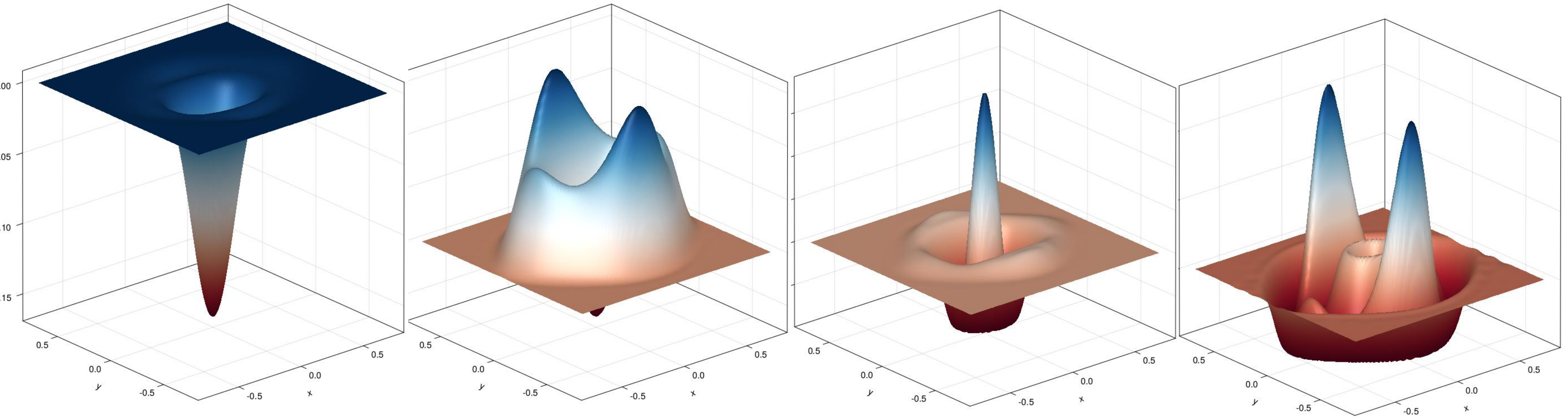
Level Set Learning for Poincaré Plots of Symplectic Maps

Max Ruth, David Bindel

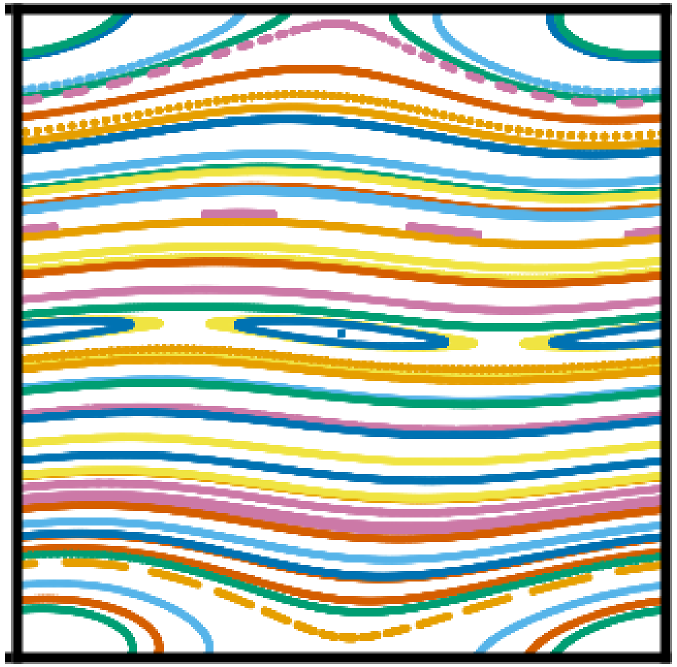


Goal

How integrable is a magnetic field?

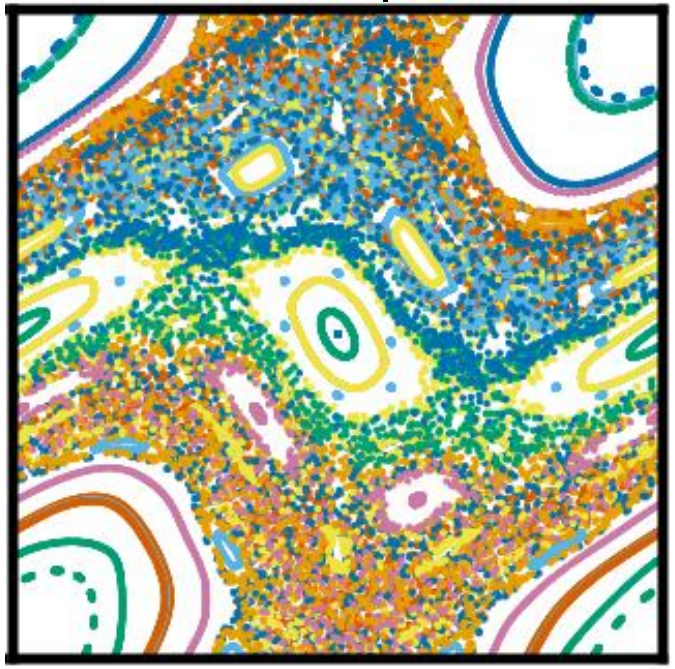
"Integrable"

Standard map $k = 0.2$



"Non-integrable"

Standard map $k = 1.2$



Previous work

- **Periodic orbits:** Greene's Residue
- **Invariant Circles:**
  - Quadratic Flux Minimization
  - Boozer Least Squares
  - Parameterization Method
- **Volume:** Anisotropic Diffusion
- **Trajectories:**
  - Weighted Birkhoff Average
  - Volume of chaos
  - Birkhoff Reduced Rank Extrapolation

- Greene's residue: J. M. Greene, Journal of Mathematical Physics, 9 (1968), pp. 760–768.
- QFMin: R.L. Dewar and J.D. Meiss, Physica D 57 (1992) 476
- BoozerLS: A. Giuliani, F. Wechsung, A. Cerfon, M. Landreman, and G. Stadler, PoP, 30 (2023), p. 042511
- Parameterization: A. Haro and R. de la Llave, Disc. Cont. Dyn. Sys., B6(6):1261–1300, 2006
- Heat Eq: E. J. Paul, S. R. Hudson, and P. Helander, JPP, 88 (2022), p. 905880107
- WBA: S. Das, Y. Saiki, E. Sander, and J.A. Yorke. Quantitative quasiperiodicity. Nonlinearity, 30(11):4111, 2017
- Volume of chaos: J.D. Meiss, Reviews of Modern Physics 64 (3) (1992), p. 795–848
- Birkhoff RRE: Me, see Simon's hour slides

Goal

How integrable is a magnetic field?

