No pressure gradient along } \mathbf{B} \\ \mathbf{J}\cdot\nabla p = 0 \Rightarrow & \text{Current flows within surfaces.} \end{cases}$

(2) $\nabla\times\mathbf{B}=\mu_0\,\mathbf{J}$

(3) $\nabla\cdot\mathbf{B}=0$



Introduce poloidal magnetic flux function $\psi(R,Z)$ and co-ord. system (R, ϕ, z). In <u>axisymmetry</u> Eq. (1), (2) become **Grad-Shafranov equation:**

$$\boxed{\nabla\cdot\frac{1}{R^2}\nabla\psi = -\frac{\mu_0 J_\phi}{R} = -\mu_0 p'(\psi) - \frac{\mu_0^{\,2}}{R^2} f(\psi) f'(\psi)}$$

second order PDE for field and currents.

With $f(\psi)$ a toroidal flux function

$f(\psi) = RB_\phi(\psi,R)/\mu_0$

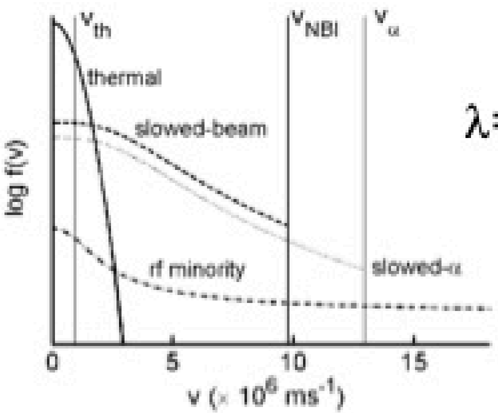
- To solve: prescribe $p'(\psi)$, $f(\psi)\,f'(\psi)$ and boundary
- Solve numerically by current-field iteration:

compute $J_\phi \rightarrow$ solve for $\psi(R,Z)$

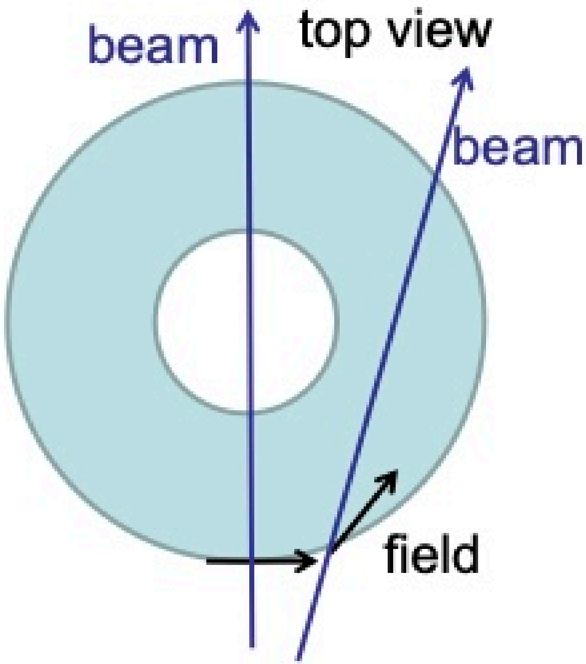
"MHD with anisotropy in velocity, pressure"

- Pressure different parallel and perpendicular to field due mainly to *directed* neutral beam injection

Illustrative 1D slice



$\lambda = v_{||}/v$



$$n = \int_0^\infty \int_{-1}^1 \hat{f}(E,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}E$$

$$nu_\parallel = \int_0^\infty \int_{-1}^1 v_\parallel \hat{f}(E,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}E$$

$$p_\parallel = m\int_0^\infty \int_{-1}^1 (v_\parallel - u_\parallel)^2 \hat{f}(E,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}E$$

$$p_\perp = \frac{m}{2}\int_0^\infty \int_{-1}^1 v_\perp^2 \hat{f}(E,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}E.$$

$\longrightarrow$

Generalised Grad-Shafranov equation with more constraints, PDE can be elliptic, hyperbolic or parabolic

Existence in 3D: Grad's Conjecture

- Harold Grad, The Physics of Fluids **10**, 137 (1967); doi: 10.1063/1.1761965
- Restated by P. Constantin *J. Plasma Phys.* (2021), *vol.* 87, 905870111

Let $T \subset \mathbb{R}^3$ be a domain with smooth boundary. The three-dimensional magneto hydrostatic (MHS) equations on T read

$$J \times B = \nabla P + f, \quad \text{in } T, \tag{1.1}$$

$$\nabla \cdot B = 0, \quad \text{in } T, \tag{1.2}$$

$$B \cdot \hat{n} = 0, \quad \text{on } \partial T, \tag{1.3}$$

where $J = \nabla \times B$ is the current, f is an external force and P is the pressure. The solution

Problem. Given a toroidal domain T, construct a function $\psi : T \to \mathbb{R}$ with nested flux surfaces and a divergence-free vector field ξ which does not generate an isometry of $\mathbb{R}^3$ and is tangent to ∂T, so that (1.13), (1.9), the nonlinear constraints (1.16), (1.17) and $\xi \cdot \nabla \psi = 0$ all hold.

Do smooth solutions for ψ, ξ exist?

Grad's conjecture: The only smooth solutions to (1.1)-(1.3) possessing a "good" flux function have a Euclidean symmetry.

In general 3D (toroidal) MHD fields are chaotic

- Field lines can be described as a 1½ DOF Hamiltonian

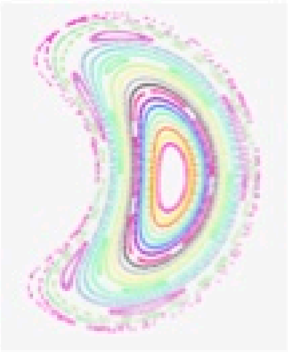
$$\boxed{\frac{d\theta}{d\zeta}=\frac{\mathbf{B}\cdot\nabla\theta}{\mathbf{B}\cdot\nabla\zeta}=\frac{\partial\Psi}{\partial\Phi},\qquad \frac{d\Phi}{d\zeta}=\frac{\mathbf{B}\cdot\nabla\Phi}{\mathbf{B}\cdot\nabla\zeta}=-\frac{\partial\Psi}{\partial\theta}}$$

$H = \Psi(\Phi,\theta,\zeta)$, $t = \zeta$
$p = \Phi$, $q = \theta$

- If axisymmetric, Hamiltonian is *autonomous* (ie. $\Psi(\Phi,\theta,\zeta)\rightarrow\Psi(\Phi,\theta)$)

ι= irrational: **B** ergodically passes through all points in magnetic surface.

