Isodrastic fields

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In collaboration with
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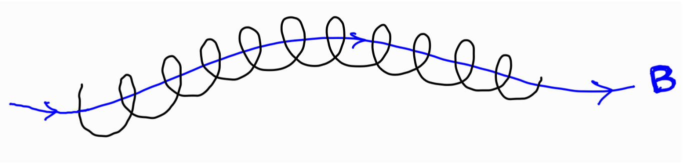
Nonlinear dynamics of guiding-center motion

Lorentz force

$$\dot{\mathbf{v}} = \frac{q}{m}\mathbf{v} \times \mathbf{B}(x)$$
$$\dot{\mathbf{x}} = \mathbf{v}$$



A charged particle (electron or ion) drifts along the magnetic field line as it gyrates (fast relative to its drift motion) around it.

- On large time-scales, compared to the gyro-oscillations, the charged particle motion in a strong magnetic field can be written as a two degrees of freedom Hamiltonian due to the adiabatic invariance of the magnetic moment, $\mu = \frac{mv_\perp^2}{2|B|}$.
- The guiding center (GC) Hamiltonian is derived by averaging the fast gyro-oscillations over the slow motion along the fieldline.

NONLINEAR DYNAMICS OF GUIDING-CENTER MOTION

- To first order in μ, the guiding center motion has a Hamiltonian formulation[1]: phase space is position: $X \in \mathbb{R}^3$ and parallel velocity: $v_{||} \in \mathbb{R}$ with parameters: e, m, μ.
- The Hamiltonian is

$$H = \frac{1}{2} m v_{||}^2 + \mu |B(X)|$$

  with the closed 2-form, $\omega = e\beta + md(v_{||} b^\flat)$, where β is the magnetic flux 2-form, $\beta = B_z dx \wedge dy + B_x dy \wedge dz + B_y dz \wedge dx$, and $b^\flat$ is the 1-form, $b \cdot dX$.
- The Hamiltonian and the closed 2-form generates the dynamics $(\dot{X}, \dot{v}_{||}) = V$ by solving $i_V \omega = -dH$ for V (except where $\tilde{B}_{||} = 0$, defined below, at which ω is degenerate). The solution can be written as

$$\dot{X} = \left(v_{||} \widetilde{B} + \frac{\mu}{e} b \times \nabla |B| \right) / \widetilde{B}_{||}$$

$$\dot{v}_{||} = -\frac{\mu}{m} \frac{\widetilde{B}}{\widetilde{B}_{||}} \cdot \nabla |B|,$$

  where $\widetilde{B} = B + \frac{m}{e} v_{||} c, \; c = \text{curl } b, \;$ and $\widetilde{B}_{||} = \widetilde{B} \cdot b$.

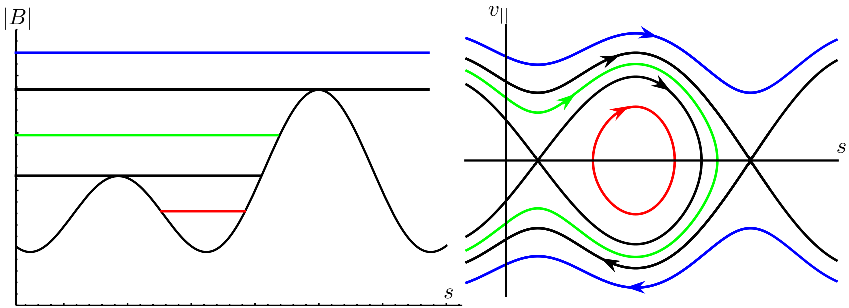
[1] Littlejohn, J. Math. Phys. 20, 2445–2458 (1979)

Nonlinear dynamics of guiding-center motion

- For low energy, guiding-center motion is along fieldlines and particles bounce at points $|B| = E/\mu$. For high energies, guiding-centers drift across fieldlines to leading order.
- Drift across fieldlines implies guiding-center of particles see a time-varying $|B|$ profile.

