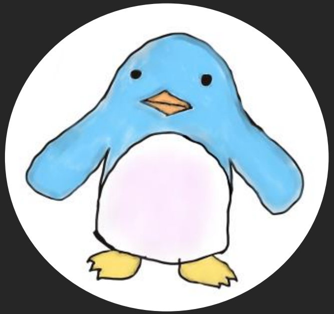

Naoki Sato

Expertise
solve theoretical fusion problems by novel math approaches

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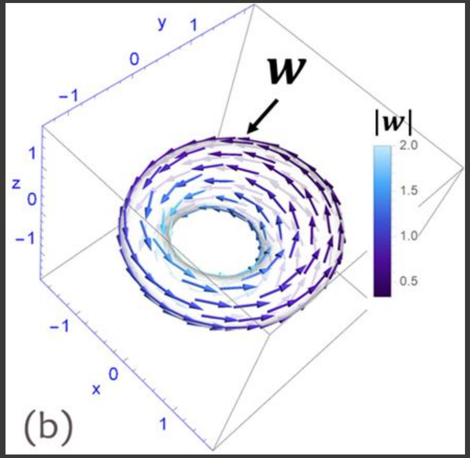
Grad conjecture workshop

What I want to discuss

3D MHD Equilibria & QS



3D Anisotropic MHD Equilibrium
JMP **64** 081505 (2023)

What I can discuss

- Self-Organization & Turbulence
- Magnetohydrostatics, Grad conjecture & QS
- Hamiltonian Mechanics
- PDE theory

Simons Hidden Symmetries and Fusion Energy Collaboration Australian Retreat
2023/12/12

A Reduced Ideal MHD System for Nonlinear Magnetic Field Turbulence in Plasmas with Approximate Flux Surfaces

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Introduction

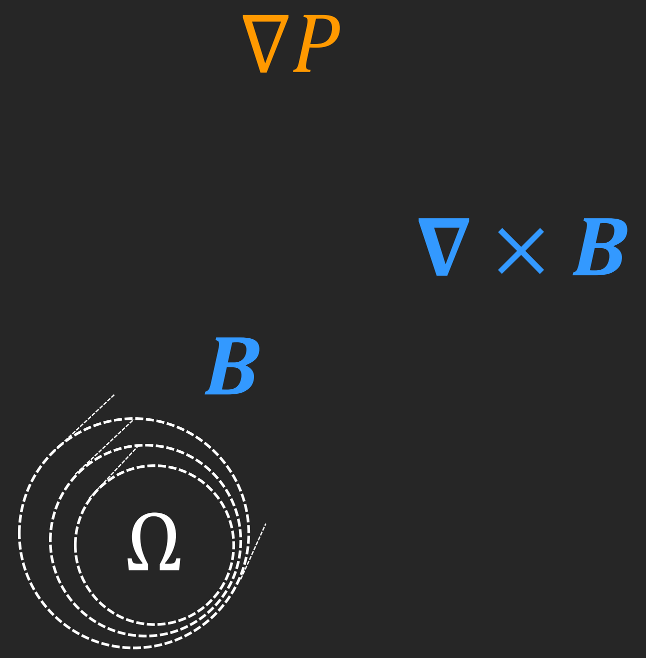
This study is concerned with the **dynamics of the magnetic field around a MHD equilibrium**

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \mu_0 \nabla P, \qquad \nabla \cdot \boldsymbol{B} = 0 \qquad \text{in } \Omega. \tag{1}$$

Here, $\boldsymbol{B}(\boldsymbol{x})$ is a three-dimensional vector field with Cartesian components B_i, $i = 1,2,3$, defined in a smooth toroidal domain $\Omega \subset \mathbb{R}^3$, μ_0 the vacuum permeability, and $P(\boldsymbol{x})$ the equilibrium pressure field.

The dynamics around (1) is governed by the **ideal MHD** equations in Ω,

$$\begin{aligned}
&\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla P,\\
&\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E},\\
&\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u}),\\
&\nabla \cdot \boldsymbol{B} = 0.
\end{aligned} \tag{2}$$



Here, $\boldsymbol{u}(\boldsymbol{x},t)$, $\boldsymbol{B}(\boldsymbol{x},t)$, $\rho(\boldsymbol{x},t)$, $P(\boldsymbol{x},t)$, and $\boldsymbol{E}(\boldsymbol{x},t)$ are velocity field, magnetic field, mass density, pressure, and electric field. Eq. (2) can be closed with the aid of an equation of state and the electron momentum equation.

Motivation

① **Magnetic Turbulence in proximity of MHD Equilibria**

② **3D MHD equilibria existence & stability**

3D = not invariant under some Euclidean isometry

Objectives

Toy Model of Magnetic Turbulence

- Obtain reduced equations preserving the Hamiltonian structure of ideal MHD (2) and describing the nonlinear evolution of the magnetic field in proximity of MHD equilibria (1) and in a physical regime relevant for stellarator plasmas

New Schemes for MHD Equilibria Existence & Stability

- Formulate dissipative and iterative schemes to construct nontrivial 3D MHD equilibria in toroidal domains
- Elucidate stability properties of such equilibria

Generalized Ohm's Law

Electron fluid momentum equation:

$$m_e n_e\left(\frac{\partial \boldsymbol{u}_e}{\partial t}+\boldsymbol{u}_e\cdot\nabla\boldsymbol{u}_e\right)=-en_e(\boldsymbol{E}+\boldsymbol{u}_e\times\boldsymbol{B})-\nabla P_e. \tag{3}$$

