Inhomogeneous Baxter Model at Criticality

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Baxter 2025 Exactly Solved Models and Beyond: Celebrating the life and achievements of Rodney James Baxter

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Generalized Ferroelectric Model on a Square Lattice

By R. J. Baxter

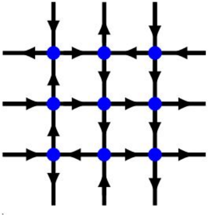
A completely general square-lattice ferroelectric model is considered, in which the configuration energies have arbitrary values at each vertex. It is found that the partition function can be evaluated by straightforward extensions of the method used for the regular ice, F- and KDP-models, provided three sets of conditions are satisfied. These ensure that the two ordered F-model states are degenerate (in the limit of a large lattice), so we conclude that the application of any staggered field which breaks this degeneracy also renders the problem insoluble by the present methods.

The partition function for these ordered states is obtained explicitly and found to be given quite simply as a product of certain parameters associated with each vertex. Finally some conjectures are made concerning the spontaneous staggered polarization.

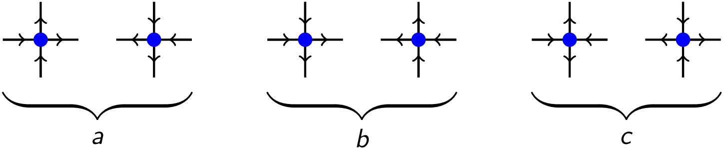
(Baxter considered the case $0 < q < 1$; in this talk we assume $|q| = 1$)

Six vertex model [Lieb'1967; Sutherland, Yang, Yang'1967]

$$Z_{6\mathrm{V}} = \sum_{\mathrm{config}} \exp\Big(-\frac{E}{k_b T}\Big) =$$



Boltzmann weights subject to ice rule



- Parameterization

$$a = q^{-1}\zeta - q\,, \qquad b = \zeta - 1\,, \qquad c = (q^{-1} - q)\,\sqrt{\zeta}$$

ζ – spectral parameter

q – anisotropy parameter

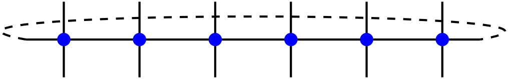
- R matrix

$$R(\zeta|q) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \in \mathrm{End}\big(\mathbb{C}^2 \otimes \mathbb{C}^2\big)$$

- Yang-Baxter equation

$$R_{12}(\zeta_1/\zeta_2)\, R_{13}(\zeta_1/\zeta_3)\, R_{23}(\zeta_2/\zeta_3) = \underbrace{R_{23}(\zeta_2/\zeta_3)\, R_{13}(\zeta_1/\zeta_3)\, R_{12}(\zeta_1/\zeta_2)}_{\in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)}$$

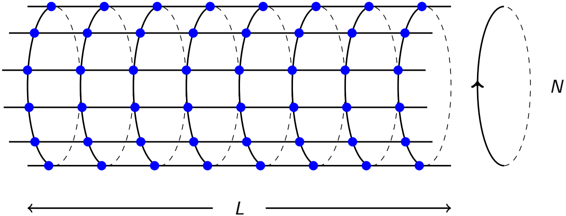
$\mathbb{T}(\zeta) =$



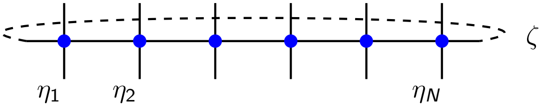
$$\mathbb{T}(\zeta) = \underbrace{\mathrm{Tr}_0\big(R_{01}(\zeta)\, R_{02}(\zeta)\, \ldots\, R_{0N}(\zeta)\big)}_{\in \mathrm{End}(\mathbb{C}_1^2 \otimes \mathbb{C}_2^2 \otimes \ldots \otimes \mathbb{C}_N^2)} : \qquad \mathrm{YBE} \implies \big[\mathbb{T}(\zeta), \mathbb{T}(\zeta')\big] = 0$$