Nonlinear dynamics of guiding-center motion

- Different classes of motion correspond to different set of wells of $|B|$.
  - passing: $|B| < E/\mu$ along the whole fieldline so the guiding center moves unidirectionally along it.
  - bouncing (one-sided or two-sided): $|B| < E/\mu$ in an interval (s_1, s_2) along the fieldline with $|B| = E/\mu$ and $|B|' \neq 0$ at both ends, so the guiding center bounces periodically between s_1 and s_2.
  - marginal: if $|B|' = 0$ at a point where $|B| = E/\mu$ then the guiding center takes infinite time to reach it.



Example of field strength along a field line and resulting ZGCM. Marginal cases are shown in black.

NONLINEAR DYNAMICS OF GUIDING-CENTER MOTION

- For confinement, it is **undesirable to let bouncing trajectories change class of guiding center motion**: leads to large and pseudo-random changes and drifts away from plasma region to leave the device.
- Transition between classes of motion degrades confinement and leads to energy loss. Thus, we want to avoid transitions.
- One proposed solution is omnigenity.[2]
  - This assumes the existence of flux function $\psi : B \cdot \nabla\psi = 0$ (flux functions are nested surfaces obtained as magnetohydrostatic equilibrium)
  - Constrains bouncing particle to stay near the flux surface to leading order: $\langle v_d \cdot \nabla\psi \rangle = 0$.
  - Prevents transitions between classes via generic homoclinics.
- We would like to relax the omnigenity requirement of existence of flux function while keeping its confinement properties.

[2] Cary, Shasharina, Phys. Plasmas 4, 3323–3333 (1997); Hall, McNamara, Phys. Fluids 18, 552–565 (1975)

But first, we need some definitions

- For (two-sided) bouncing motion there is a second adiabatic invariant L, called “longitudinal”, whose asymptotic expansion starts with $\int_{s_1}^{s_2} mv_{\|}\, ds$.
  Using energy conservation, for $\mu > 0$ the **second adiabatic invariant can be written as** $L = \sqrt{m\mu}\, j$ with
  $$j = \int_{s_1}^{s_2} \sqrt{2(h - |B|)}\, ds,$$
  where $h = E/\mu$ and the bounce points are at arclengths s_1, s_2.

- A key role is played in the reduced dynamics by **the set Σ of critical points of $|B|$ along fieldlines**, $\Sigma = \{x \in \mathbb{R}^3 : |B(x)|' = 0\}$, where, $'$ denotes derivative with respect to arclength along a fieldline.

- Subdivide Σ into the disjoint union $\Sigma = \Sigma^+ \cup \Sigma^0 \cup \Sigma^-$, according as $|B|'' > 0, = 0$ or < 0, respectively.

- By the implicit function theorem, $\Sigma^\pm$ are C^{r-1} surfaces (with possibly several components). For $r \geq 3$, Σ^0 is generically a C^{r-2} curve (with possibly several components) and generically forms the common boundary of $\Sigma^\pm$.

Weakly isodrastic: approximate treatment of preventing transitions

Definition 1

A magnetic field B is weakly isodrastic *if the marginal cases are never reached from non-marginal ones by the first-order reduced dynamics.*[3]

[3] Burby, MacKay, Naik (2023) *Nonlinearity 36 5884*

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- A field, B, is **weakly isodrastic** if the level curves of $|B|$ and $\mathcal{J}_\sigma$ coincide on Σ^- where $\mathcal{J}_\pm$ are $\mathcal{J}$, for GC in the two directions from Σ^-, and $|B|$ is constant along Σ^0, where Σ is the set of critical points of $|B|$ along fieldlines and $\Sigma = \Sigma^+ \cup \Sigma^0 \cup \Sigma^-$ as $|B|'' > 0, = 0, < 0$.