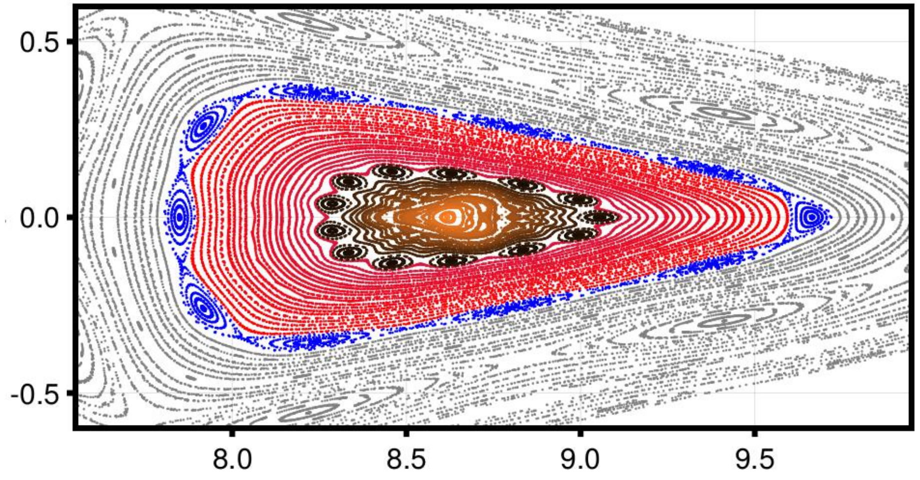
Goal

~~***How integrable is a magnetic field?***~~

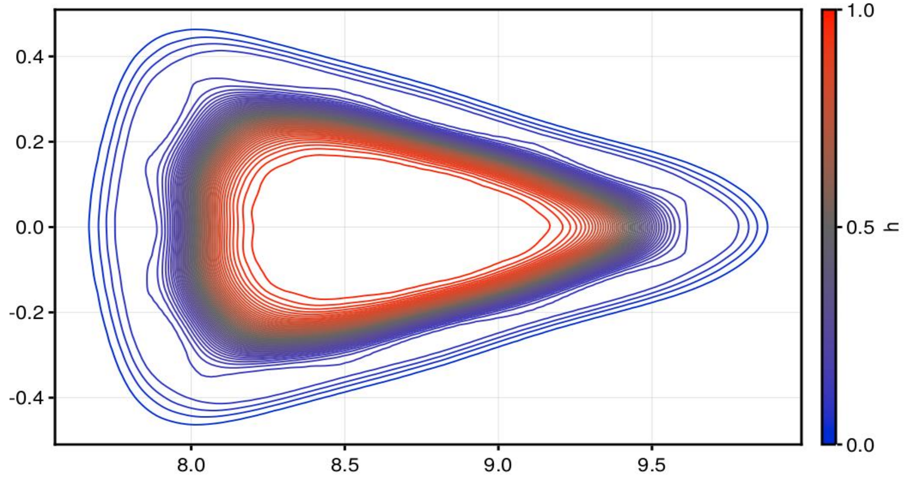
How can we *quickly* and *robustly* measure the integrability of a *volume* of magnetic field?

- Route: Attempt to find a flux label for the volume

Poincare Plot



Label Function



Outline

1. Background / Motivation
2. **Approximately Invariant Functions (as defined by me)**
3. Method
4. Results
5. Conclusion

Setup

- We consider a *symplectic map* $F : X \to X$
  - E.g. from solving $\dot{x} = B(x)$ and intersecting with a poloidal cross section X
  - Or, the standard map with $X = \mathbb{T} \times \mathbb{R}$

- On the state space, we define *observables* to be of the form $h : X \to \mathbb{R}$

- A function h is *invariant* if it obeys the relationship

$$h(x) = h\big(F(x)\big)$$

  - E.g. the Hamiltonian is always invariant

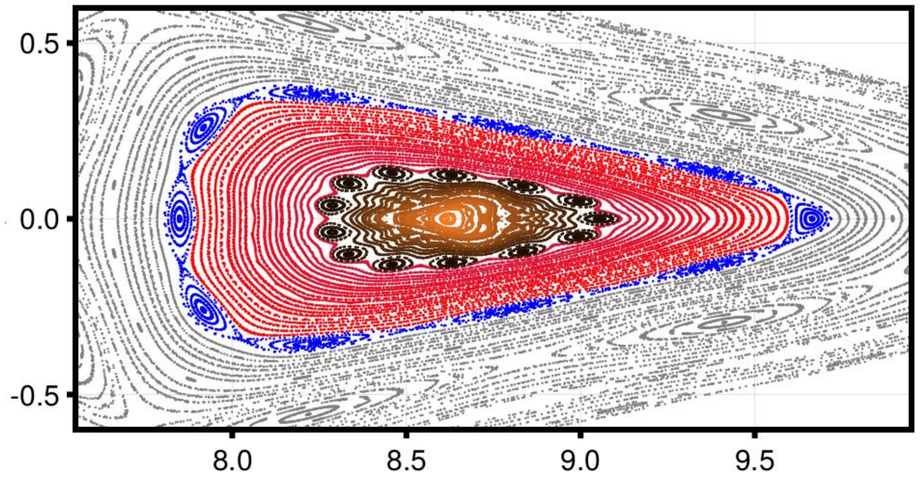
Approximate Invariance

- Let $\Omega \subset X$. We consider h *approximately invariant* on Ω if

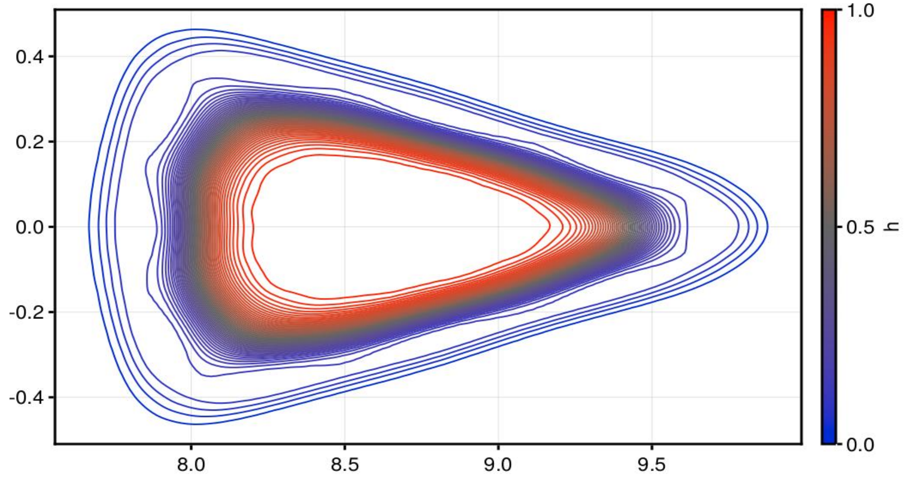
$$E_{Inv} = \|h - h \circ F\|_{L^2(\Omega)}^2 \ll \|\nabla h\|_{L^2(\Omega)}^2$$

- The norm on the right ignores the constant function
- **Our goal is to find an approximately invariant function**

Poincare Plot



Label Function



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Kernel Approximation

- A kernel is any function $K : X \times X \to \mathbb{R}$
- E.g.