Existence of **large commuting family of operators** simplifies spectral problem for $\mathbb{T}(\zeta)$ and enables application of powerful methods to compute partition function

$$Z_{6\mathrm{V}} = \mathrm{Tr}\big(\mathbb{T}(\zeta)\big)^L$$



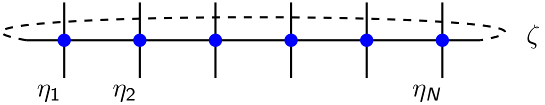
Same argument follows through for

$\mathbb{T}(\zeta|\eta_1,\ldots,\eta_N;q) =$ 

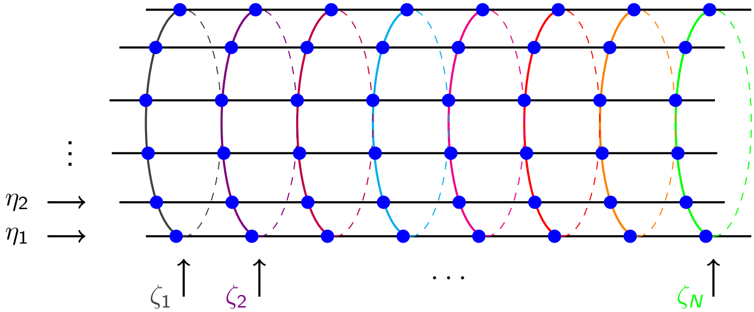
$$\mathbb{T}(\zeta|\eta_1,\ldots,\eta_N;q) = \mathrm{Tr}_0\big(R_{01}(\zeta/\eta_1)\,R_{02}(\zeta/\eta_2)\,\ldots\,R_{0N}(\zeta/\eta_N)\big)\,,$$

$\{\eta_m\}$ – inhomogeneities

$$\mathrm{YBE} \quad\Longrightarrow\quad \big[\mathbb{T}(\zeta),\mathbb{T}(\zeta')\big]=0$$

$\mathbb{T}(\zeta|\eta_1,\ldots,\eta_N;q) =$ 

$$Z_{6\mathrm{V}} = \mathrm{Tr}\Big(\prod_{i=1}^{L}\mathbb{T}(\zeta_i|\eta_1,\ldots,\eta_N;q)\Big)$$

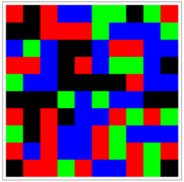


Defines multiparametric integrable statistical system!

Inhomogeneous six vertex model – applications

$$Z_{\text{Potts}} = \sum_{\text{config}} \exp\Big(\sum_{\langle i,j\rangle} J_{i,j}\,\delta_{s_i,s_j}\Big)$$

$$(s_\ell = 1, 2, \ldots, Q)$$



Homogeneous six-vertex model maps into Potts model with [Baxter '73]

$$\sqrt{Q} = q + q^{-1} > 0\,, \qquad J_{i,j} = J\ :\ \mathrm{e}^{J} - 1 = \sqrt{Q}$$

Inhomogeneous six-vertex model (proper BCs) and

$$\zeta_m = \pm \mathrm{i}\zeta\,, \qquad \eta_m = \pm \mathrm{i} \quad \text{for} \quad m \text{ odd/even}$$

maps into Potts model with [Baxter, Kelland, Wu '75]

$$J_{i,j} = \begin{cases} J_1 & \langle i,j\rangle \text{ horizontal neighbours} \\ J_2 & \langle i,j\rangle \text{ vertical neighbours} \end{cases}, \qquad (\mathrm{e}^{J_1} - 1)\,(\mathrm{e}^{J_2} - 1) = Q$$

Inhomogeneous six vertex model – applications

$$|q| = 1\,, \qquad\qquad \zeta_m = \eta_m = \begin{cases} \Lambda & m \text{ odd} \\ \Lambda^{-1} & m \text{ even} \end{cases}$$

model provides lattice regularization of massive Thirring/sine-Gordon model [Japaridze, Nersesyan, Wiegmann '83; Destri, de Vega '87]

$$\mathcal{A} = \frac{1}{8\pi}\int \mathrm{d}t\mathrm{d}x\,\Big((\partial_\alpha\phi)^2 + \mu\cos(2\phi/\mathtt{R})\Big)$$

with $\mathtt{R} = \sqrt{\frac{\pi}{2\gamma}}$ and $\mu \propto N^{2(1-\frac{\gamma}{\pi})}\ \Lambda^{-1}|_{N,\Lambda\to\infty}$

Critical behavior ($R_C = \infty$)

- Classical statistical system $\longrightarrow$ 1D spin chain.
- 1D critical spin chains: low energy spectrum organizes into conformal towers [Cardy '86]

$$E \asymp Ne_\infty + \frac{2\pi v_F}{N}\left(-\frac{c}{12} + \Delta + \bar{\Delta} + \mathtt{L} + \bar{\mathtt{L}}\right) + \dots$$

Δ, $\bar{\Delta}$ conformal dimensions, $\mathtt{L}, \bar{\mathtt{L}} \geq 0$ levels, c central charge.