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WEAKLY ISODRASTIC: APPROXIMATE TREATMENT OF PREVENTING TRANSITIONS

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Theorem 1

(a) If magnetic field B is weakly isodrastic then for both directions along the field, dh and d$\mathcal{J}$ are linearly dependent at every point of $\Sigma^{-\prime}$;
(b) If B is weakly isodrastic and Σ^0 is a smooth curve without heteroclinic cases, then h and $\mathcal{J}$ are constant on connected components of Σ^0;
(c) If for both directions, $\mathcal{J}$ is constant on components of level sets of h then B is weakly isodrastic.

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Weakly isodrastic: approximate treatment of preventing transitions

- Theorem above leads to a quantification of failure to be isodrastic. The failure of contours of h and $\mathcal{J}$ to coincide on Σ^- can be measured by the 2-form $dh \wedge d\mathcal{J}$. This is most simply described by comparing it to the magnetic flux-form β, which is a nondegenerate top-form on Σ^-. Thus there is a function $\mathcal{M}$ on Σ^- such that

$$dh \wedge d\mathcal{J} = \mathcal{M}\beta.$$

- $\mathcal{M}$ will be identified as a "Melnikov function" for the FGCM dynamics (in the exact treatment). But for now, to compute $\mathcal{M}$, if Σ^- is given locally as the graph $z = Z(x, y)$ of a function in Cartesian coordinates then

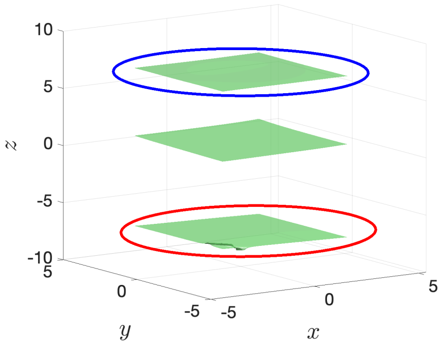
$$\mathcal{M} = \frac{h_{,x}\mathcal{J}_{,y} - h_{,y}\mathcal{J}_{,x}}{B_z - B_x Z_{,x} - B_y Z_{,y}},$$

  where subscripts after a comma indicate partial derivatives.

- "Dimensional proof": The function $\mathcal{M}$ has units of square root of field strength divided by length, but it is natural to multiply $\mathcal{M}$ by the factor $\sqrt{m\mu}$ to turn $\mathcal{J}$ into L. The quantity $\sqrt{m\mu}\,\mathcal{M}$ is an inverse time, so represents the rate of transition between classes of GC motion.

Illustration of failure to be weakly isodrastic

- We consider the field due to two coils (red and blue) with higher current in the coil below than the top and axisymmetry is perturbed by rotating one with respect to another.



(Left) Showing $\Sigma^{\pm}$ as green planes with Σ^{-} inside the coils and Σ^{+} outside the coils near $z = 0$,

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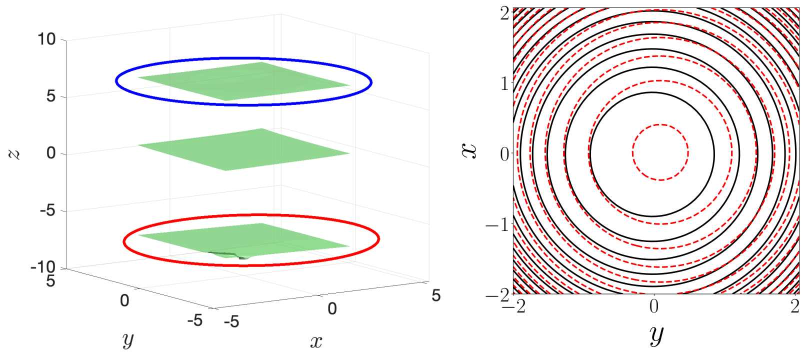

