Hausdorff measure for limsup sets

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Diophantine Approximation

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Theorem (Dirichlet 1842)

For any real number α and $N \in \mathbb{N}$, there exists integers p, q with $1 \leq q \leq N$ such that

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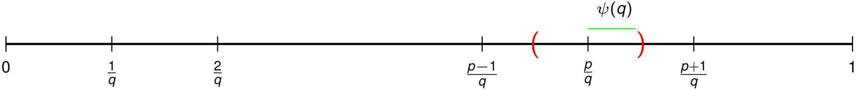
$$W(\psi) \;=\; \bigcap_{N=1}^{\infty} \bigcup_{q=N}^{\infty} \bigcup_{p=0}^{q} \left\{\alpha \in [0, 1) : \left|\alpha - \tfrac{p}{q}\right| < \psi(q)\right\} = \limsup_{q \to \infty} A_q(\psi).$$



Borel-Cantelli Lemmas

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Lemma (Borel-Cantelli, 1909)

Let $E_1, E_2, \ldots$ be a sequence of events in some probability space.

$$\text{If} \quad \sum_{n=1}^{\infty} \Pr(E_n) < \infty, \quad \text{then} \quad \Pr\left(\limsup_{n\to\infty} E_n\right) = 0.$$

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Example

$A_q(\psi) = E_q = (0, 1/q)$, $\quad \sum_{q=1}^{\infty} \mathcal{L}(E_q) = \infty$. But

$$\limsup_{t \to \infty} E_t = \bigcap_{t=1}^{\infty} \bigcup_{q=t}^{\infty} E_q = \bigcap_{t=1}^{\infty} (0, 1/t) = \emptyset$$

Classical Diophantine Approximation

$$W(\psi) = \bigcap_{N=1}^{\infty} \bigcup_{q=N}^{\infty} \bigcup_{p=0}^{q} \left\{\alpha \in [0,1) : \left|\alpha - \frac{p}{q}\right| < \psi(q)\right\} = \limsup_{q\to\infty} A_q(\psi).$$

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Lebesgue measure criterion

Theorem (Khintchine, 1924)

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function. Then

$$\mathcal{L}(W(\psi) \cap \mathbb{I}) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} q\psi(q) < \infty \\ 1 & \text{if} \quad \sum_{q=1}^{\infty} q\psi(q) = \infty \;\; \psi \textit{ is decreasing.} \end{cases}$$

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- $\psi_1(q) = \frac{1}{q^2}$; *then* $\sum_{q=1}^{\infty} q\psi_1(q) = \infty \Longrightarrow \mathcal{L}(W(\psi_1) \cap \mathbb{I}) = 1.$
- $\psi_2(q) = \frac{1}{q^{10}}$; *then* $\sum_{q=1}^{\infty} q\psi_2(q) < \infty \Longrightarrow \mathcal{L}(W(\psi_2) \cap \mathbb{I}) = 0$

Uniform Approximation

Definition (Dirichlet improvable sets)

Let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be a non-increasing function with $t_0 \geq 1$ fixed.

$$D(\psi) := \left\{ x \in \mathbb{R} : \begin{array}{c} \exists\, N : \text{ the system } |qx - p| < \psi(t),\ |q| < t \\ \text{has a non trivial integer solution for all } t > N \end{array} \right\}$$

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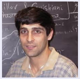
Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t. Then



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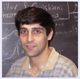


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Example

$$\mathcal{L}(D(\psi)^c) = \begin{cases} 0, & \text{if} \quad \psi(t) = \frac{1}{t}\left(1 - \frac{1}{\log t (\log\log t)^{2+\epsilon}}\right) \text{ for any } \epsilon > 0; \\ \text{full}, & \text{if} \quad \psi(t) = \frac{1}{t}\left(1 - \frac{1}{\log t (\log\log t)^{2}}\right). \end{cases}$$

Hausdorff Measure and Dimension

Definition

Let $s > 0$ and $\{B_i\}$ be a countable collection of Euclidean balls with $diam(B_i) \leq \rho$ such that $X \subset \bigcup_i B_i$.

$$\mathcal{H}^s_\rho(X) = \inf\left\{\sum_i (diam(B_i))^s \ : \ \{B_i\} \text{ is a } \rho\text{-cover for } X\right\},$$

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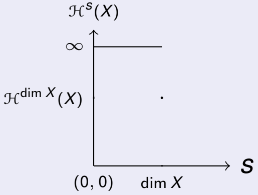

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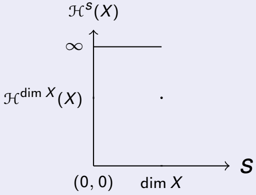
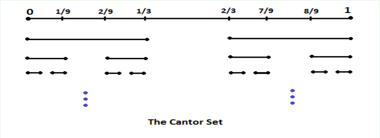


Example

Let K be the middle third Cantor set,

$$\dim K = \frac{\log 2}{\log 3}$$



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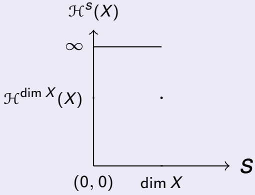

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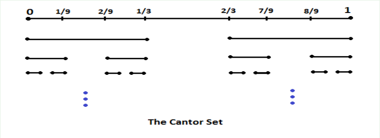
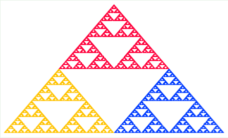


Example

The Hausdorff dimension of Sierpinksi Triangle is $\frac{\log 3}{\log 2}$ obtained by solving the equation $2^d = 3$.