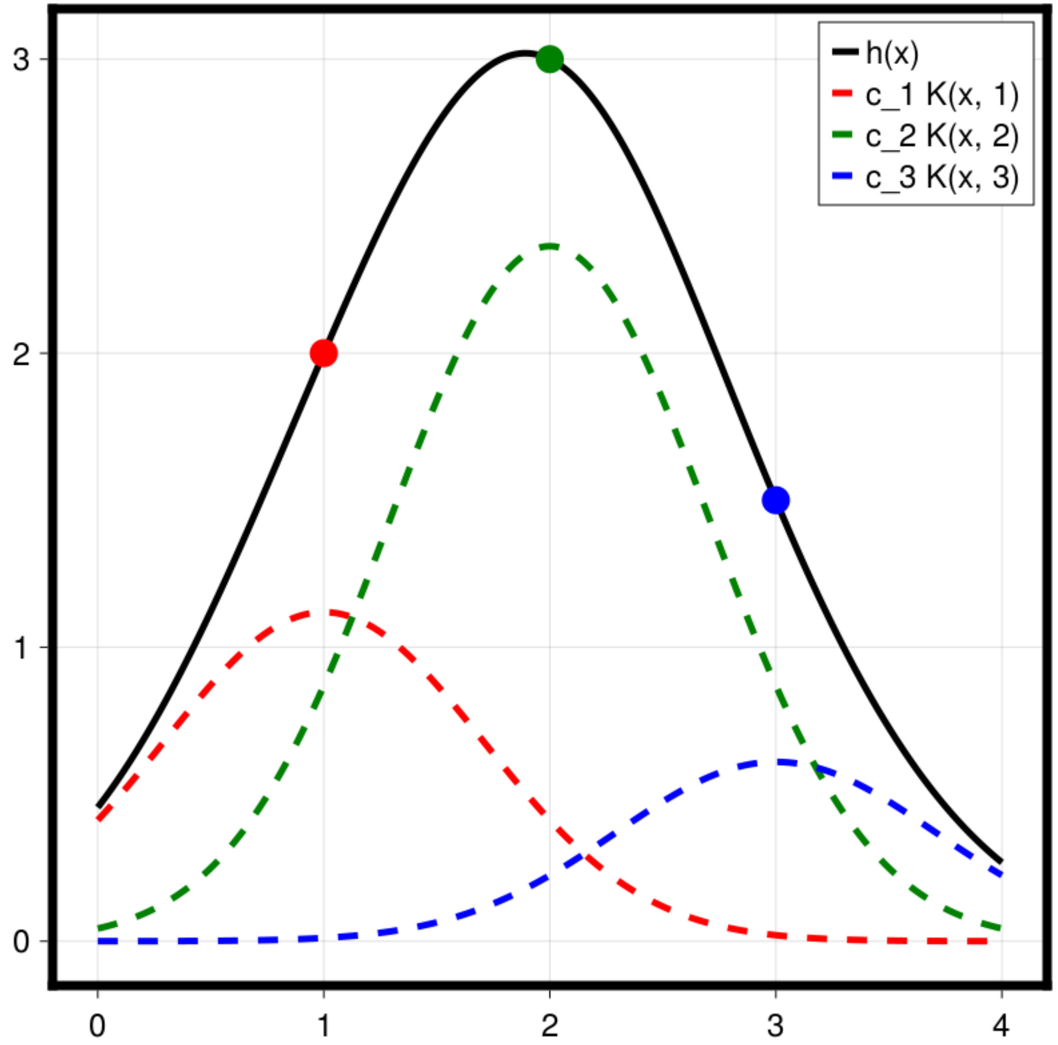
$$K(r, r') = e^{-\frac{\|r - r'\|^2}{2\sigma^2}} \text{ on } \mathbb{R}^2$$

- We represent h by kernels as

$$h(x) = \sum_j c_j K(x, x_j)$$

- where the x_j are our knots

- Kernels are good for arbitrary domains
- Positive definite kernels define Reproducing Kernel Hilbert Spaces
  - Useful for e.g. interpolation problems



Method

1. **Discretization**
2. Regularization
3. Boundary conditions
4. The problem

$$h(z_m) = \sum_{n=1}^{2N} c_n K(z_m, z_n)$$

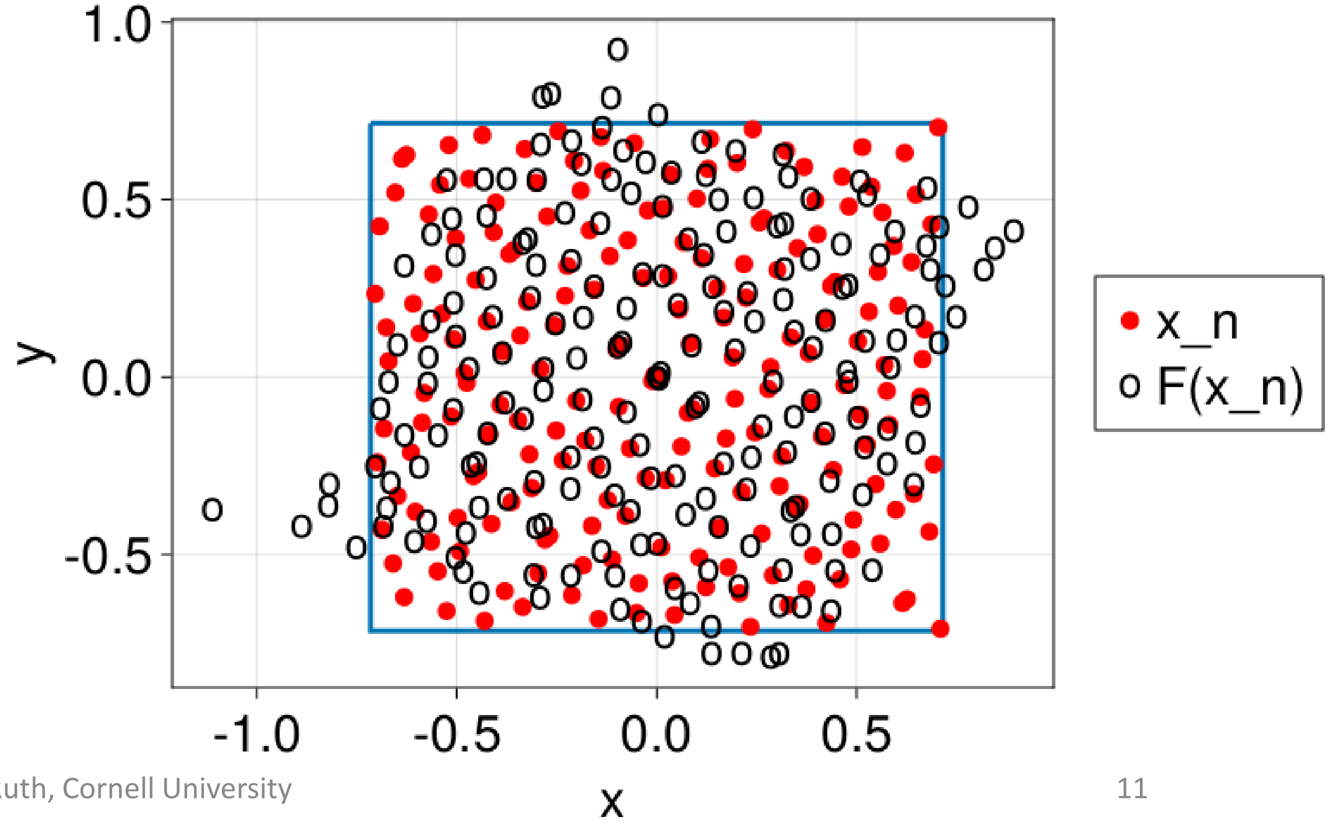
or

$$\boldsymbol{h} = K\boldsymbol{c}$$

$$G_{Inv} = \begin{pmatrix} -1 & 1 & & & & \\ & & -1 & 1 & & \\ & & & & \ddots & \ddots \end{pmatrix}$$

Steps:

1. Sample N points x_n in domain Ω
2. Evaluate $y_n = F(x_n)$ for each point
3. Concatenate $\{z_n\} = \{x_1, y_1, x_2, y_2, \dots\}$