ι = rational (m/n) : **B** lines close on each other.



Eg. **B** Poincaré sections in H-1, courtesy S. Kumar

- If non axis-symmetric , Hamiltononian is *non-autonomous* $\Psi(\Phi,\theta,\zeta)$ and the field is in general *non-integrable*.
  - Magnetic islands form at rational $\iota(\Phi)$, in which field is stochastic or chaotic, ergodically filling the island volume.
  - $B\cdot\nabla p=0$ $\therefore$ confinement lost in these islands.

Some sufficiently irrational magnetic flux surfaces survive 3D perturbation

- Kolmogorov Arnold Moser (KAM) Theory (c. 1962)

  - Perturbs an integrable Hamiltonian ψ_p within a torus (flux surface) by a periodic functional perturbation ψ_{p1}:

$$\psi_p = \psi_{p0} + \varepsilon \psi_{p1}$$

  - **KAM theory :** if flux surface are sufficiently far from resonance (q sufficiently irrational), **some** flux surfaces **survive for $\varepsilon < \varepsilon_c$**

Maximising number of good flux surfaces is the topic of advanced stellarator design S. R. Hudson *et al* Phys. Rev. Lett. 89, 275003, 2002

- Bruno and Laurence (c. 1996, Comm. Pure Appl. Maths, XLIX, 717-764,)
  Derived existence theorems for sharp boundary solutions for tori for small departure from axisymmetry.

$\Rightarrow$ stepped pressure profile solutions can exist.

Generalised Taylor Relaxation:

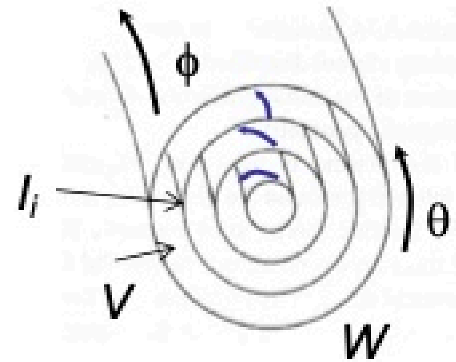
Multiple Relaxed Region MHD (MRXMHD)

R. L. Dewar

- Assume each invariant tori I_l act as ideal MHD barriers to relaxation, so that Taylor constraints are localized to subregions.

New system comprises:

- N plasma regions R_l in relaxed states.
- Regions separated by ideal MHD barrier I_l.
- Enclosed by a vacuum region R_V,
- Encased in a perfectly conducting wall W



$$W_l = \int_{R_l} \left(\frac{B_l^{\,2}}{2\mu_0} + \frac{P_l}{\gamma - 1} \right) d\tau^3$$

$$H_l = \int_V (\mathbf{A}_l \cdot \mathbf{B}_l) d\tau^3$$

Seek minimum energy state:

$$F = \sum_{l=1}^{N} \left(W_l - \mu_l H_l / 2 \right)$$

$$\begin{array}{ll}
P_l : & \nabla \times \mathbf{B} = \mu_l \mathbf{B} \\
 & P_l = \text{constant} \\
I_l : & \mathbf{B} \cdot \mathbf{n} = 0 \\
 & [[P_l + B^2/(2\mu_0)]] = 0 \\
V : & \nabla \times \mathbf{B} = 0 \\
 & \nabla \cdot \mathbf{B} = 0 \\
W : & \mathbf{B} \cdot \mathbf{n} = 0
\end{array}$$

Stepped Pressure Equilibrium Code, SPEC

[Hudson et al Phys. Plasmas 19, 112502 (2012)] Hudson

Vector potential is discretised using mixed Fourier & finite elements

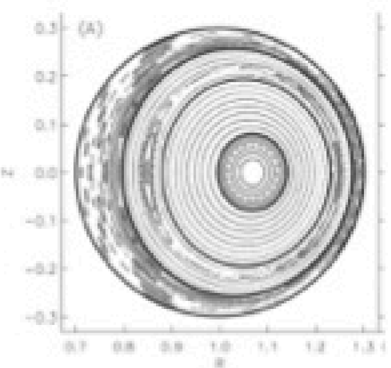
- Coordinates (s,φ, ζ), field $\boldsymbol{B} = \nabla\times\boldsymbol{A}$
- Fourier Interface geometry $R_i = \sum_{l,m,n} R_{lmn}\cos(m\vartheta - n\zeta),\ Z_i = \sum_{l,m,n} Z_{lmn}\sin(m\vartheta - n\zeta)$
- Fourier: $A_\vartheta = \sum_{m,n} \alpha(s)\cos(m\vartheta - n\varsigma)$
- Finite-element: $a_\vartheta(s) = \sum_i a_{\vartheta,i}(s)\varphi(s)$

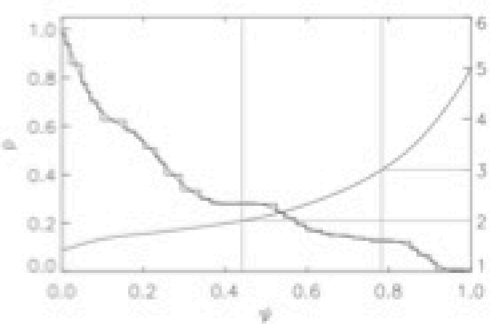
& inserted into constrained-energy functional $F = \sum_{l=1}^{N}(W_l - \mu_l H_l/2)$

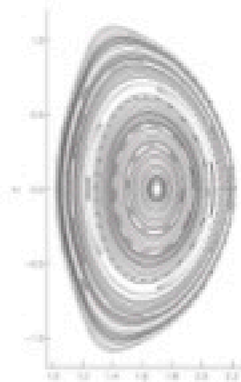
- Field in each annulus computed independently, distributed across multiple cpu's