(Left) Showing $\Sigma^{\pm}$ as green planes with Σ^{-} inside the coils and Σ^{+} outside the coils near $z = 0$, (Center) contours of $|B| = h$ on Σ^{-}, which are closed loops around a 3D saddle point and contours of $\mathscr{J}$ on Σ^{-}, for the fieldline segment starting on Σ^{-} and going downwards to the first bounce,

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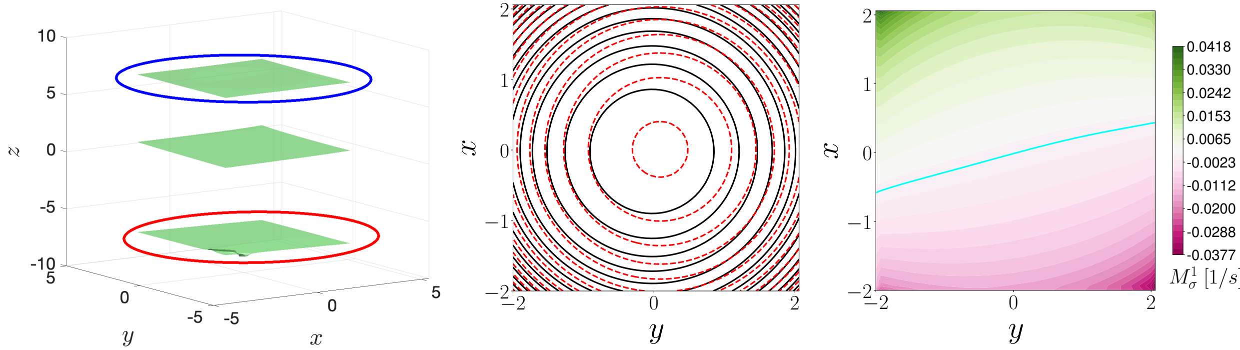

(Left) Showing $\Sigma^{\pm}$ as green planes with Σ^- inside the coils and Σ^+ outside the coils near $z = 0$, (Center) contours of $|B| = h$ on Σ^-, which are closed loops around a 3D saddle point and contours of $\mathcal{J}$ on Σ^-, for the fieldline segment starting on Σ^- and going downwards to the first bounce, (Right) failure of the h and $\mathcal{J}$ contours to coincide is measured by the Melnikov function M on Σ^- defined to be the ratio of $dh \wedge d\mathcal{J}$ to the magnetic flux form β.

Strongly isodrastic: exact treatment of preventing transitions

- The treatment of weak isodrasticity rests on assuming conservation of the adiabatic invariant j which fails near the transitions (on the separatrix).
- The key idea is that $\Sigma^- \times \{v_{||} = 0\}$ is an approximate normally hyperbolic submanifold (NHS) for guiding-centre motion.
- An NHS is an invariant submanifold such that any tangential contraction or expansion is weaker than normal contraction or expansion, respectively.

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- An NHS is an invariant submanifold such that any tangential contraction or expansion is weaker than normal contraction or expansion, respectively.
- NHS have forward and backward contracting submanifolds $W^{\pm}$ (usually called stable and unstable manifolds respectively), consisting of the set of points whose trajectory in the stated direction of time converges to the NHS.
- To prevent transitions, the strong isodrastic condition is that the relevant branches of $W^{\pm}$ coincide, forming 'separatrices': invariant submanifolds that separate motions of different types.
- Example: Separatrix in the phase space of an unperturbed pendulum separates librating and rotating motion. There the NHS is just a saddle point in 2D, but the same idea extends to higher dimensions (2D NHS in 4D in our case).

Strongly isodrastic: exact treatment of preventing transitions

- A magnetic field is **strongly isodrastic** if $\Sigma^{-0} \times \{0\}$ continues to a maximal invariant submanifold N^{-0} with boundary for guiding-center motion for a range of $\sqrt{\tilde{\mu}} > 0$, which can be decomposed into normally hyperbolic N^- and its boundary N^0, the relevant branches of the contracting submanifolds, $W^\pm$, of N^- coincide, and $\tilde{H}$ is constant along N^0 where $\tilde{\mu} = \frac{m}{e^2}\mu$, $\tilde{H} = \frac{H}{\mu}$.