Invariance

- Approximate invariance becomes

$$E_{Inv} = \sum_{n=1}^{N} |h(z_{2n}) - h(z_{2n-1})|^2$$
$$= \boldsymbol{h}^T G_{Inv}^T G_{Inv} \boldsymbol{h}$$

Method

1. Discretization
2. **Regularization**
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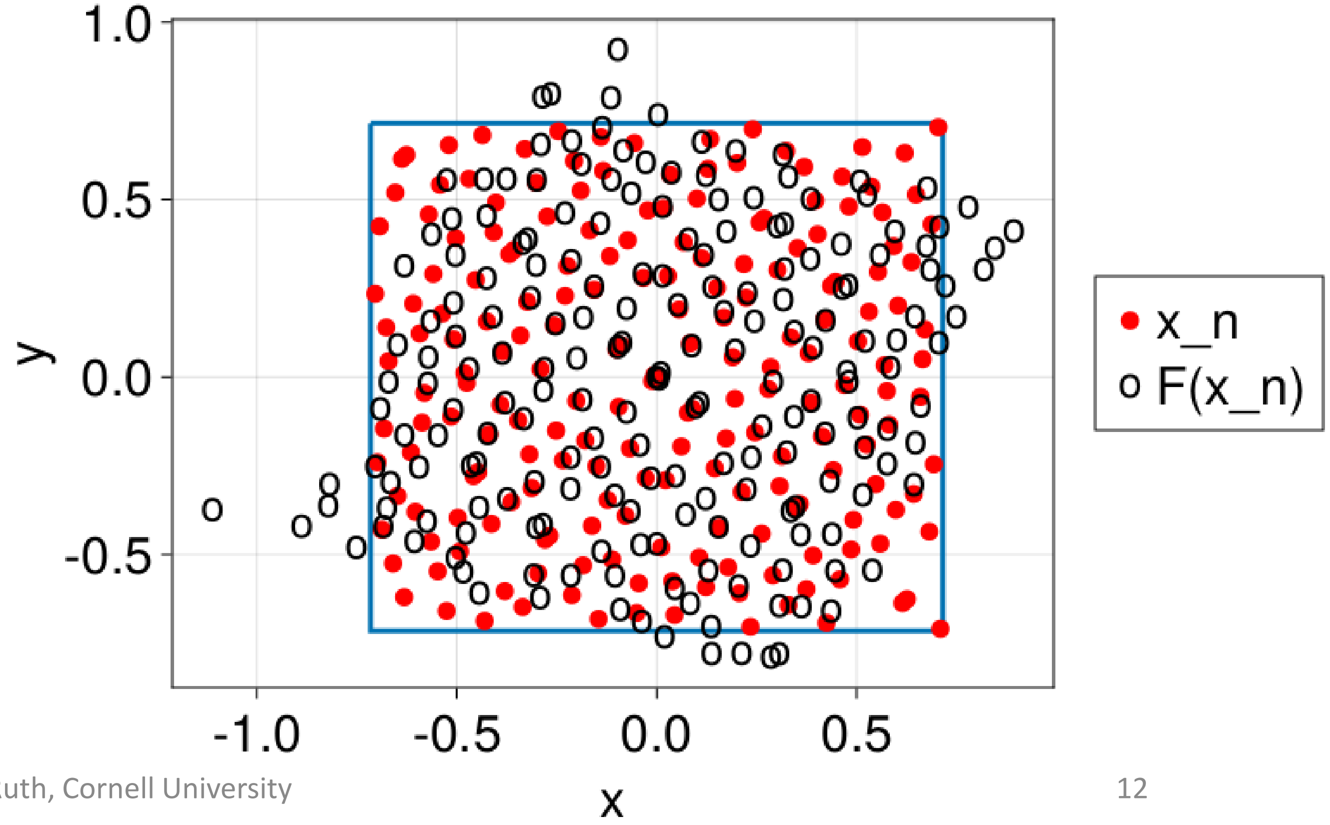
Invariant: $E_{Inv} = \frac{1}{N}\sum_{n=1}^{N} |h(z_{2n}) - h(z_{2n-1})|^2 = \frac{1}{N}\boldsymbol{h}^T G_{Inv}^T G_{Inv} \boldsymbol{h}$

$$h(z_m) = \sum_{n=1}^{2N} c_n K(z_m, z_n)$$
$$\boldsymbol{h} = K\boldsymbol{c}$$

- Counting equations:
  - h parameterized by $2N$ elements $\boldsymbol{c}$
  - Invariance condition energy rank N
- N invariant functions; regularize for smoothness
- For this, use reproducing kernel Hilbert space norm

$$E_K = \boldsymbol{h}^T K^{-1} \boldsymbol{h} = \boldsymbol{c}^T K \boldsymbol{c}$$

- RKHS norms are *smoothing*



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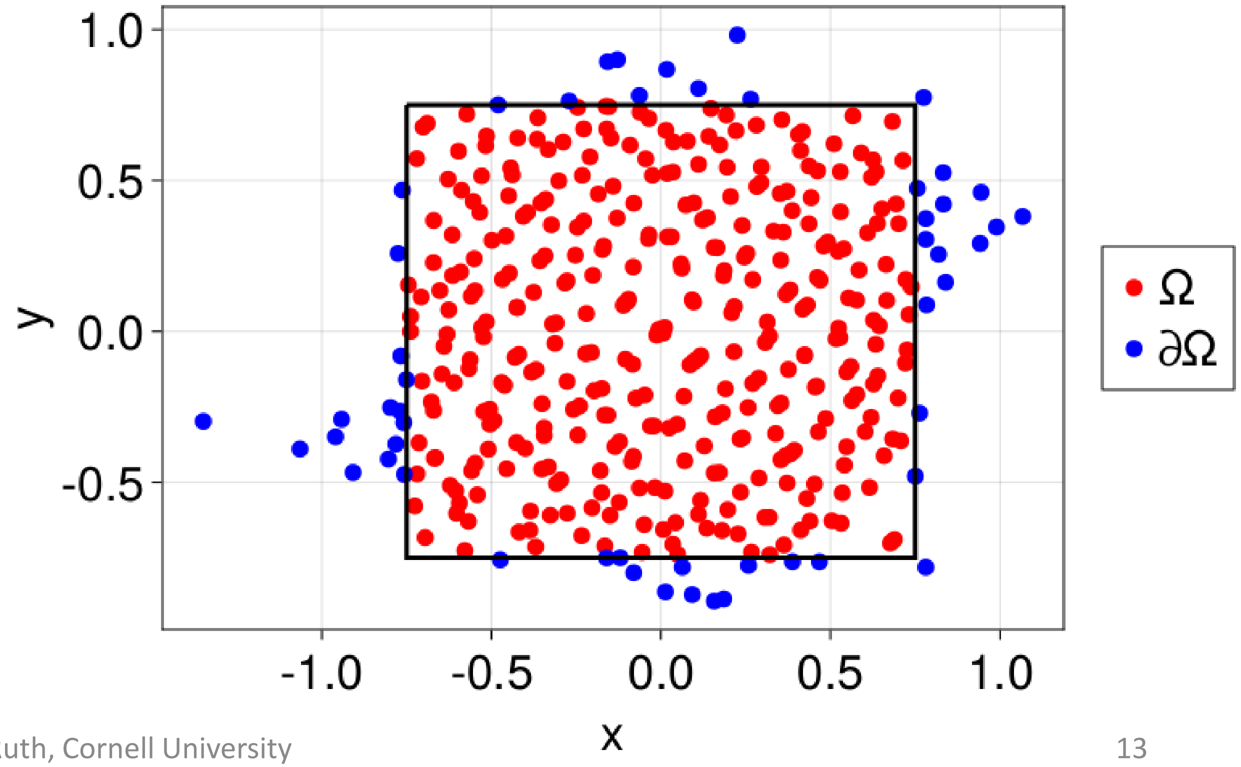
Smooth: $E_K = \boldsymbol{h}^T K^{-1}\boldsymbol{h}$