As an example:

- Homogeneous 6v model $\longrightarrow$ XXZ spin-1/2 chain
- For q being unimodular, i.e.

$$q = \mathrm{e}^{\mathrm{i}\gamma}$$

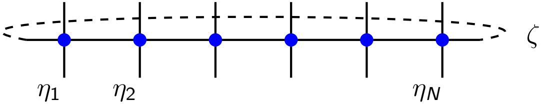
the 6v model/XXZ spin chain is critical
- Scaling limit is governed by compact boson

$$\phi \sim \phi + 2\pi \mathtt{R}$$

with compactification radius $\mathtt{R} = \sqrt{\frac{\pi}{2\gamma}}$

[Luther, Peschel'1975, Kadanoff, Brown'1979]

Critical inhomogeneous six vertex model

$\mathbb{T}(\zeta|\eta_1,\ldots,\eta_N;q) =$ 

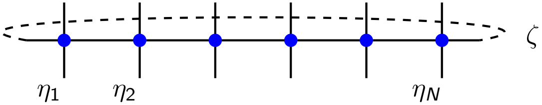
- As in the homogeneous case, it is expected that for the general inhomogeneous six-vertex model at criticality,

$$q = \mathrm{e}^{\mathrm{i}\gamma} \qquad (0 < \gamma \le \pi)$$

- However, the inhomogeneities η_m must satisfy certain conditions.

r-site periodicity

$\mathbb{T}(\zeta|\eta_1,\ldots,\eta_N;q) =$



We impose restrictions:

- Take N divisible by $r = 1, 2, 3, \ldots$ and consider r - site periodic model with

$$\eta_{m+r} = \eta_m$$

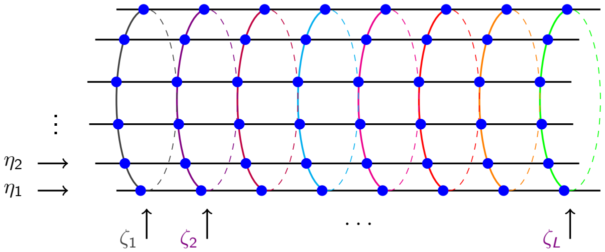
  Scaling limit $N \to \infty$ with r fixed ($\Longrightarrow$ translational invariance in CFT)

- Without loss of generality, we can set

$$\prod_{m=1}^{r} \eta_m = 1$$

Except for the cases $r = 1$ and $r = 2$, a comprehensive understanding of criticality in the r-site periodic inhomogeneous six-vertex model remains unavailable.

Staggered ($r = 2$) six vertex model

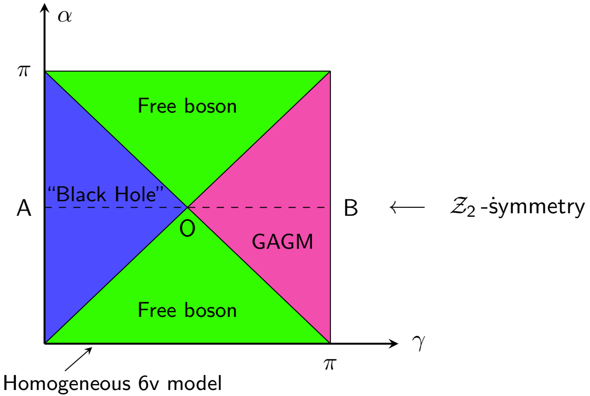


L, N even and

$$\eta_m = \begin{cases} \eta_1 & \text{for} \quad m \text{ odd} \\ \eta_2 & \text{for} \quad m \text{ even} \end{cases}, \qquad \zeta_m = \zeta \eta_m \,, \qquad \eta_1 \eta_2 = 1$$

$$\begin{array}{lll} \text{Non}-\text{critical}: & \eta_1 = \eta_2^{-1} = \Lambda \in \mathbb{R} & \gamma \in (0, \pi) \\ \text{Critical}: & \eta_1 = \eta_2^{-1} = \mathrm{e}^{\mathrm{i}\alpha} & \alpha, \gamma \in (0, \pi) \end{array}$$

