Inverse problems for power sums

September 4, 2023

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4. Differential equations
5. Zeros of L-functions and estimates for primes

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• This talk will mainly be focused on applications of Turán’s bounds to *L*-functions.

Fundamental estimates for power sums

• Turán's first main theorem states: For any complex numbers $b_1, \ldots, b_n, z_1, \ldots, z_n$ with each $|z_i| \geq 1$ and any integer m we have

$$\max_{m+1 \leq \ell \leq m+n} \left| \sum_{i=1}^{n} b_i z_i^{\ell} \right| \geq \left(\frac{n}{2e(m+n)} \right)^n \left| \sum_{i=1}^{n} b_i \right| .$$

• Turán's second main theorem states: For any complex numbers $b_1, \ldots, b_n, z_1, \ldots, z_n$ with $\max_i |z_i| = 1$ and any integer m we have

$$\max_{m+1 \le \ell \le m+n} \left| \sum_{i=1}^{n} b_i z_i^{\ell} \right| \ge \left(\frac{n}{8e(m+n)} \right)^n \min_k \left| \sum_{i=1}^{k} b_i \right| .$$

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$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

- $(s-1)\zeta(s)$ is an entire function of order 1 and hence can be expressed as a Hadamard product over zeros:

$$(s-1)\zeta(s) = \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}.$$

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• This conjecture (probably) won't be established any time soon. Power sums allow us to say interesting things about the primes from partial knowledge about the zeros.

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• If k is chosen large enough, the sum on the LHS of the above is dominated by ρ close to s_0:

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By Jenson's formula, there's not too many ρ close to s_0, hence we're in a position to apply Turán's fundamental theorems.

Power sums and the zeros of ζ

• The Guinand-Weil explicit formula states that for any sufficiently nice function f:

$$\sum_{\rho} \widehat{f}(i(\rho - 1/2)) \approx \sum_{p} \frac{\log p}{p^{1/2}} f(\log p).$$

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• Let $f^{(k)}$ denote the k-fold convolution of f and apply the above with $f \to f^{(k)}$ to get

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Again this puts us in a situation where Turán's inequalities may be applied.

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2. Connect the point count above to a sum over powers of zeros of the corresponding L-function.
3. Combine (1),(2) above (as $n \to \infty$) with a lower bound inequality for power sums (possibly along a subsequence of $n \to \infty$.)

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3. Quantitative versions of the Sato-Tate conjecture (Lemke Oliver, Thorner IMRN 2017)
4. Work of Maynard and Pratt (arxiv:2206.11729) who show if the Riemann hypothesis fails in a specific way then we may obtain new estimates for the number of primes in short intervals.

New problem:

• Suppose we have a sequence of complex numbers $b_1, \ldots, b_n, z_1, \ldots, z_n$ whose power sums

$$\max_{m+1 \le \ell \le m+n} \left| \sum_{i=1}^{n} b_i z_i^{\ell} \right| ,$$

are close to the lower bounds given by Turán's first and second main theorems.

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2. If so, can we use these results to say something along the lines of: If the Riemann hypothesis fails, then it has to fail in some specific kind of way?????

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1. Can we say anything about the structure of the $z_i's$ (b_i's)???????
2. If so, can we use these results to say something along the lines of: If the Riemann hypothesis fails, then it has to fail in some specific kind of way?????
3. Or, if some consequence of the Riemann hypthoesis is false then the zeros of the zeta function off the critical line have to have a certain kind of structure???

An example:

One of the easiest results about power sums, due to Turán is as follows:

Theorem

Let $z_1, \ldots, z_n$ be complex numbers on the unit circle. Then

$$\max_{1 \le \ell \le n} \left| \sum_{j=1}^{n} z_j^{\ell} \right| \ge 1.$$

There is equality if and only if the z_j's form verticies of a regular $(n+1)$-gon.

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Theorem

With notation and conditions as above, for each

$$\frac{2CL}{n} \le U \le L$$

there exists $u \le U$ and at least $n/4$ values of j and integers a_j such that

$$\left| \theta_j - \frac{a_j}{u} \right| \le \frac{1}{u}\left(\frac{U}{L}\right).$$

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tells us that the frequencies θ_j are good substitutes for the set of additive characters when using the circle method to detect solutions to equations.

- Choosing our equations suitably and using duality, we can then extract information about the θ's.

Proof:

Let U, V be parameters satisfying

$$UV \sim L$$

and consider

$$S = \sum_{u=1}^{U} \sum_{j=1}^{n} \left| \sum_{v=1}^{V} e^{2\pi i u v \theta_j} \right|^2 .$$

Expanding the square and interchanging summation:

$$S = \sum_{v_1, v_2 = 1}^{V} \sum_{u=1}^{U} \left(\sum_{j=1}^{n} e^{2\pi i u (v_1 - v_2) \theta_j} \right) .$$

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If $v_1 = v_2$ then the inner sum over j is size n. For all other values of $v_1 \neq v_2$, by assumption:

$$\left| \sum_{j=1}^{n} e^{2\pi i u (v_1 - v_2) \theta_j} \right| \leq C.$$

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$$S = \sum_{u=1}^{U} \sum_{j=1}^{n} \left| \sum_{v=1}^{V} e^{2\pi i u v \theta_j} \right|^2.$$

and use the (heuristic) approximation

$$\sum_{m \leq M} e^{2\pi i \theta m} \approx \begin{cases} M & \text{if} \quad \|\theta\| \leq 1/M \\ 0 & \text{otherwise} \end{cases}$$

We get

$$\frac{nU}{2} \le \sum_{1 \le u \le U} \#\{1 \le j \le n \, : \, \|\theta_j u\| \le 1/V\}.$$

By the pigeonhole principle, there exists some $u \le U$ such that for a set $\mathcal{J} \subseteq \{1, \ldots, n\}$ of size $\ge n/4$ we have

$$\left| \theta_j - \frac{a_j}{u} \right| \le \frac{U}{uL},$$

for each $j \in \mathcal{J}$. This completes the proof.

Thank you for your attention.