- Finally, we use boundary conditions to
  1. Remove constant functions
  2. Handle non-invariant Ω
  3. Introduce "knowledge"
- Define boundary value $h_{bd} : X \to \mathbb{R}$ and weight $w_{bd} : X \to [0,1]$

$$E_{bd} = \sum_{n=1}^{2N} w_{bd}(z_n)|h(z_n) - h_{bd}(z_n)|^2$$



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Smooth: $E_K = \boldsymbol{h}^T K^{-1}\boldsymbol{h}$

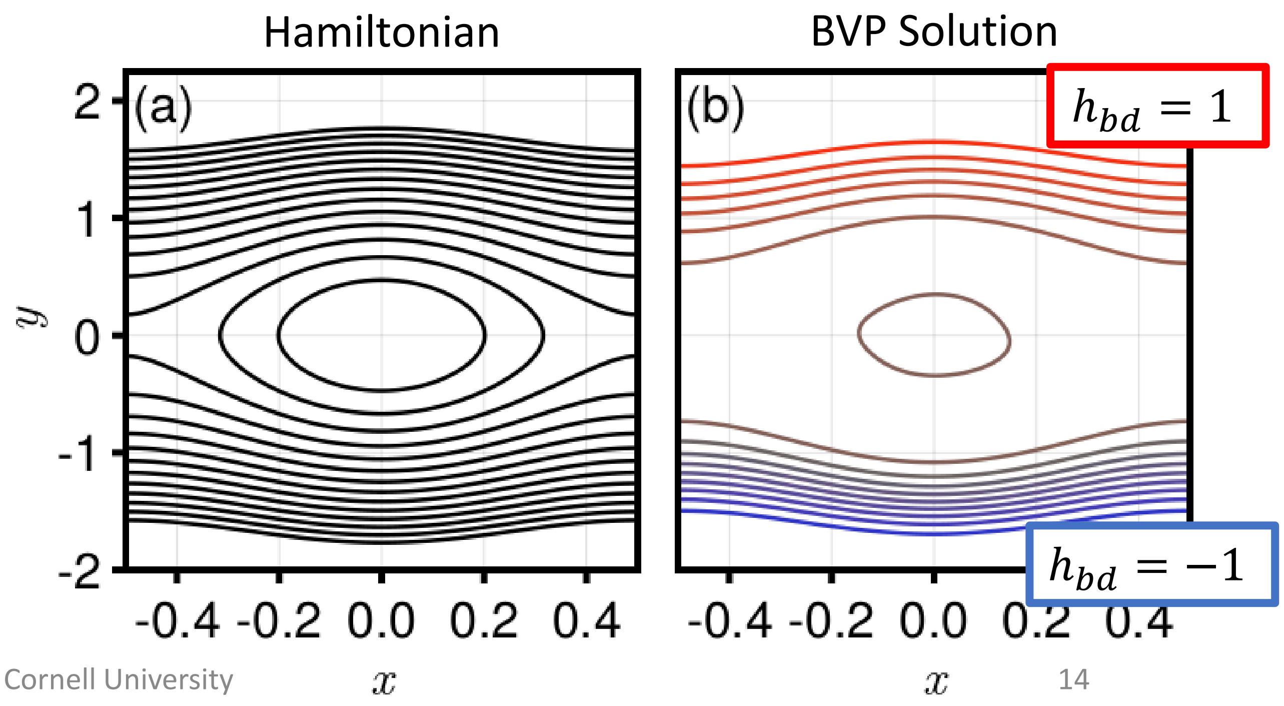
Boundary: $E_{bd} = \sum_{n=1}^{2N} w_{bd}(z_n)|h(z_n) - h_{bd}(z_n)|^2 = (\boldsymbol{h} - \boldsymbol{h}_{bd})^T W_{bd}(\boldsymbol{h} - \boldsymbol{h}_{bd})$

Example: Nonlinear pendulum $\dot{x} = y, \dot{y} = -\sin(2\pi x)$
Trajectories evolved for time $T = \sqrt{2}$
$N = 100$, $\epsilon = 10^{-6}$, Gaussian kernel w/ width $\sigma = 0.5$

Problem 1: Least Squares BVP

- Good for transport from one barrier to another

$$R = \min_{\boldsymbol{c}\in\mathbb{R}^{2N}} E_{Inv} + E_{bd} + \epsilon E_K$$



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- Good for transport from one barrier to another

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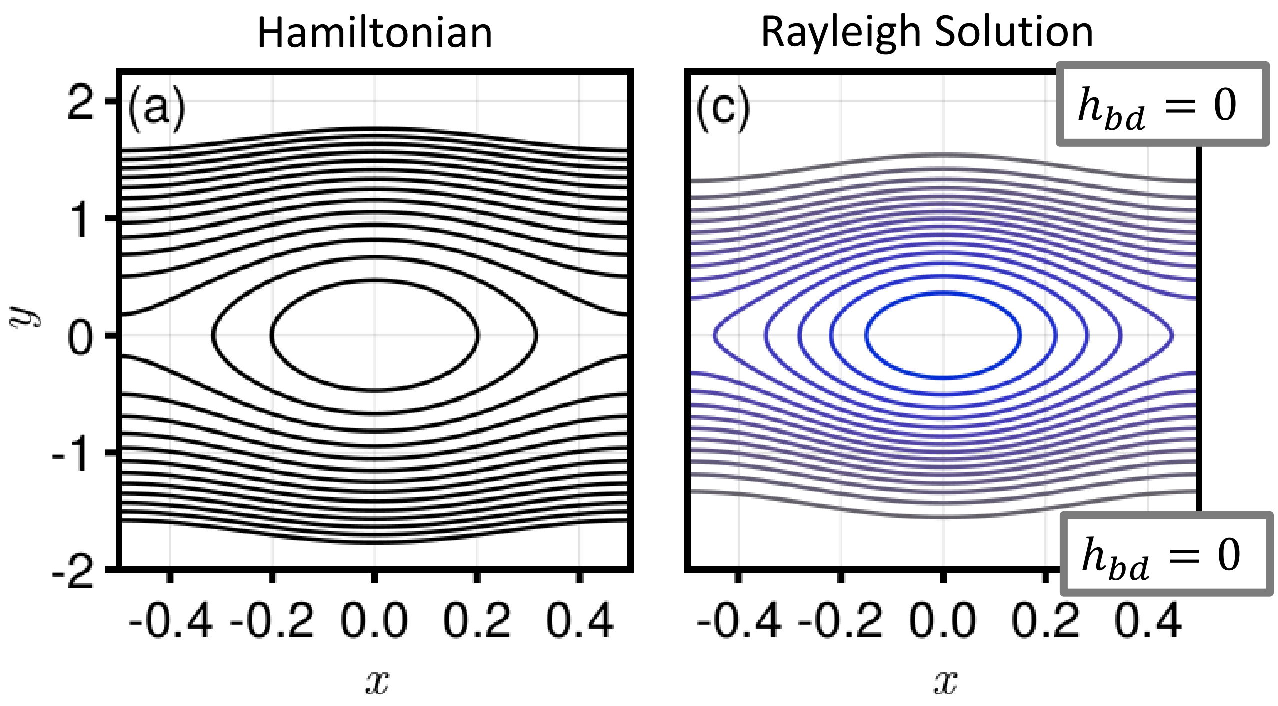
Problem 2: Rayleigh Quotient

