Topology Meets Number Theory

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by

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In the short period available to me I plan to discuss 7 topological concepts, 6 number theory concepts, and one measure theory concept.

First the topological concepts:

(1) homeomorphism;

(2) dense set;

(3) G_δ-space;

(4) analytic space;

(5) Hausdorff dimension;

(6) Cantor space, $\mathbb{G}$;

(7) space of irrational numbers $\mathbb{P}$.

Now the Number Theory concepts

(1) transcendental number;

(2) algebraically independence;

(3) transcendence basis;

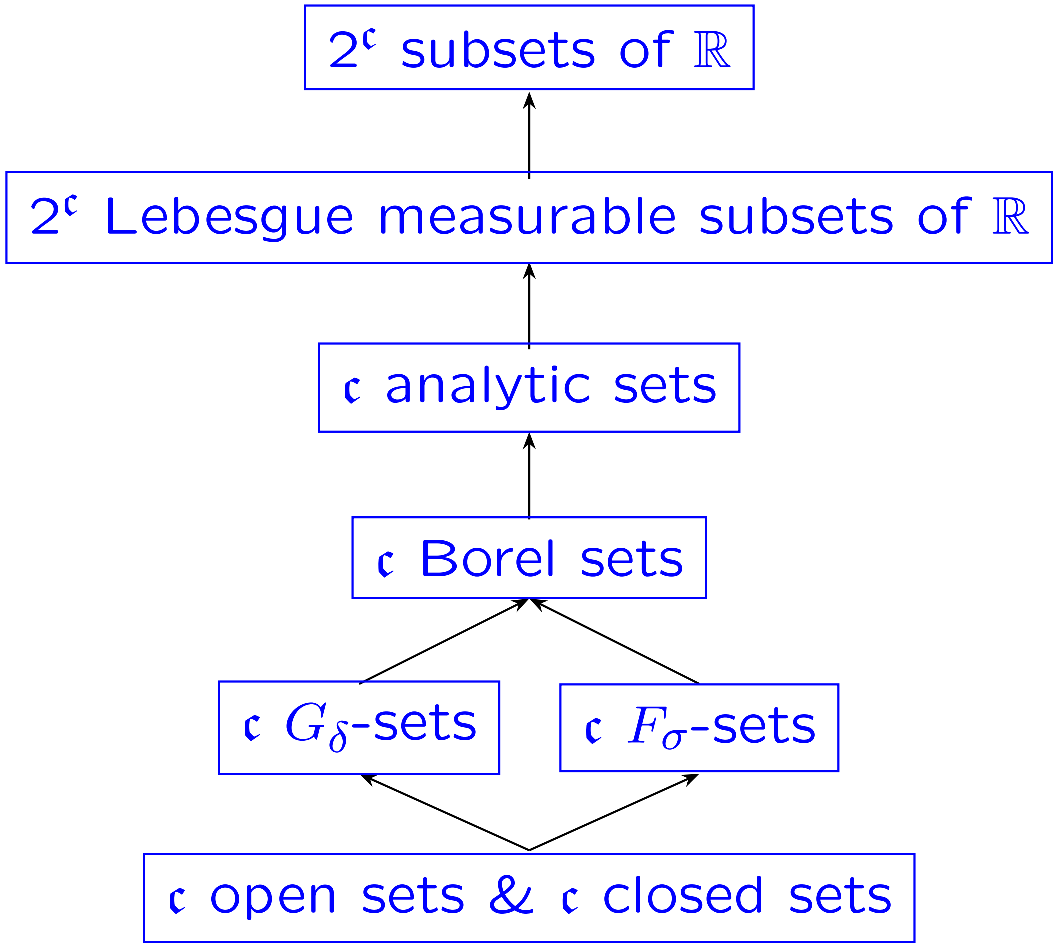
(4) irrationality exponent;

(5) Liouville numbers $\mathcal{L}$;

(6) Mahler sets $\mathbb{A}$, S, T, U.

and the measure theory concept of

Lebesgue measure.


$2^{\mathfrak{c}}$ subsets of $\mathbb{R}$
$2^{\mathfrak{c}}$ Lebesgue measurable subsets of $\mathbb{R}$
$\mathfrak{c}$ analytic sets
$\mathfrak{c}$ Borel sets
$\mathfrak{c}$ G_δ-sets
$\mathfrak{c}$ F_σ-sets
$\mathfrak{c}$ open sets & $\mathfrak{c}$ closed sets

- G_δ-set : countable intersection of open sets
- F_σ-set: countable union of closed sets
- Borel-set: can be constructed from open sets using countable intersections, countable unions, and relative complements ($B \setminus A$ is relative complement of A in B)
- analytic set: continuous image of $\mathbb{P}$ or equivalently continuous of a Borel set

Definition 1. Let X be a subset of $\mathbb{R}$. The set X is said to have **the Erdős property** if for each $r \in \mathbb{R}$ there exist $x_1, x_2 \in X$ such that $r = x_1 + x_2$. The set X is said to have **the multiplicative Erdős property** if for every $s \in \mathbb{R}, s > 0$ there exist $x_3, x_4 \in X$ such that $s = x_3 \cdot x_4$.