- When Σ^0 does not exist and hence no N^0, so strong isodrasticity is just the coincidence of $W^\pm$.

Strongly isodrastic: exact treatment of preventing transitions

- For a mirror machine, there is a non-degenerate saddle point of $|B|$ near the centre of each coil. For the gradient field $\nabla|B|$, each of them has one-dimensional downhill subspace and two-dimensional uphill subspace. They give unstable equilibrium points (saddle-centres) of guiding-centre dynamics with $v_{||} = 0$.
- They are each surrounded by a family of periodic orbits of guiding-centre motion, called Lyapunov orbits, which form the 2D centre manifold of the equilibrium point. The periodic orbits are hyperbolic (hyp. PO) and the centre manifold is normally hyperbolic.
- This is a case of a general phenomenon for Hamiltonian systems with an index-one saddle, understood by Conley and McGhee in the context of celestial mechanics.
- The forward contracting submanifold of the periodic orbit at given energy separates trajectories that bounce from those that pass over the saddle. The flux of energy-surface volume passing over the saddle at given energy is the action of the corresponding periodic orbit.

ILLUSTRATION OF STRONGLY ISODRASTIC

- We consider the axisymmetric vacuum field

$$B^z = 1 - a\cos kz\, I_0(kr), \quad B^r = -a\sin kz\, I_1(kr), \quad B^\phi = 0$$

with I_j being modified Bessel functions.

- Then, we add a similar vacuum field of twice the period to break the reflection symmetry about $z = 0$, so

$$B^z = 1 - a\cos kz\, I_0(kr) - a\eta\sin\tfrac{kz}{2} I_0(\tfrac{kr}{2}), \quad B^r = -a\sin kz\, I_1(kr) + a\eta\cos\tfrac{kz}{2} I_1(\tfrac{kr}{2}),$$

with $\eta \in (0, 4)$, thereby making the upper saddle weaker than the lower one so that we can study transitions involving passing through the top alone, as for the two-coil example.

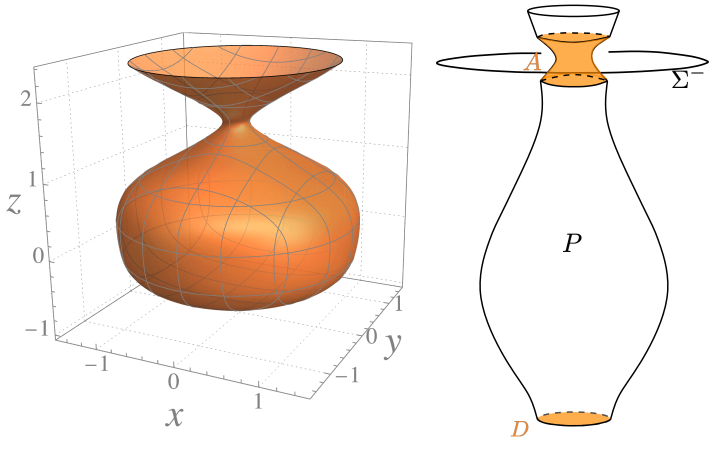
- The saddles are at $z = \pm\pi/k$ and have $|B|_\pm = 1 + a(1 \mp \eta)$, respectively. Σ^- is the two planes $z = \pm\pi/k$.

- Then we break axisymmetry by adding (in contravariant components)

$$B^\phi = -\varepsilon xk\sin kz, \quad B^z = \varepsilon y\cos kz.$$

Illustration of strongly isodrastic

- We choose $k = 2.0$, $a = 0.5$, $\eta = 1.5$, $\tilde{\mu} = \frac{m}{e^2}\mu = 10^{-2}$ for the following illustrations.