Force balance solved using multi-dimensional Newton method

- Interface geometry adjusted to satisfy force balance $\mathbf{F}[\xi] = \{[\![p + B^2/2]\!]_{m,n}\} = 0$





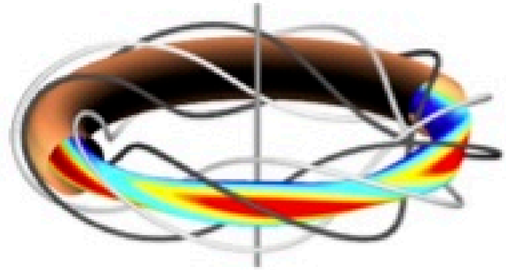


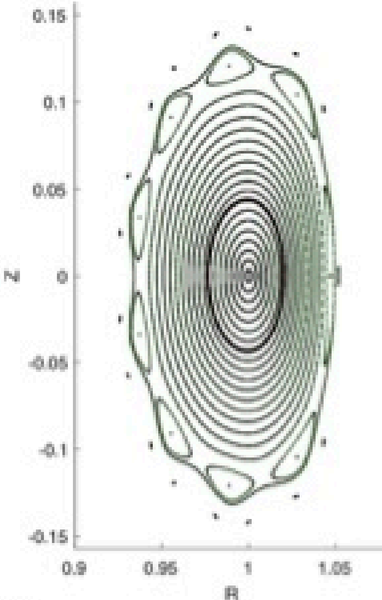
Free-Boundary SPEC, Zernike basis

[Hudson et al Plasma Phys. Control. Fusion 62 (2020) 084002]

SPEC extended to handle free-boundary

- Vacuum field computed in $R_v \in R_D \backslash R_P$, with D ‘computational boundary’,
- Virtual-casing principle is used iteratively to compute the normal field on D.

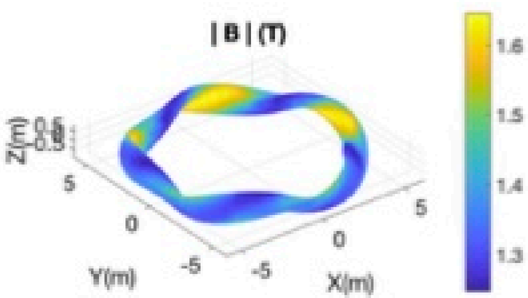




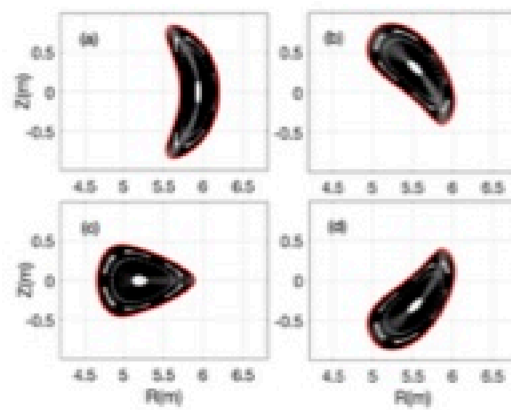
Zernike radial basis function

[Z S Qu, D Pfefferle *et al* 2020 *Plasma Phys. Control. Fusion* **62** 124004

- Improved choice for reference axis to prevent surfaces overlapping
- Employ Zernike radial basis functions near axis
- Accelerated code by O(1000).



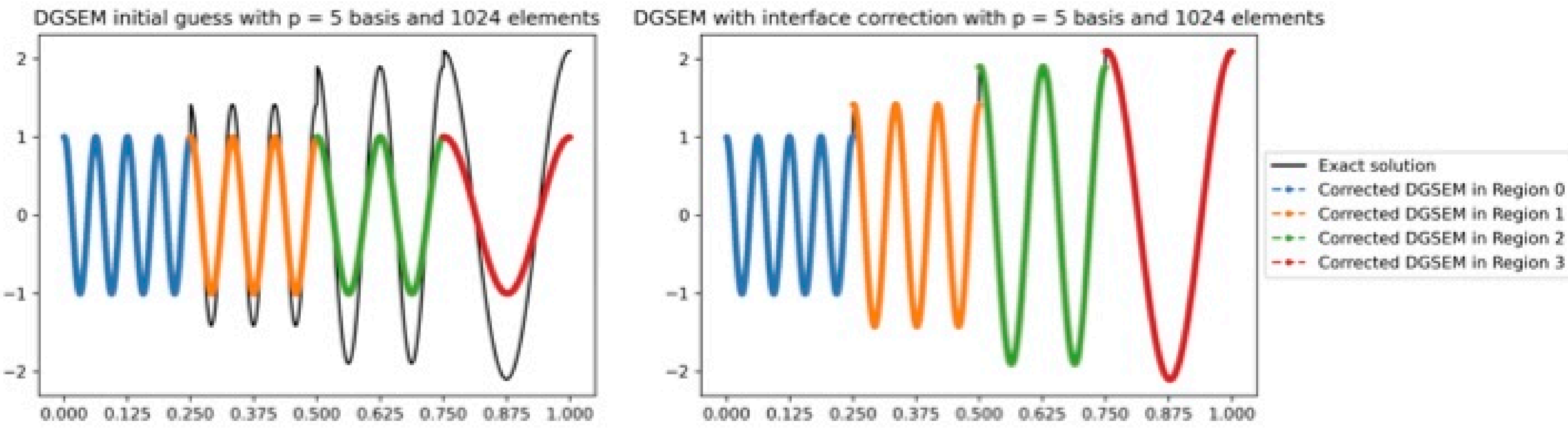
W7-X OP1.1 boundary



Discontinuous Galerkin approach

Tang, Duru

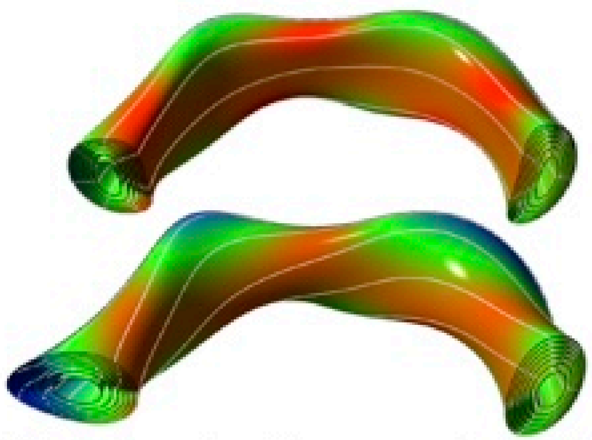
- Attempt to develop a more capable and robust numerical scheme for the MRxMHD model & improve the ability of the current SPEC code to handle complex geometry.
- 1D with jumps in p, B