In 1962 Paul Erdős proved the surprising result that though the set $\mathcal{L}$ of all Liouville numbers is a small set in that its Lebesgue measure is zero, and its s-dimensional Hausdorff measure, for $s > 0$, is zero, it has the the Erdős and multiplicative Erdős properties.

Theorem 1. If $\boldsymbol{X}$ is any dense $\boldsymbol{G_\delta}$-subset of $\mathbb{R}$ then it has the Erdős property and the multiplicative Erdős property. In particular this is the case for $\boldsymbol{\mathcal{L}}$.

It follows from the Baire Category Theorem that the intersection of any two (or even a countable number of) dense G_δ-subsets of $\mathbb{R}$ is a G_δ-subset of $\mathbb{R}$.

Theorem 2. [CM] Every dense $\boldsymbol{G_\delta}$-subset of $\mathbb{R}$ contains an uncountable subset of $\boldsymbol{\mathcal{L}}$.

A purely topological property of a set $\Longrightarrow$ it contains an uncountable number of transcendental numbers.

Theorem 3. Every dense G_δ-subset of $\mathbb{R}$ is homeomorphic to $\mathbb{P}$. In particular, this is the case for the set $\mathcal{T}$ of all transcendental numbers and $\mathcal{L}$.

Observe that set $\mathbb{P}$ contains the set $\mathcal{L}$ and the cardinality of $\mathbb{P} \setminus \mathcal{L}$ is $\mathfrak{c}$. This immediately gives us:

Theorem 4. [CM] Every dense G_δ-subset X of $\mathbb{R}$ contains a dense G_δ-subset Y of $\mathbb{R}$ such that the set $X \setminus Y$ has cardinality $\mathfrak{c}$.

This answers a question of Erdős.

Erdős searched for a proper subset of $\mathcal{L}$ which has the Erdős property. From Theorem 4 we know that $\mathcal{L}$ contains a chain $L_1, L_2, \ldots, L_n, \ldots$ such that

$$\mathcal{L} \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \ldots$$

with each L_n being a dense G_δ-subset of $\mathbb{R}$ and so having the Erdős property. So there is no smallest set with the Erdős property. Indeed as $\mathcal{L} \setminus L_1$ has cardinality $\mathfrak{c}$, if Y is any of the $2^{\mathfrak{c}}$ subsets of $\mathcal{L} \setminus L_1$, then $L_1 \cup Y$ has the Erdős property.

$\boxed{\text{\textbf{Theorem 5.} [CM] There exist } \mathbf{2^{\mathfrak{c}}}\text{-subsets of } \boldsymbol{\mathcal{L}} \text{ with the Erdős property.}}$

s

The Gelfond-Schneider Theorem says: *if a and b are (complex) algebraic numbers with $a \neq 0, 1$ and b not a rational number, then a^b is a transcendental number.* In 2023 Diego Marques and Marcelo Oliveira extended this to when b is a Liouville number. By contrast:

Theorem 6. [CM] If $\boldsymbol{s}$ is any postive real number with $\boldsymbol{s \neq 1}$, then there exist $\boldsymbol{a, b \in \mathcal{L}}$, with $\boldsymbol{a, b > 0}$, such that $\boldsymbol{s = a^b}$.
Indeed, if $\boldsymbol{X}$ is any dense $\boldsymbol{G_\delta}$-subset of $\mathbb{R}$, then $\boldsymbol{s = {x_1}^{x_2}}$, for some $\boldsymbol{x_1, x_2 \in X}$.

In his influential book"Transcendental Number Theory" Alan Baker introduces the chapter on Mahler's Classification as follows: "*A classification of the set of transcendental numbers into three distinct aggregates, termed S-, T-, and U-numbers, was introduced by Mahler in 1932, and it has proved to be of considerable value in the general development of the subject.*"

Given a polynomial $P(X) \in \mathbb{C}[X]$, recall that the height of P, denoted by $H(P)$, is the maximum of the absolute values of the coefficients of P.
Given a complex number ξ, a positive integer n, and a real number $H \geq 1$, we define the quantity

$$w_n(\xi, H) = \min\{|P(\xi)| \colon P(X) \in \mathbb{Z}[X],\ H(P) \leq H,\\ \deg(P) \leq n,\ P(\xi) \neq 0\}.$$

Furthermore, we set

$$w_n(\xi) = \limsup_{H \to \infty} \frac{-\log w_n(\xi, H)}{\log H} \quad \text{and}$$

$$w(\xi) = \limsup_{n \to \infty} \frac{w_n(\xi)}{n}.$$

With the above notation in mind, Kurt Mahler partitions the set $\mathbb{R}$ as follows:

Definition 2. Let ξ be a real number. The number ξ is

(i) an $\mathbb{A}$-**number** if $w(\xi) = 0$,

(ii) an $\boldsymbol{S}$-**number** if $0 < w(\xi) < \infty$,

(iii) a $\boldsymbol{T}$-**number** if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any $n \geq 1$,

(iv) a $\boldsymbol{U}$-**number** if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for all $n \geq n_0$, for some positive integer n_0.