- Good when only the stellarator region is known

$$\lambda = \min_{\boldsymbol{c}\in\mathbb{R}^{2N}} \frac{E_{Inv} + E_{bd} + \epsilon E_K}{||\boldsymbol{h}||^2}, \qquad h_{bd} = 0$$



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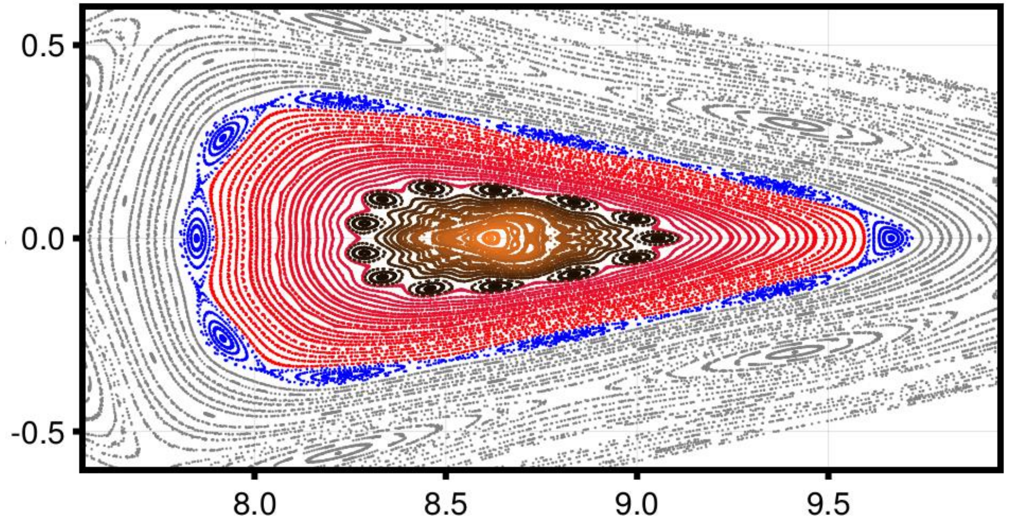
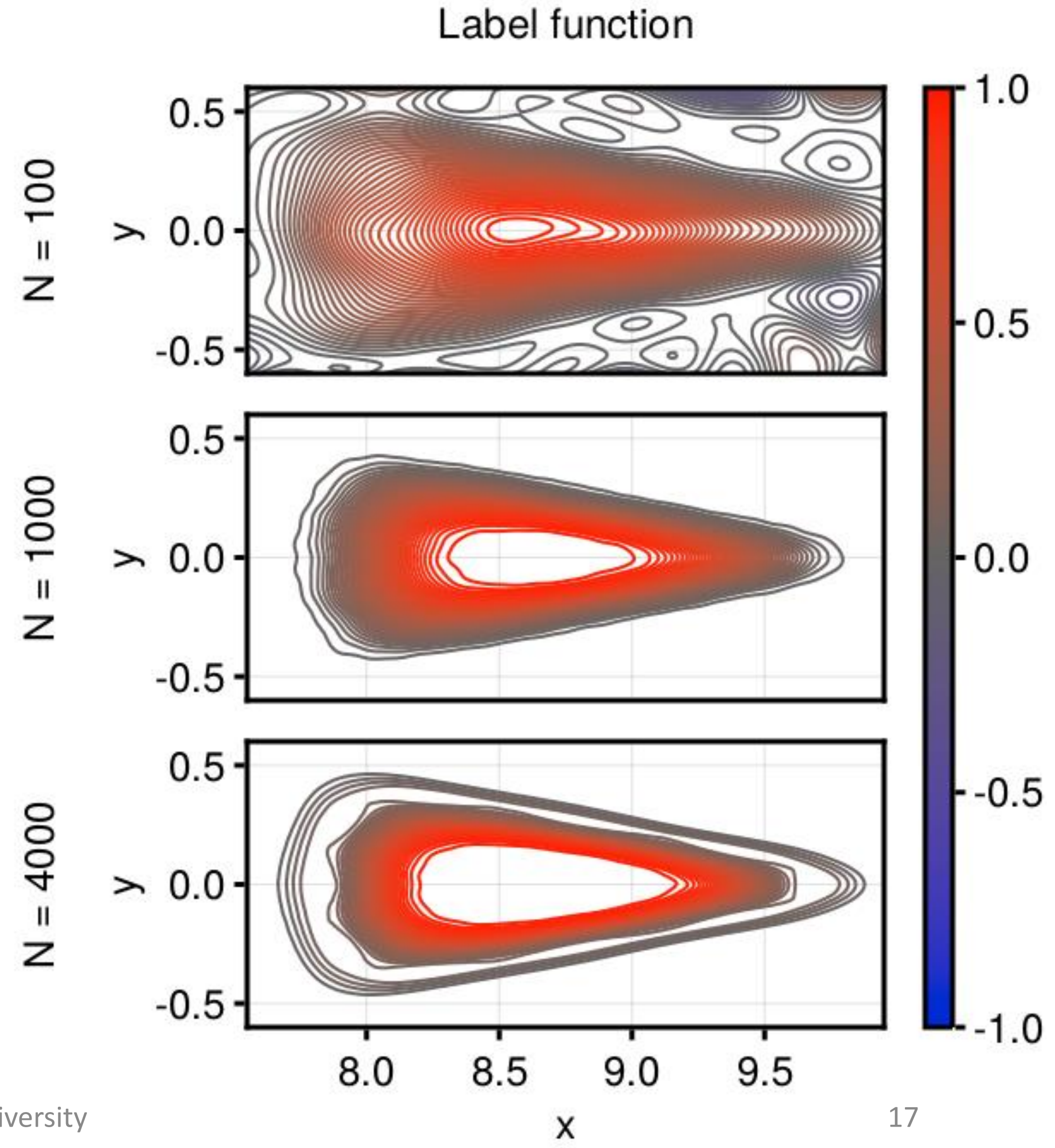
Example: Stellarator