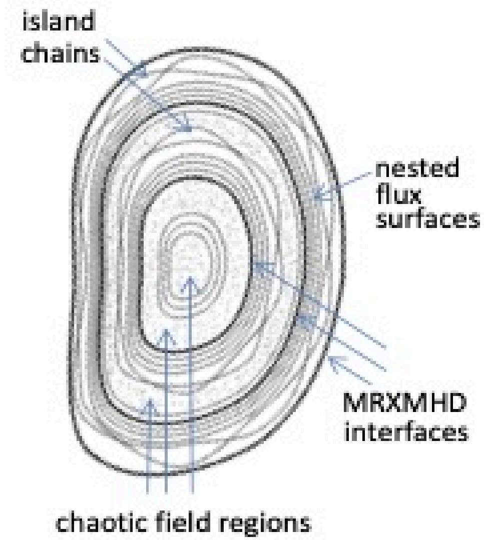
	Exact [p + 1/2 B^2]	UnCorrected	Corrected
Interface 1	0.0	-0.5	-0.00038976952476421634
Interface 2	0.0	-0.8	4.117509425860533e-05
Interface 3	0.0	-0.4	1.0349951263310952e-05

What is the “standard” physics approach?

- Current 3D MHD solvers built on premise that volume is foliated with toroidal magnetic flux surfaces (eg. VMEC[1]), &/or adapts magnetic grid to try to compensate (PIES[2])
- Can not rigorously solve ideal MHD – error usually manifest as a lack of convergence[3,4].



[CTH stellarator, Hanson et al, IAEA 2012]



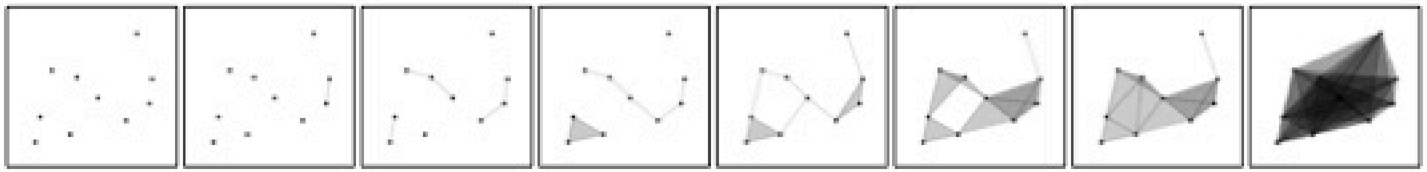
[1] S. P. Hirshman and Whitson, Phys. Fluids **26**, 3553 1983.
[2] A. H. Reiman and H. S. Greenside, J. Comput. Phys. **75**, 423 1988.
[3] H. J. Gardner and D. B. Blackwell, Nucl. Fusion **32**, 2009 1992.
[4] S.R. Hudson, M.J. Hole and R.L. Dewar, Phys. Plasmas **14**, 052505 (2007)

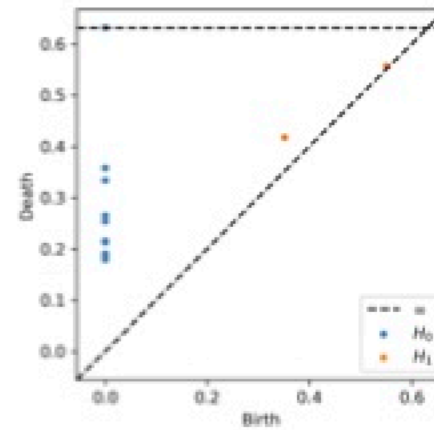
Field line classification with persistent homology

Bohlsen

Question: Can we determine the class of a field line (stochastic, KAM torus, island chain, etc.) from only the Poincare section automatically?

Answer: Yes, with persistent Homology





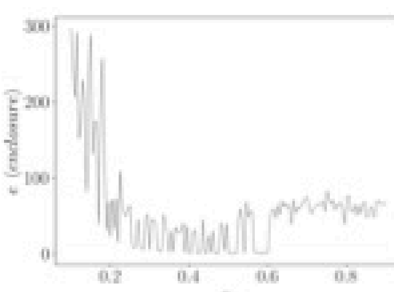
Form sequence of simplicial complexes by progressively connecting points which are further and further apart.

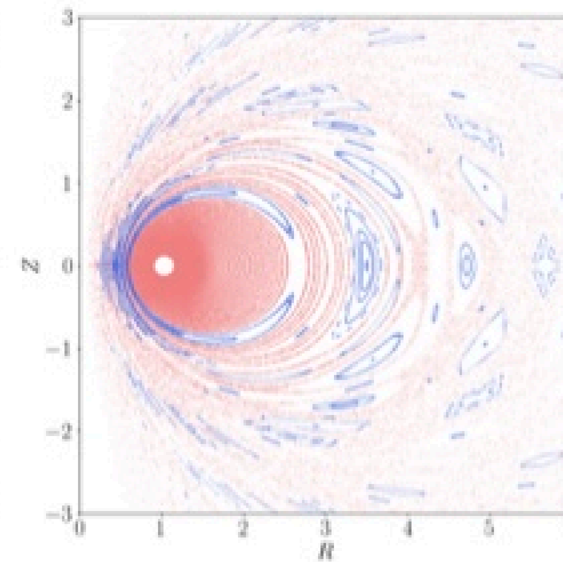
Add triangles as soon as possible (Vietoris-Rips filtration). Homology classes are born and die (display on Persistence Diagram).

Multiplicative persistence of the last H_1 class (enclosure) is very small for islands and high elsewhere.

Gives an automatic classifier which can be used to make a map of the field line class.