(Left) The region accessible to guiding centres for the axisymmetric case: $\varepsilon = 0.0$, $E = 1.135$, $B_+ = 1.125$, (Center) In a fusion device, the accessible region is restricted to $P : r \leqslant R(z)$ for some small function R,

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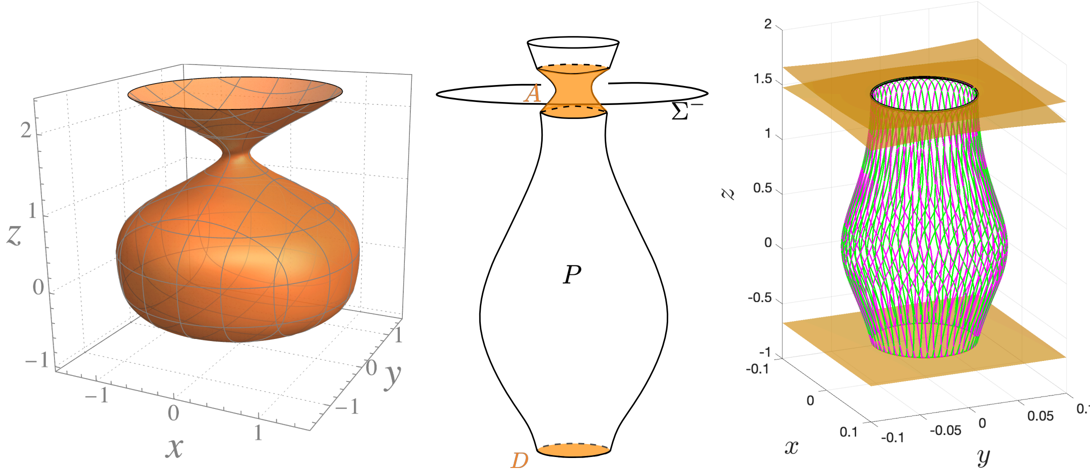

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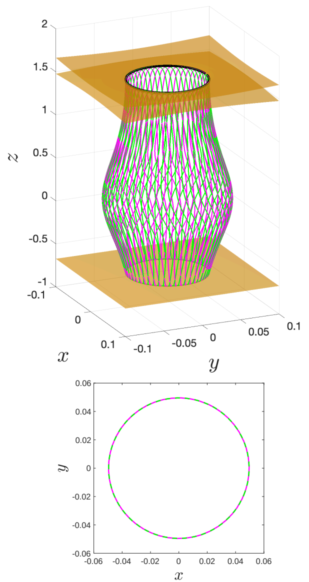
ILLUSTRATION OF FAILURE TO BE STRONGLY ISODRASTIC

- (Top row) Projection to the physical space of the hyp. PO and contracting manifolds and (Bottom row) traces of the first bounces of the contracting manifolds.



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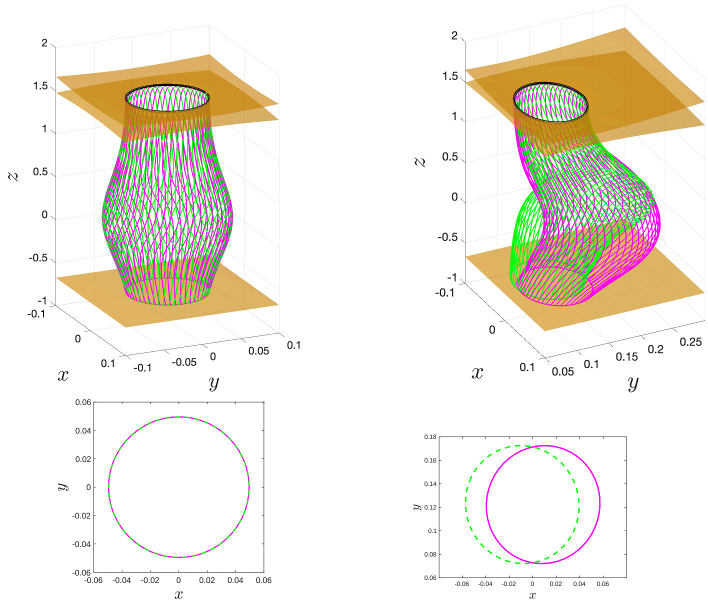