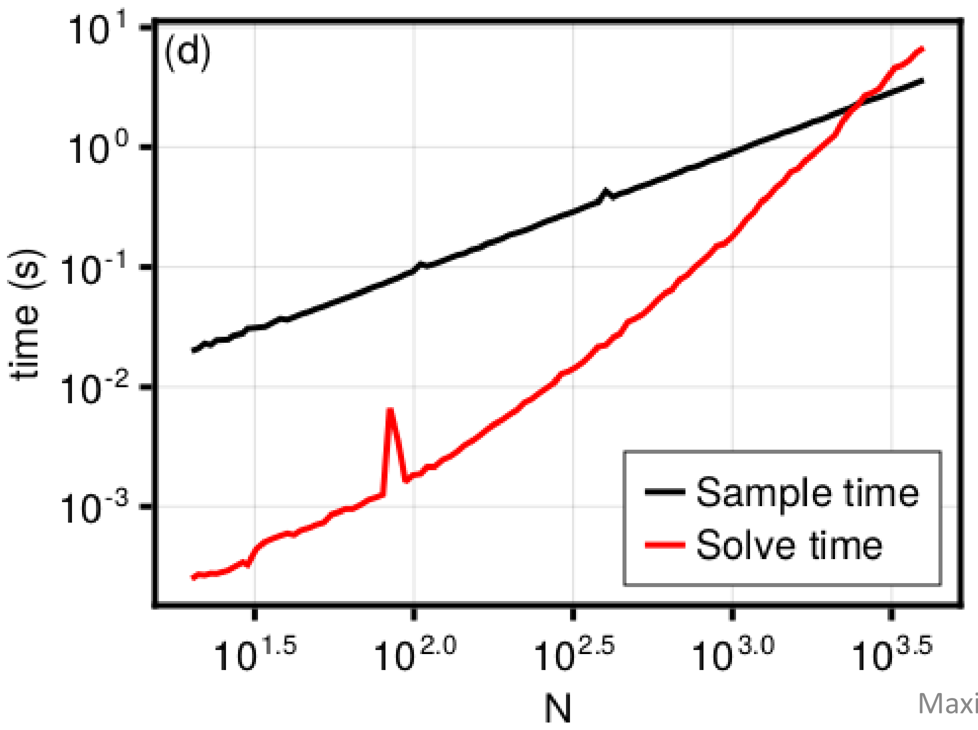
Example:
- Stellarator with islands return map
- Eigenvalue problem
- Vary N from 100 to 4000
- $\epsilon = 10^{-8}$
- Gaussian kernel w/ width $\sigma = 2.43/\sqrt{N}$

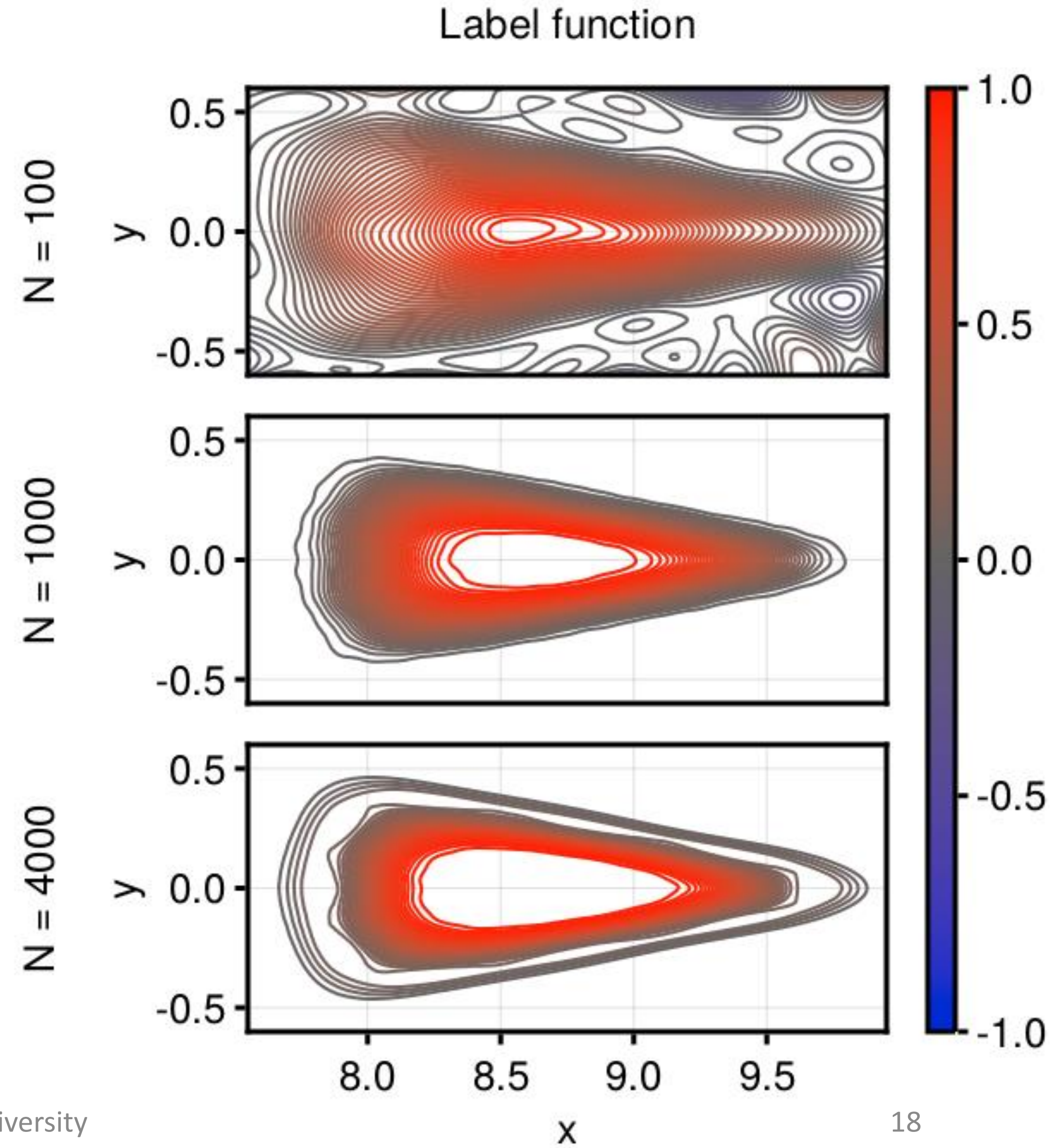




Single-Core Timing

Example:
- Stellarator with islands return map
- Trajectory solve time scales linearly
- Eigenvalue problem scales cubically
- $N = 1000 \sim 1$ sec
- $N = 4000 \sim 15$ sec