Procedure should generalise to problem of phase space orbit identification in high dimensions.



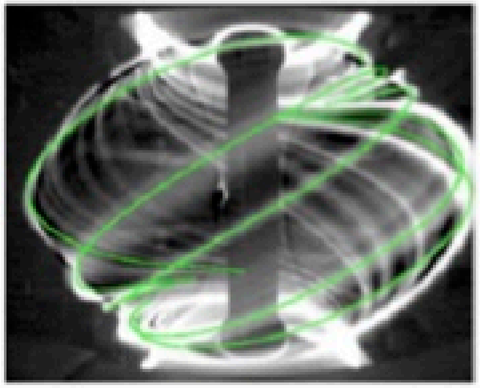


Tokamak Stability Zoo

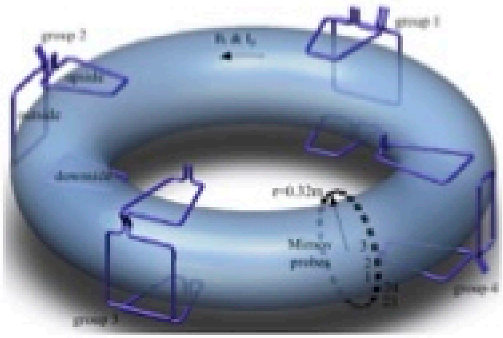
A whole zoo of modes. Can divide them as:

- Most-serious (disruptive): e.g. external modes such as the (n, m) = (1,1) *external kink*, driven by gradients in pressure and current density

- Serious but tolerable (performance-limiting):
    - *Sawteeth*, internal kink, (n, m) = (1,1) – reconnection of core. Periodic collapse of central temperature
    - Alfven eigenmodes, wave-particle resonance driven. Loss of fast particle confinement
    - *Edge-Localised Modes* (ELMs), which occur for moderately high *m* and *n*.



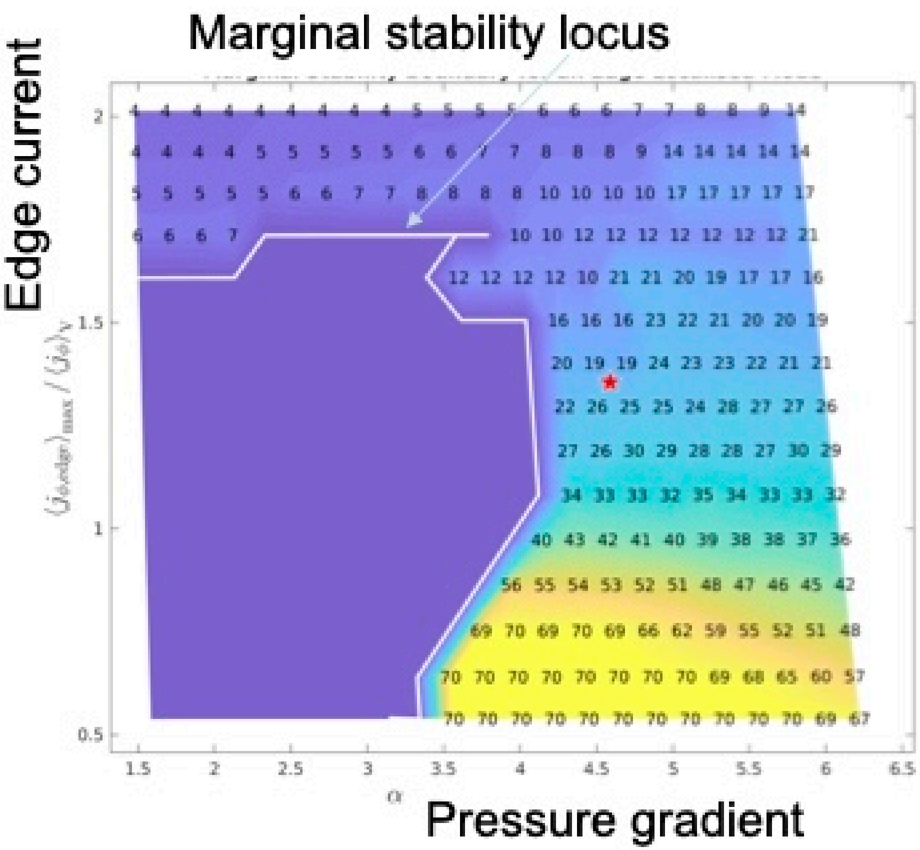
ELM mitigation / suppression demonstrated by application of resonant magnetic perturbation coils, that deliberately perturb edge