The $\mathbb{A}$-numbers are the algebraic numbers and there exist an infinity of $\mathbb{A}$-numbers, S-numbers, U-numbers and T-numbers.

The set $\mathcal{L}$ of Liouville numbers is a proper subset of the set of U-numbers.

It was an open question for 36 years on whether the set of T-numbers is non-empty. It was answered in 1970 in the positive by Wolfgang M. Schmidt who won the Frank Nelson Cole Prize in Number Theory for work on Diophantine Approximation.

The following theorem of Mahler records a fundamental property of the Mahler classes.

Theorem 7. If $\xi, \eta \in \mathbb{R}$ are algebraically dependent then they belong to the same Mahler class.

Theorem 8. [H. Ki (2022)] Each of the Mahler sets is a Borel set.

The next beautiful theorem is very easily proved using Mahler classes and Theorem 7.

Theorem 9. [CM] For any (complex) U-number α, in particular for α any Liouville number, all of the following are transcendental numbers: e^{α}, $\log_{\mathrm{e}} \alpha$, $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, $\sinh \alpha$, $\cosh \alpha$, $\tanh \alpha$ **and the inverse functions** evaluated at α of the listed trigonometric and hyperbolic functions, noting that wherever multiple values are involved, every such value is transcendental.

The following powerful theorem combines the main result in 2002 of the paper "Hausdorff dimension, analytic sets and transcendence" by Gerald A Edgar and a standard result from topology.

Theorem 10. If X is an uncountable analytic subset of $\mathbb{R}$, then it has a subspace homeomorphic to $\mathbb{G}$. In particular, X has cardinality $\mathfrak{c}$.
If Y is an analytic subset of $\mathbb{R}$ with finite positive Hausdorff dimension, then it has cardinality $\mathfrak{c}$ and contains a maximal algebraically independent subset of $\mathbb{R}$ (that is a transcendence basis for $\mathbb{R}$).

Theorem 11. [CM] Let $\boldsymbol{X}$ be an analytic subset of $\mathbb{R}$ having finite positive Hausdorff dimension. Then the intersection of $\boldsymbol{X}$ with each Mahler set $\boldsymbol{S}$, $\boldsymbol{T}$, and $\boldsymbol{U}$ is infinite.

The following theorem of Vojtēch Jarník dates back almost 100 years to 1929.

Theorem 12. The set of real numbers of irrationality exponent equal to 2 has full Lebesgue measure. The set of real numbers of exponent $\boldsymbol{m \in (2, \infty)}$ has Lebesgue measure $\mathbf{0}$ and Hausdorff dimension equal to $\boldsymbol{\frac{2}{m}}$.

It is routine exercise to prove that each set of real numbers of exponent $m \in (2, \infty)$ is analytic.

Our next result shows that the mysterious Mahler set T, as well as the sets S and U contain a homeomorphic copy of $\mathbb{G}$ (and of $\mathbb{P}$) and so have $\mathfrak{c}$ elements, This also extends a result in 2000 of Yann Bugeaud.

Theorem 13. [CM] If for each $\boldsymbol{m} \in (\mathbf{2}, \infty)$, $\boldsymbol{E_m}$ is the set of real numbers of irrationality exponent equal to $\boldsymbol{m}$, then $\boldsymbol{E_m}$ has infinite intersection with each Mahler set $\boldsymbol{S}$, $\boldsymbol{T}$, and $\boldsymbol{U}$.
Further, each of the Mahler sets $\boldsymbol{S}$, $\boldsymbol{T}$, and $\boldsymbol{U}$ contains an element of irrationality exponent $\boldsymbol{m}$ for each $\boldsymbol{m} \in (\mathbf{2}, \infty)$ and so $\boldsymbol{S}$, $\boldsymbol{T}$, and $\boldsymbol{U}$ each have a subspace homeomorphic to $\mathbb{G}$ (and a subspace homeomorphic to $\mathbb{P}$) and so have cardinality $\mathfrak{c}$.

Finally we state two results about the middle-third Cantor set $\mathbb{G}$.

Theorem 14. [CM] For each $\boldsymbol{m} \in [\mathbf{2}, \boldsymbol{\infty})$, let $\boldsymbol{E_m}$ be the set of real numbers of irrationality exponent equal to $\boldsymbol{m}$. Then $\mathbb{G} \cap \boldsymbol{E_m}$ has cardinality $\mathfrak{c}$ and a subspace homeomorphic to $\mathbb{G}$.
For the Mahler sets $\boldsymbol{S}$, $\boldsymbol{T}$, and $\boldsymbol{U}$, each of the sets $\mathbb{G} \cap \boldsymbol{S}$, $\mathbb{G} \cap \boldsymbol{T}$, and $\mathbb{G} \cap \boldsymbol{U}$ is (infinite and) a dense subset of $\mathbb{G}$.

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3. Chalebgwa, T.P. and Morris, S.A., (2023/2024), Topology Meets Number Theory, (to appear)


Thank You