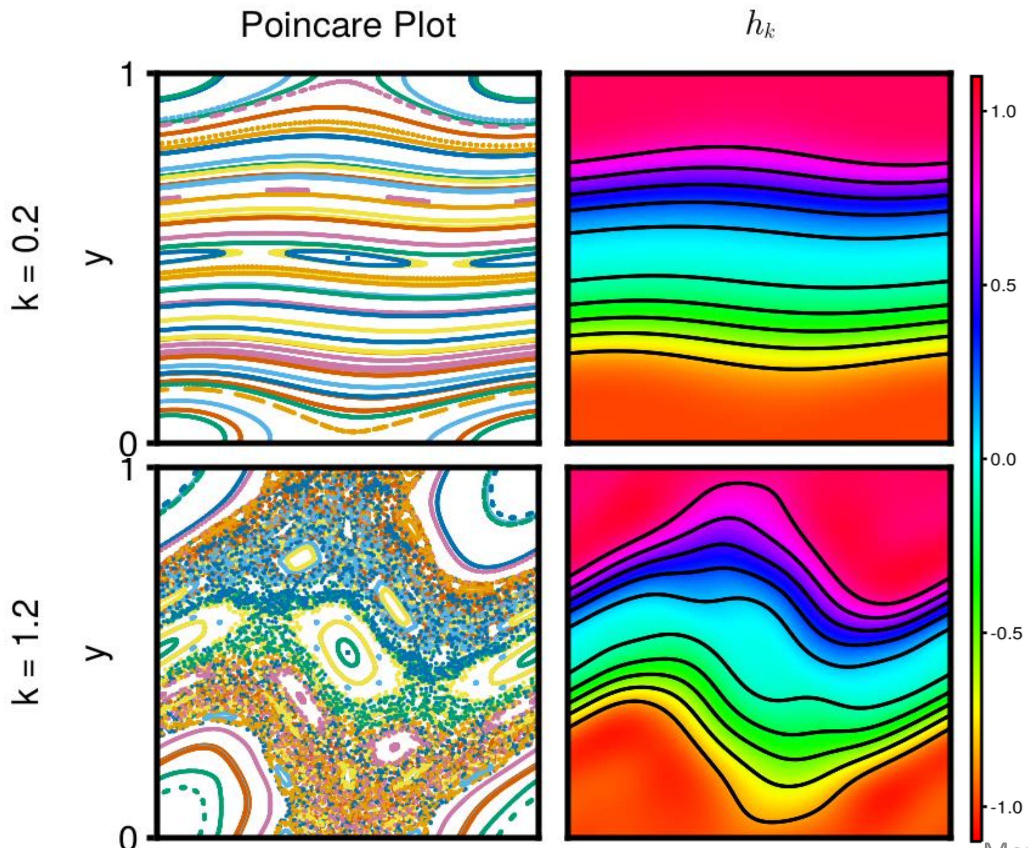


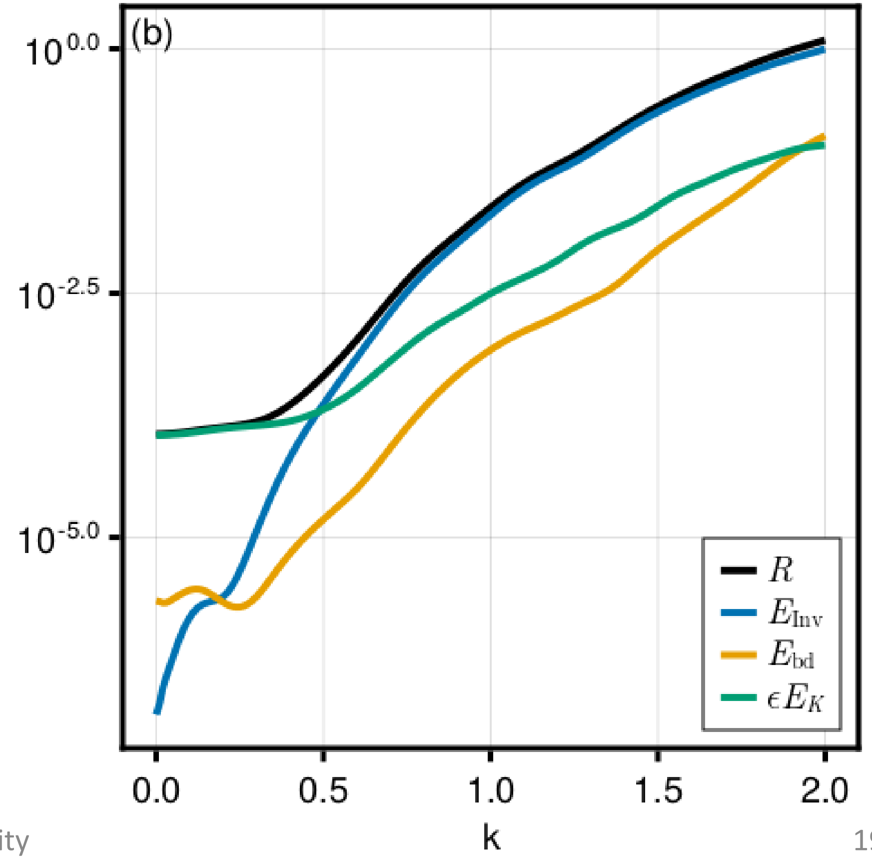
Example: Standard Map

Example:
- Chirikov standard map

$$x' = x + y', \qquad y' = y - \frac{k}{2\pi}\cos(2\pi x)$$

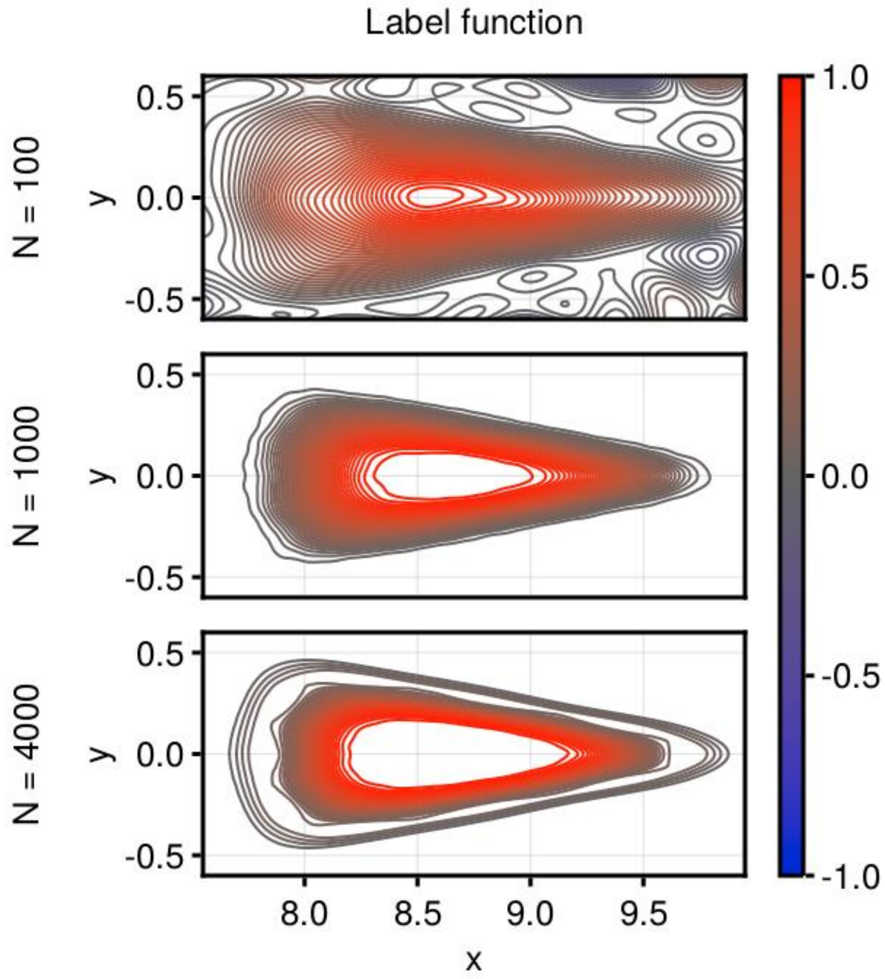
- k controls the size of perturbation
- Boundary value problem, $\epsilon = 10^{-5}$, $N = 1000$
- Squared sine exponential kernel, $\sigma = 0.2$
- Vary k from 0 to 2.0





Conclusion

- We provide two mesh-free methods for finding approximately invariant functions
- Level sets of functions approximately give invariant tori
- Residual gives level of belief in the function (a measure of integrability)
- The measure is cheap to evaluate and smooth in the level of chaos
- Kernel width and regularization allow for tuning



M. Ruth and D. Bindel, arXiv:2312.00967 (2023)

Code at https://github.com/maxeruth/SymplecticMapTools.jl, or **]add SymplecticMapTools** in Julia