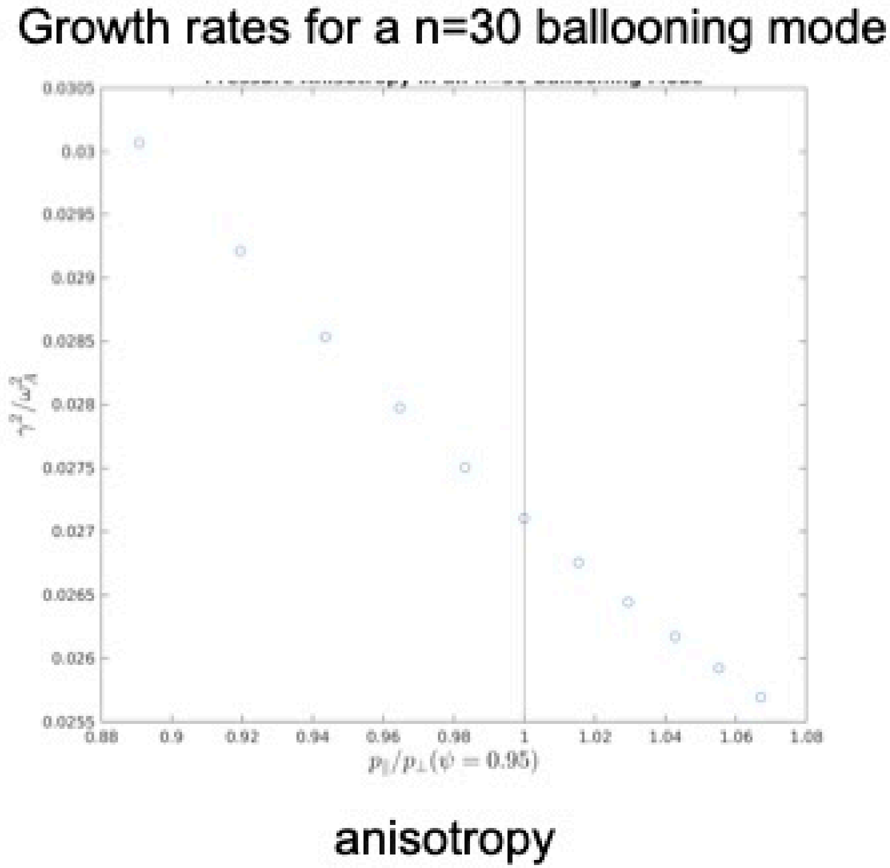


The effect of plasma anisotropy on ELMs

Doak

ELMs comprise instabilities driven by high currents (peeling modes) and steep pressure gradients (ballooning modes) that form in the edge pedestal





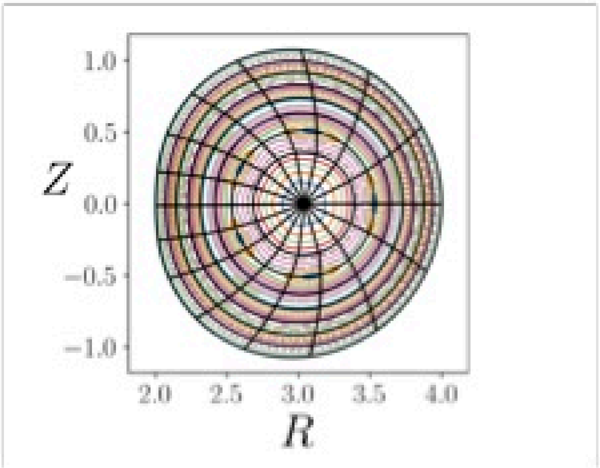
Growth rates γ of linear perturbations

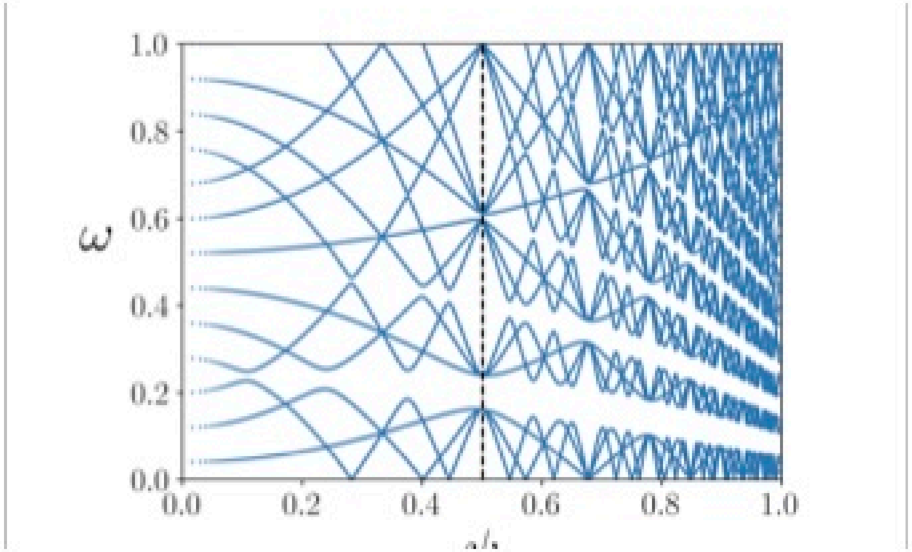
$\xi(t) = \xi e^{(\gamma - i\omega)t}$

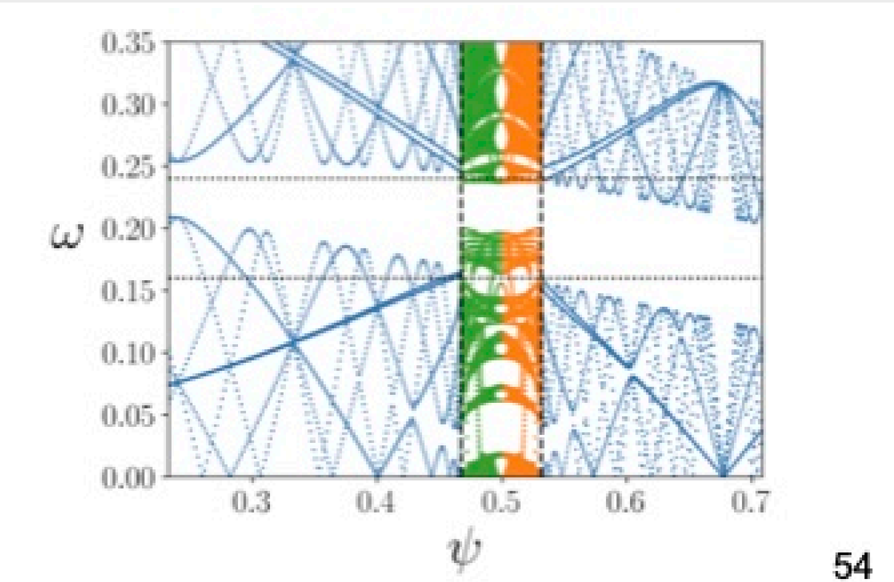
Kinetic Alfvén continuum with magnetic islands

Thomas

- Alfven waves exist in a continuum of frequencies
- Z. Qu showed how this continuum is modified by magnetic islands using ideal MHD
- We want to extend this to include non-ideal effects of finite Larmor radius