Phase diagram for $\eta_1 = \eta_2^{-1} = e^{i\alpha}$, $q = e^{i\gamma}$ & $\alpha, \gamma \in (0, \pi)$



- Line AO: [(Ikhlef), Jacobsen, Saleur'05;'06,'11; Frahm, Martins'12; Candu, Ikhlef'13; Bazhanov, Kotousov, Koval, SL'19,'20]

  Whole BH region [Frahm, Seel'13]

- Line OB: [Ikhlef, Jacobsen, Saleur'09]

  Whole GAGM region [Kotousov, SL'21](compact boson + 2 Majorana fermions)

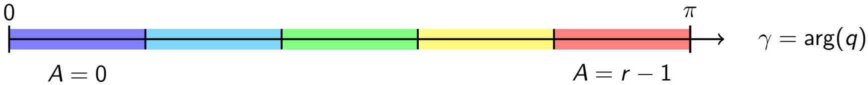
Model with r-site periodicity

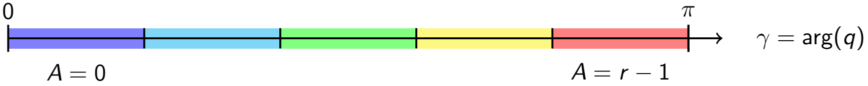
- Extra $\mathcal{Z}_r$ symmetry by setting

$$\eta_\ell = (-1)^r\, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}\,(2\ell-1)} \qquad\qquad (\ell = 1, \ldots, r)$$

- System critical when $q = \mathrm{e}^{\mathrm{i}\gamma}$ but the critical behaviour described differently in each sector

$$\frac{\pi}{r}\, A < \gamma < \frac{\pi}{r}\,(A+1)\,.$$





- Red region (GAGM) with

$$\eta_m \approx (-1)^r \, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2m-1)} \qquad \pi\left(1-\tfrac{1}{r}\right) < \gamma < \pi \qquad (A = r-1)$$

governed by r compact bosons (deformation of $\big[\widehat{\mathfrak{su}_1(2)}\big]^{\otimes r}$)
[Kotousov, SL'21]

- Blue region (BH for $r = 2$) [Kotousov, SL'23]

$$\eta_m = (-1)^r \, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2m-1)} \qquad \text{and} \qquad 0 < \gamma < \tfrac{\pi}{r} \qquad (A = 0)$$

(i) Continuous component in spectrum for r even

(ii) Infinite degeneracy of conformal primary states for $r \geq 3$!

Bethe Ansatz equations

Integrability: eigenstates of the 1D spin-$\frac{1}{2}$ chain Hamiltonian labeled by solutions of algebraic system [Baxter'1971]

$$\left(\frac{1+q^{+r}\,\zeta_m^r}{1+q^{-r}\,\zeta_m^r}\right)^{N/r} = -\mathrm{e}^{2\mathrm{i}\pi\mathrm{k}}\,q^{2S^z}\prod_{j=1}^{\frac{N}{2}-S^z}\frac{\zeta_j-q^{+2}\,\zeta_m}{\zeta_j-q^{-2}\,\zeta_m}$$

with

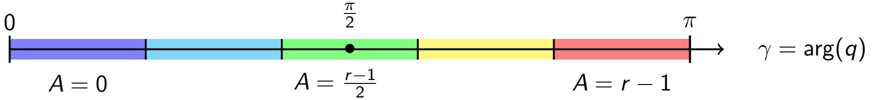
$$\mathcal{E}=\sum_{m=1}^{\frac{N}{2}-S^z}\frac{2\mathrm{i}r\left(q^r-q^{-r}\right)}{\zeta_m^r+\zeta_m^{-r}+q^r+q^{-r}}$$

$$\mathrm{Quasi-periodic\ BC}\ :\quad \sigma^{\pm}_{N+m}=\mathrm{e}^{\pm 2\pi\mathrm{i}\mathrm{k}}\,\sigma^{\pm}_m\ ,\qquad \sigma^z_{N+m}=\sigma^z_m$$

Spectrum at $N\gg 1$ can be studied by finding solutions of BA equations corresponding to low energy states