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Let ψ *be an approximating function. Let* f *be a dimension function such that* $q^{-1}f(q) \to \infty$ *as* $q \to 0$ *and* $q^{-1}f(q)$ *is decreasing. Then*

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An increasing, continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+ : f(r) \to 0$ as $r \to 0$.

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Jarník theorem $\Longrightarrow$ Jarník–Besicovitch theorem.

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Question

How to prove $\mathcal{H}^f(W(\psi)) > 0$?

A general strategy

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Lemma (Mass Distribution Principle)

Let μ be a probability measure supported on a subset F of X. Suppose there are positive constants $c > 0$ and $\epsilon > 0$ such that

$$\mu(U) \leq cf(\mathrm{diam}(U))$$

for all sets U with $\mathrm{diam}(U) \leq \epsilon$. Then $\mathcal{H}^f(F) \geq \mu(F)/c$.

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If this can be done, then by the mass distribution principle, it follows that

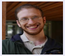
$$\mathcal{H}^f(F) \geq \mathcal{H}^f(\mathcal{K}) \geq c^{-1}.$$

Then since c is arbitrary, it follows that $\mathcal{H}^f(F) = \infty$.

A general principle

Theorem (H.–Simmons, PAMS 2019)

Fix $\delta > 0$, let $(B_i)_i$ be a sequence of open sets in an Ahlfors δ-regular metric space X, and let f be a dimension function such that



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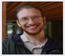

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(*) *For every ball* $B_0 \subset X$ *and for every* $N \in \mathbb{N}$, *there exists a probability measure* $\mu = \mu(B_0, N)$ *with* $\mathrm{Supp}(\mu) \subset \bigcup_{i \geq N} B_i \cap B_0$, *such that for every ball* $B = B(x, \rho) \subset X$, *we have*

$$\mu(B) \lesssim \max\left(\left(\frac{\rho}{\mathrm{diam} B_0}\right)^{\delta}, \frac{f(\rho)}{C}\right). \tag{3}$$

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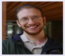

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Then for every ball B_0,

$$\mathcal{H}^f\left(B_0 \cap \limsup_{i \to \infty} B_i\right) \gtrsim C.$$

In particular, if the hypothesis (*) holds for all C, then $\mathcal{H}^f\left(B_0 \cap \limsup_{i \to \infty} B_i\right) = \infty$.

Mass Transference Principle

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Theorem (Beresnevich–Velani, Ann. Math. 2006)

Let $X \subset \mathbb{R}^d$ be Ahlfors δ-regular. Let $(B_i)_{i\in\mathbb{N}}$ be a sequence of balls in X with $rad(B_i) \to 0$ as $i \to \infty$. Let f be a dimension function such that $r \mapsto r^{-\delta} f(r)$ is monotonic. Suppose that for every ball $B \subset X$

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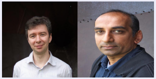

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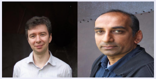

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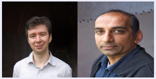

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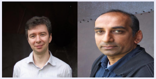

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A Jarnik type criterion for uniform approximation

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Definition (Dirichlet improvable sets)

Let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be a non-increasing function with $t_0 \geq 1$ fixed.

$$D(\psi) := \left\{ x \in \mathbb{R} : \begin{array}{c} \exists\, N : \text{ the system } |qx - p| < \psi(t),\ |q| < t \\ \text{has a non trivial integer solution for all } t > N \end{array} \right\}$$

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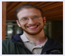

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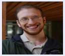

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Corollary (H.–Kleinbock–Wadleigh–Wang, Mathematika 2018)




Let ψ be a non-increasing positive function with $t\psi(t) < 1$ for all large t, and let f be an essentially sub-linear dimension function. Then

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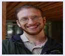

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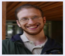

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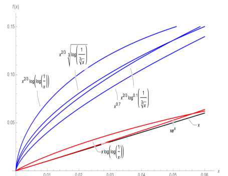

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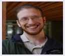

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$$\Psi(Q^x) \asymp \Psi(Q), \tag{4}$$

where the implied constant depends only on x. Then

$$\mathcal{H}^f(D^c(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_q q \log(\Psi(q)) f\left(\frac{1}{q^2 \Psi(q)}\right) < \infty; \\ \infty & \text{if} \quad \sum_q q \log(\Psi(q)) f\left(\frac{1}{q^2 \Psi(q)}\right) = \infty. \end{cases}$$