Provably stable method for magnetic field aligned anisotropic diffusion

Muir

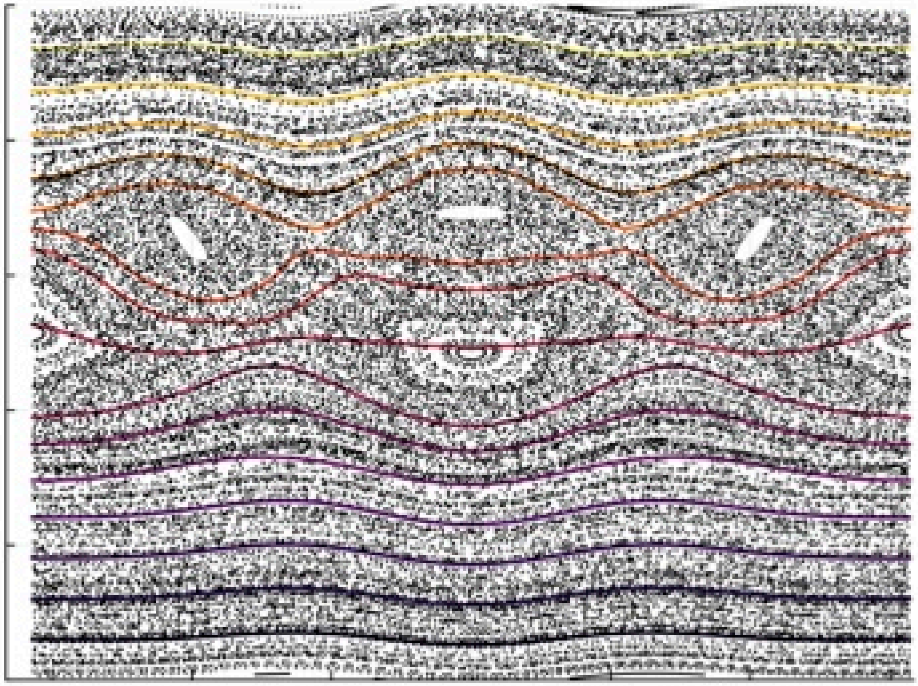
$$\frac{\partial u}{\partial t} = \nabla \cdot (\kappa_\perp \nabla_\perp + \kappa_\parallel \nabla_\parallel) u$$

where $\nabla_\parallel = \mathbf{B}\frac{\mathbf{B}\cdot\nabla}{\|\mathbf{B}\|^2}$, $\nabla_\perp = \nabla - \nabla_\parallel$ and $\kappa_\parallel/\kappa_\perp \sim 10^{10}$.

- Multiscale due to ratio of $\kappa_\parallel/\kappa_\perp$
- Numerical error quickly overwhelms small perpendicular diffusion
- Grid aligned with magnetic field often impossible
- Numerical method:
  - preserves IBP
  - Weak imposition of boundary conditions
  - Provably numerically stable

Dean Muir, Kenneth Duru, Matthew Hole, Stuart Hudson (PPPL)

Poincarè section with overlaid solution contours



Structure preserving discretisations for collision operations

Jeyakumar

Collaboration with David Pfefferlé UWA and IPP Michael Kraus

THE UNIVERSITY OF WESTERN AUSTRALIA

Max-Planck-Institut für Plasmaphysik

- Adding dissipation (e.g. collisions) to the Vlasov equation:

$$\underbrace{\partial_t f + \mathbf{v} \cdot \nabla_x f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f}_{\text{Hamiltonian}} = \underbrace{C[f](\mathbf{v})}_{?}$$

- Examples of collision operators:

$$C[f](v) = \nu \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v} + vf\right),$$

$$C[f](v) = \frac{\partial}{\partial v} \cdot \int Q(v - v')\left(f(x, v')\frac{\partial f(x, v)}{\partial v} - f(x, v)\frac{\partial f(x, v')}{\partial v'}\right) \mathrm{d}v'.$$

Aim: to solve this equation while preserving structures (Hamiltonian/ Poisson), invariants etc. in particle-in-cell simulations

exact flow explicit Euler symplectic Euler

Exact and numerical flow for the pendulum problem

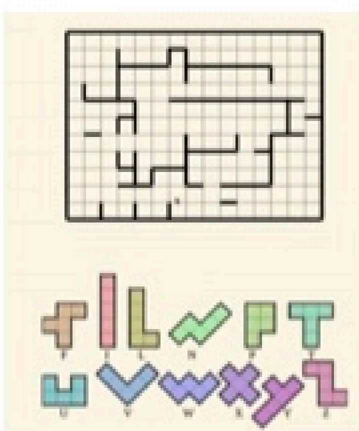
AI for Analysis of Physics Codes and Instrument Models

Gretton

- We have developed new AIs for circumstances where we encounter
  a. hard combinatorial puzzles -- e.g., can this code/controller/model yield bad results?
  b. real-time operational constraints -- e.g., dynamic control of instrument at 1khz.

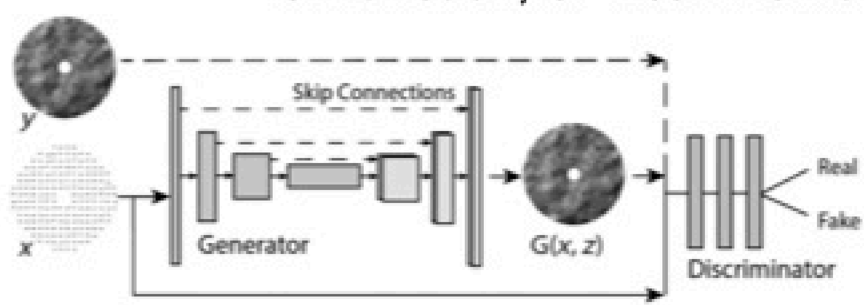
a. Hard combinatorial puzzle

Burgess *et al.*, AAAI2023/PRICAI2022



b. Real-time constraint in instrumentation motivates AI *network-based* phase estimate

Smith *et al.*, SPIE/JATIS 2023



Fusion Plasma Initiatives

Strategic Planning for Fusion Science in Australia:
Australian ITER Forum, 2005 - present

IAEA Consultancy Meeting on Assessment of Cooperation Areas in Fusion Education and Capacity Building (6-9 June 2022)

Host " Mathematics Sol Terrae", April 2022: topic of higher mathematics related to high performance computing in the fields of the solid Earth (geophysics), land-atmosphere carbon exchange, and solar physics.