Free Fermions point

Let $r = 1, 3, 5, \ldots$



FF point

$$q = \mathrm{i} \qquad \left(\gamma = \tfrac{\pi}{2}\right)$$

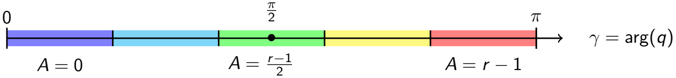
$$\mathrm{BAE} \ : \quad \left(\frac{1 + \mathrm{i}^{+r}\, \zeta_m^r}{1 + \mathrm{i}^{-r}\, \zeta_m^r}\right)^{N/r} = -\mathrm{e}^{2\mathrm{i}\pi \mathtt{k}}\, (-1)^{S^z}$$

The classification of the Bethe states remains unchanged in the green domain

$$\tfrac{\pi}{r}\, A < \gamma < \tfrac{\pi}{r}\, (A+1) \quad \text{with} \quad A = \tfrac{r-1}{2}$$

(similar to the case $r = 1$)

CFT near FF point



- For $\gamma = \frac{\pi}{2}$, the chiral CFT space is built from r copies of chiral Dirac fermions:

$$\psi_a \quad : \quad \psi_a(z)\psi_b^\dagger(0) = \frac{\delta_{ab}}{z} + O(1)$$

- Equivalently, this theory can be described by a level-1 Kac–Moody current algebra $\widehat{\mathfrak{su}_1(r)}$

$$J^{(m)} = \sum_{a,b} \psi_a^\dagger \mathtt{t}_{ab}^{(m)} \psi_b$$

together with a compact boson field

$$\partial\phi = \frac{1}{\sqrt{r}} \sum_a \psi_a^\dagger \psi_a \ , \quad \phi \sim \phi + 2\pi \mathtt{R} \quad \text{with} \quad \mathtt{R} = 1$$

- For $\gamma = \frac{\pi}{2}(1 - \delta)$ with $|\delta| < \frac{1}{r}$

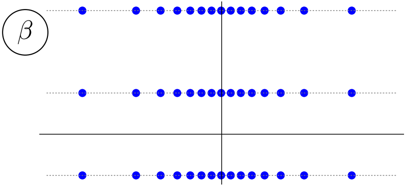
$$\mathtt{R} = \frac{1}{\sqrt{1 - r\delta}} \qquad \left(|\delta| < \frac{1}{r}\right)$$

The $\mathcal{Z}_r$ invariant case: Bethe roots

The ground state Bethe roots arrange a simple pattern

$$\arg(\zeta) = \frac{\pi}{r}\,\big(2\,(a-1) - A\big) \qquad \Big(A = \tfrac{r-1}{2}\ ,\ a = 1, \ldots r\Big)$$

$\zeta = \mathrm{e}^{2\beta}$



$\Im m(\beta) = \frac{\pi}{2}$

$\Im m(\beta) = +\frac{\pi}{6}$

$\Im m(\beta) = -\frac{\pi}{6}$

The limits

$$E_m^{(a)} = \lim_{\substack{N\to\infty \\ m-\text{fixed}}} \left(\frac{N}{N_0}\right)^{\frac{1}{r}+\delta} \zeta_m^{(a)}$$

exist and are non-vanishing (N_0 is a conveniently chosen constant).

- At FF point ($q=\mathrm{i}$)

$$-\mathrm{e}^{2\mathrm{i}\pi\mathtt{k}}\,(-1)^{S^z}=\left(\frac{1+\mathrm{i}^{+r}\ \zeta_m^r}{1+\mathrm{i}^{-r}\,\zeta_m^r}\right)^{N/r}\to \mathrm{e}^{\mathrm{i}\pi\,(-1)^A E_m^r}\qquad \left(A=\tfrac{r-1}{2}\right)$$

$$E_m^{(a)}=\lim_{\substack{N\to\infty\\ m-\mathrm{fixed}}}\left(\tfrac{2N}{\pi}\right)^{\frac{1}{r}}\ \zeta_m^{(a)}$$

- For the vacuum state

$$E_m^{(a)}=\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2a-A)}\ \big(2m-1+2\mathtt{k}\big)^{\frac{1}{r}}\qquad (a=1,\ldots,r;\ \ m=1,2,\ldots)$$