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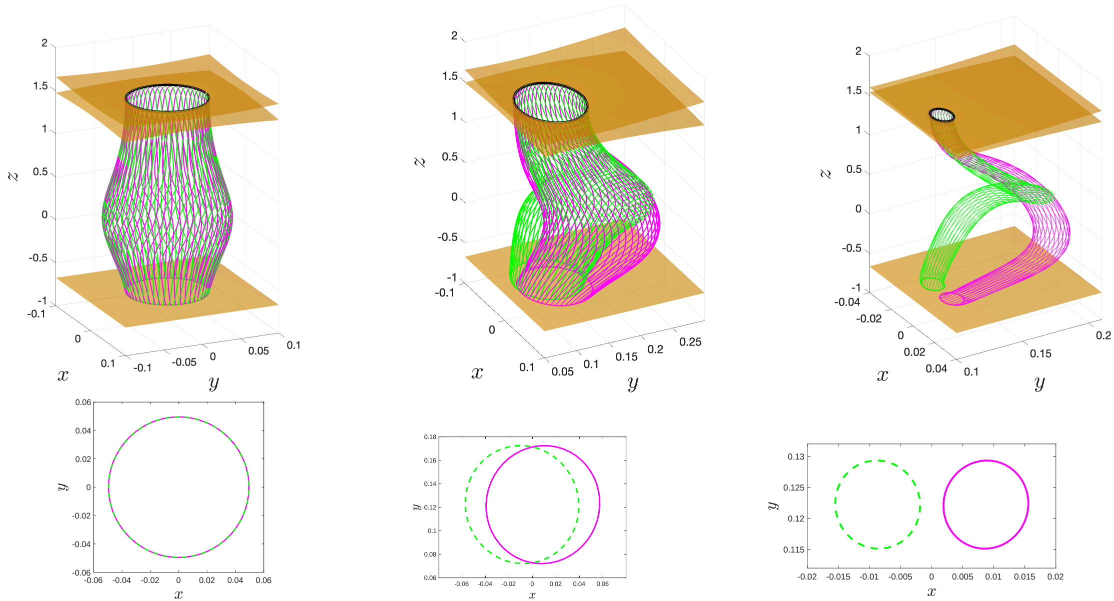

(Left) Axisymmetric: $\varepsilon = 0.0, E = 1.135, B_+ = 1.125$, (Center) $\varepsilon = 0.1, E = |B|_+ + 10^{-3}, |B|_+ = 1.11887$, and (Right) $\varepsilon = 0.1, E = |B|_+ + 2 \times 10^{-5}, |B|_+ = 1.11887$.

Summary and Outlook

- We also derive a measure of the failure of strong isodrasticity, "Melnikov function".
- Elizabeth Paul (Columbia University) has implemented the approximate treatment of isodrasticity in SIMSOPT. We aim to make it a concrete module soon!
- Isodrastic fields can improve confinement by suppressing transitions between classes of guiding-center motion.
- Isodrasticity does not require a flux function, so imposes less constraint on optimizing MHD equilibria and device configurations than omnigenity (which is in turn weaker than quasisymmetric).
- This makes isodrastic magnetic fields more realisable in practice.

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Isodrastic magnetic fields for suppressing transitions in guiding-centre motion

J W Burby[3,1], R S MacKay[3,2] and S Naik[3,2]

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We show isodrasticity is a necessary and sufficient condition for $O(\epsilon^2)$ transition flux, exact treatment (strong isodrastic) and more examples.

- On going work: field and coil design of STAR_Lite, introducing isodrasticity in the coil design.

Thank you for your attention.

https://github.com/Shibabrat/isodrastic

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