Sketch of the proof

$$G(\Psi) = \left\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(q_n) \ \text{ for i.m. } n \in \mathbb{N}\right\},$$

$$\begin{aligned} G(\Psi) =& \left\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(q_n) \ \text{ for i.m. } n \in \mathbb{N}\right\} \\ \subseteq& \left\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(a_n q_{n-1}) \ \text{ for i.m. } n \in \mathbb{N}\right\} \\ \subseteq& \bigcup_{n=N}^{\infty} \bigcup_{a_1,\ldots,a_n} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}) \\ =& \mathcal{A}_1(\Psi) \cup \mathcal{A}_2(\Psi). \end{aligned}$$

Where

$$\mathcal{A}_1(\Psi) = \bigcup_{n=N}^{\infty} \bigcup_{a_1,\ldots,a_n} \bigcup_{a_n \leq \Psi(q_{n-1})} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}),$$

$$\mathcal{A}_2(\Psi) = \bigcup_{n=N}^{\infty} \bigcup_{a_1,\ldots,a_n} \bigcup_{a_n > \Psi(q_{n-1})} \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1, \ldots, a_n, a_{n+1}).$$

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$$J_n(a_1,\ldots,a_n) := \bigcup_{a_{n+1} > \frac{\Psi(a_n q_{n-1})}{a_n}} I_{n+1}(a_1,\ldots,a_n,a_{n+1}).$$

$$|J_n(a_1,\ldots,a_n)| \leq \frac{1}{\Psi(a_n q_{n-1}) a_n q_{n-1}^2}.$$

Let $Q > 1$ and $Q < q_{n-1} \leq 2Q$. Then

$$|J_n(a_1,\ldots,a_n)| \ll \frac{1}{\Psi(a_n Q) a_n Q^2}.$$

Hence, the cost of the cover when $a_n < \Psi(q_{n-1})$, is

$$\sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{aQ^2\Psi(aQ)}\right).$$

In the case $a_n > \Psi(q_{n-1})$, the cost of the cover is given by

$$f\left(\frac{1}{Q^2\Psi(Q)}\right).$$

Sketch of the proof

Since $Q > 1$, it follows that for each window $[Q, 2Q]$, there are at most Q^2 cylinders I_n of length comparable (up to a constant) to Q^{-2}. Multiplying the cost of the cover given above by Q^2 which are the number of such intervals, and then summing over all the windows $Q = 2^k$, we have

$$\sum_{Q=2^k;k\geq 1} Q^2 \sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{aQ^2\Psi(aQ)}\right) + \sum_{Q=2^k;k\geq 1} Q^2 f\left(\frac{1}{Q^2\Psi(Q)}\right).$$

Applying Cauchy's condensation test on the second term, and rewriting the first term gives the total cost as

$$\sum_{\substack{k\geq 1\\ Q=2^k}} \sum_{\substack{j\geq 1, A=2^j\\ A<\Psi(Q)}} Q^2 A f\left(\frac{1}{Q^2 A\Psi(QA)}\right) + \sum_q qf\left(\frac{1}{q^2\Psi(q)}\right).$$

Sketch of the proof

Theorem

Let $(u_a)_{a \in E}$ be the Gauss iterated function system. For each finite word $\omega \in E^*$ and $a \leq \Psi(Q_\omega)$ let

$$S_{\omega a} = u_{\omega a}([0, a/\Psi(Q_\omega a)]).$$

Let f be a dimension function such that $\sum_{\omega,a} f(\operatorname{diam} S_{\omega a})$ diverges. Then

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First we show that $\limsup_{\omega,a} S_{\omega a} \subseteq G(\Psi)$.

$$\sum_{\omega,a} f(\operatorname{diam} S_{\omega a}) \asymp \sum_{Q=2^k;k\geq 1} Q^2 \sum_{a=1}^{\Psi(Q)} f\left(\frac{1}{aQ^2\Psi(aQ)}\right) \asymp \sum_{k=1}^{\infty} \sum_{j<\log_2 \Psi(2^k)} 2^{2k+j} f\left(\frac{2^{-(2k+j)}}{\Psi(2^{k+j})}\right).$$

Fix $B_0 \subset [0,1]$ and $N \in \mathbb{N}$, and we will construct the measure $\mu = \mu(B_0, N)$ such that the hypothesis (*) in Theorem 0 holds.

Applications: Recurrence sets

Let $(X, \mathcal{B}, \mu, T, d)$ be a metric measure preserving system (m.m.p.s.). the Poincaré recurrence theorem implies that μ-almost every $x \in X$ is recurrent in the sense that

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$$\mathcal{R}(\psi) = \{x \in X : d(T^n x, x) < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Theorem (H. 2023)

Let T be a beta or Gauss dynamical system. Let f be a dimension function such that

$$f(ax) \asymp a^s f(x) \quad \forall x \leq a^\epsilon$$

for some $s, \epsilon > 0$. Then

$$\sum_n e^{nP(s)} f(\psi(n)) = \infty \quad \Longleftrightarrow \quad \mathcal{H}^f(\mathcal{R}(\psi)) = \infty$$


THANK YOU!!!
FOR YOUR TIME AND ATTENTION