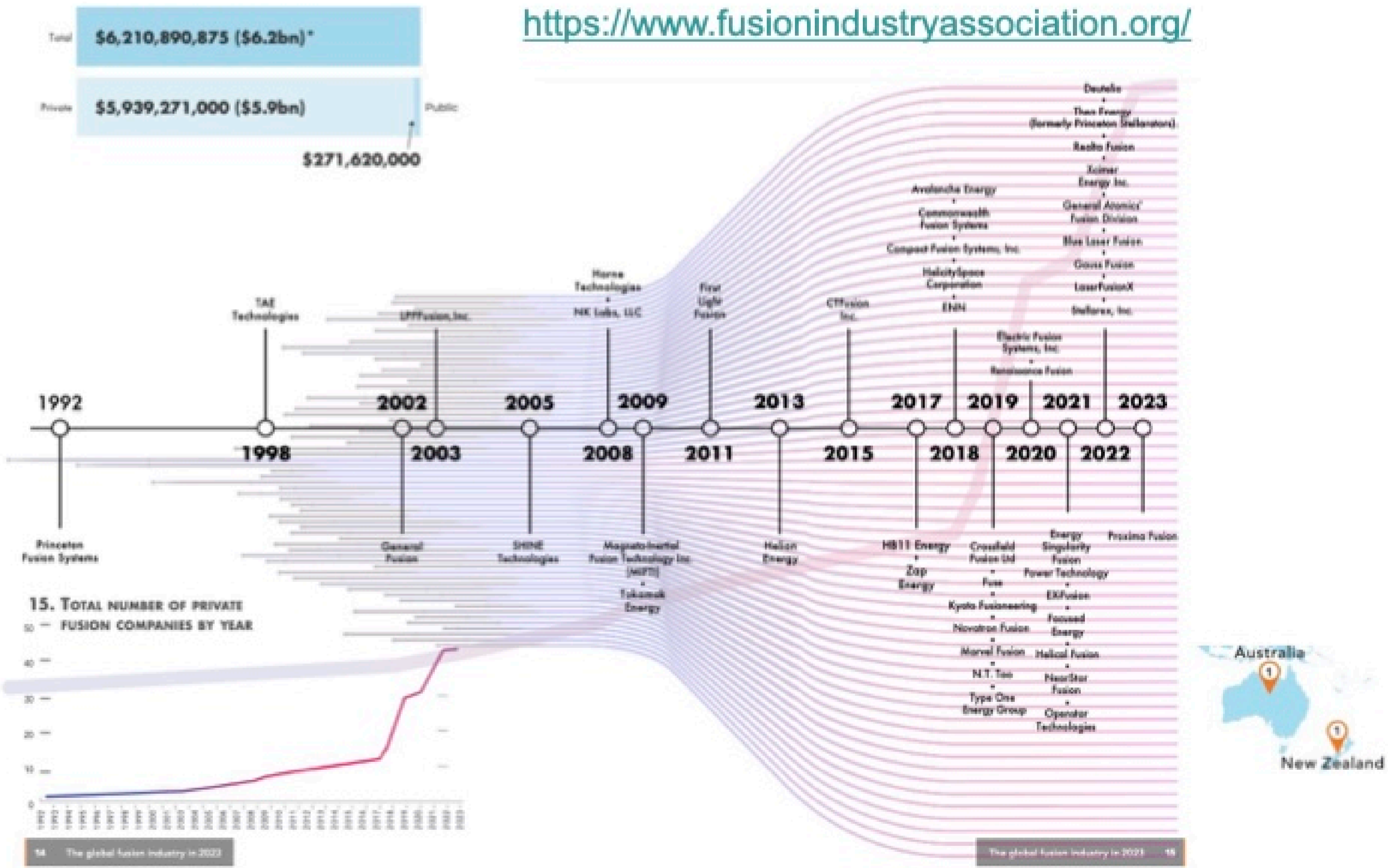
ANU-ITER Global Partnership
Investment from ANU, MSI, CECS, COS, Comp. Science, Engineering
PhD top-up and internship scheme to undertake projects relevant to ITER

Simons Hidden Symmetries and Fusion Energy Collaboration
Australian Retreat: 4 – 15 December 2023, ANU

https://maths.anu.edu.au/news-events/events/simons-hidden-symmetries-and-fusion-energy-collaboration-australian-retreat

Fusion Private Industry Investment

https://www.fusionindustryassociation.org/



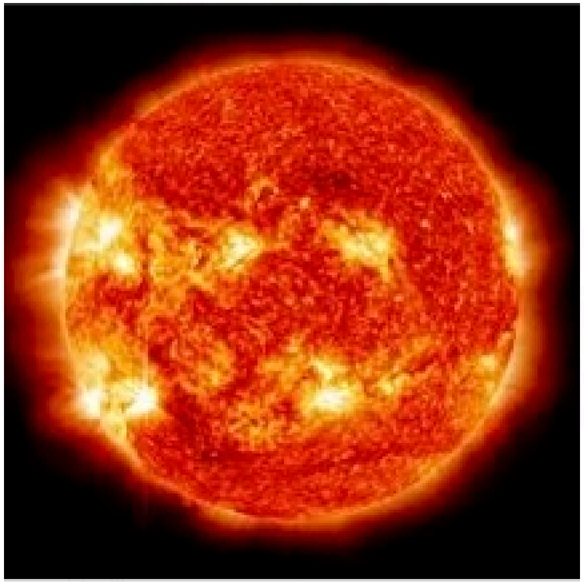
Summary

- Introduction to Toroidal Magnetic Confinement
- Fusion reactor 101: Physics / Engineering constraints of a power plant
- The role of ITER
- Alternative Toroidal Magnetic Confinement Concepts
- Like high energy particle physics, experimental burning plasma physics is big science. Collaboration is the future.
- Generator of interesting mathematical and computational challenges.
- Private investment may create opportunity…

Implications *when* Fusion Power realised

On Earth, fusion could provide:

- Large-scale energy production:
  Essentially limitless fuel
  No greenhouse gases
  Intrinsic safety
  Cannot be weaponised
  No long-lived radioactive waste
- Industrial heat



Ubiquitous, near limitless clean power could:

- Lift the developing world out of poverty
- End (energy) wars
- Power large-scale clean water through desalination
- Enable CO_2 removal from the atmosphere
- Enable extra-solar space travel….

SimCity fusion power plant