- The scaling limits $E_m^{(a)}$ are expressed in terms of the spectrum of the harmonic oscillator:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\ell(\ell+1)}{x^2}+x^2+E^r\right)\Psi=0$$

$$\ell+\tfrac{1}{2}=2\mathtt{k}$$

The vacuum ODE away from FF point

- For $q = \mathrm{i}\ \mathrm{e}^{\frac{\mathrm{i}\pi}{2}\delta}$: $E_m^{(a)}$ are expressed in terms of the spectrum of the anharmonic oscillator [Dorey, Tateo'98; Bazhanov, Zamolodchikov, SL'98]

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\ell(\ell+1)}{x^2} + x^{2\alpha} + E^r \right) \Psi = 0$$

$$\alpha = \frac{1 + r\delta}{1 - r\delta}\ , \qquad \ell + \frac{1}{2} = \frac{2}{1 - r\delta}\ \mathtt{k}$$

- By a change of variables, $x = \mathrm{e}^{(1-\delta r)v}$, $\psi = x^{-\frac{1}{2}}\Psi$, the ODE is brought to

$$\left[-\partial_v^2 + \mathtt{k}^2 + \mathrm{e}^{2v} + \epsilon^r\ \mathrm{e}^v \right] \psi = 0$$

where

$$\epsilon \equiv \epsilon(v) = E\ \mathrm{e}^{-v\delta}$$

The excited states ODE for $q = \mathrm{i}\, \mathrm{e}^{\frac{\mathrm{i}\pi}{2}\delta}$; $r = 3, 5, \ldots$

[Kotousov, Shebetnik, SL'25]

$$\big(-\partial_v^2 + T(v) \big)\, \psi = 0$$

$$T(v) = \mathtt{H}^2 + \mathrm{e}^{2v} + \epsilon^r\ \mathrm{e}^v + \sum_{i=1}^{L} \left(\frac{3\delta^2\, \varpi_i^2 \epsilon^2}{(\epsilon^2 - \varpi_i^2)^2} + \frac{2\delta}{\epsilon^2 - \varpi_i^2}\ \big(c_i \epsilon^2 + \varpi_i \epsilon\, \mathrm{e}^v \big) \right)$$

where

$$\epsilon = E\ \mathrm{e}^{-v\delta}\ , \qquad \mathtt{H} = \mathtt{k} + \tfrac{1-\delta r}{2r}\, S^z \qquad \big(|\delta| < \tfrac{1}{r} \big)$$

The set $\{\varpi_j, c_j\}_{j=1}^{L}$ is defined by a system of algebraic equations

$$2c_j = \varpi_j^r - 1 + 2\delta \sum_{i \neq j} \frac{\varpi_i \varpi_j}{\varpi_j^2 - \varpi_i^2} \qquad (i, j = 1, \ldots, L)$$

$$c_j^2 = \mathtt{H}^2 + \delta\, c_j - \frac{\delta^2}{4} + \sum_{i \neq j} \left(\frac{3\delta^2\, \varpi_i^2 \varpi_j^2}{(\varpi_i^2 - \varpi_j^2)^2} + \frac{2\delta\, c_i \varpi_j^2}{\varpi_j^2 - \varpi_i^2} \right)$$

For $r = 1$, the above ODEs provide a different but equivalent description of the quantum KdV integrable structure compared to the 'monster potential' ODEs from [Bazhanov, Zamolodchikov, SL'2003].

Summary

- A brief overview of the Baxter inhomogeneous six-vertex model and its relevance to understanding critical phenomena.
- Critical regimes with $0 < \gamma < \pi$ and $\eta_m \approx (-1)^r \, e^{\frac{i\pi}{r}(2m-1)}$ (including ODE/IQFT correspondence):
  - ✓ $0 < \gamma < \frac{\pi}{r}$
    - $r = 2$ related to black hole sigma models
    - r odd infinite degeneracy of states in scaling limit
    - r even continuous component also appears
  - ✓ $\pi(1 - \frac{1}{r}) < \gamma < \pi$
    - r compact bosons (deformation of $\big[\widehat{\mathfrak{su}_1(2)}\big]^{\otimes r}$)
  - ✓ $\pi \, \frac{r-1}{2r} < \gamma < \pi \, \frac{r+1}{2r}$ $(r = 3, 5, \ldots)$
    - r compact bosons (deformation of $\widehat{\mathfrak{su}_1(r)} \otimes \widehat{\mathfrak{u}(1)}$)
- The critical properties of the Baxter model are not yet fully understood.