Using quasi-neutrality, the ideal MHD variables can be related to the ion-electron two-fluid variables as

$$\boldsymbol{u}=\frac{\boldsymbol{u}_i+\delta\boldsymbol{u}_e}{1+\delta},\qquad \rho=m_i n(1+\delta),\qquad \boldsymbol{u}_e=\boldsymbol{u}-\frac{\nabla\times\boldsymbol{B}}{e\mu_0(1+\delta)n},\qquad n_i=n_e=n,\qquad \delta=\frac{m_e}{m_i}. \tag{4}$$

From (3) and (4), the electric field can be expressed as

$$\boldsymbol{E}=\boldsymbol{B}\times\boldsymbol{u}+\frac{m_i}{e\mu_0\rho}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+\frac{m_i(1+\delta)}{e\rho}\nabla P_e-\frac{m_e}{e}\left(\frac{\partial\boldsymbol{u}_e}{\partial t}+\boldsymbol{u}_e\cdot\nabla\boldsymbol{u}_e\right). \tag{5}$$

Neglecting electron inertia and assuming a barotropic electron pressure $P_e=P_e(\rho)$,

$$\nabla\times\boldsymbol{E}=\nabla\times\left[\boldsymbol{B}\times\left(\boldsymbol{u}-\frac{\kappa}{\rho}\nabla\times\boldsymbol{B}\right)\right],\qquad \kappa=\frac{m_i}{e\mu_0}. \tag{6}$$

- The first term on the right-hand of (6) side gives the ideal Ohm's law, the second term is the Hall effect.
- Eq. (6) can be used to eliminate $\boldsymbol{E}$ from the ideal MHD system (2).

Boundary Conditions

We consider the following boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n}=0, \qquad \boldsymbol{B}\cdot\boldsymbol{n}=0, \qquad \nabla\times\boldsymbol{B}\cdot\boldsymbol{n}=0, \qquad P=\text{constant} \qquad \text{on } \partial\Omega. \tag{7}$$

Here $\boldsymbol{n}$ denotes the unit outward normal to the bounding surface $\partial\Omega$.

Remarks:

- Eq. (7) implies that there is no net electric current
$$\boldsymbol{J}=\frac{1}{\mu_0}\nabla\times\boldsymbol{B}=en(\boldsymbol{u}_i-\boldsymbol{u}_e)$$
across $\partial\Omega$. This is expected to hold true as long as $\boldsymbol{u}_e$ and $\boldsymbol{u}_i$ are tangent to $\partial\Omega$.

- The boundary conditions (7) describe what we expect from a physical standpoint.

- The set of boundary conditions required for existence of solutions will be described for each set of governing equations when necessary.

Ordering I

Let $\epsilon > 0$ be a small ordering parameter, $L \sim \Omega^{1/3}$ the typical size of the system (e.g. the linear size of a stellarator), and T a reference time scale (for example, a small fraction of the confinement time scale). Assuming $\rho > 0$, we order

$$\begin{gathered} \frac{T}{L}\frac{\boldsymbol{B}}{\sqrt{\mu_0 \rho}} \sim 1, \qquad v_A = \frac{B}{\sqrt{\mu_0 \rho}} \\ \frac{T}{L}\boldsymbol{u} \sim T\nabla\times\boldsymbol{u} \sim \epsilon, \\ \frac{T}{\sqrt{\mu_0\rho}}\nabla\times\boldsymbol{B} \sim \frac{T^2}{\rho L^2}P \sim \frac{T}{|\boldsymbol{u}|}\frac{\partial \boldsymbol{u}}{\partial t} \sim \frac{T}{\rho}\frac{\partial \rho}{\partial t} \sim \frac{T}{|\boldsymbol{B}|}\frac{\partial \boldsymbol{B}}{\partial t} \sim \frac{T}{|\boldsymbol{B}|}\nabla\times\boldsymbol{E} \sim \epsilon^2. \end{gathered} \tag{8}$$

We close the ideal MHD system (2) via the equation of state (generalized Bernoulli principle)

$$\frac{T^3}{L^3}\boldsymbol{u}\cdot\left(\frac{1}{\rho}\nabla P + \frac{1}{2}\nabla \boldsymbol{u}^2\right) \sim \epsilon^4. \tag{9}$$

- When $\rho = \rho_c \in \mathbb{R}$ is constant, eq. (9) can be satisfied through the Bernoulli principle $\nabla(P + \rho_c \boldsymbol{u}^2 + h) = \boldsymbol{0}$ where h is any function such that $\boldsymbol{u}\cdot\nabla h = 0$.
- Eq. (9) arises from the plasma momentum equation when the system is steady and $\nabla\times\boldsymbol{u}$ and $\nabla\times\boldsymbol{B}$ are small (stellarator regime).

Reduction at Leading Order – Ideal MHD

Using generalized Ohm's law (6), ordering (8), generalized Bernoulli principle (9), ideal MHD (2) reduces to

$$\begin{aligned}
&\rho(\nabla\times\boldsymbol{u})\times\boldsymbol{u} = \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} - \nabla P - \frac{1}{2}\rho\nabla\boldsymbol{u}^2, && \text{Momentum}\\
&\boldsymbol{u}\cdot\left(\nabla P + \frac{1}{2}\rho\nabla\boldsymbol{u}^2\right) = 0, && \text{Eq. of state (Gen. Bernoulli)}\\
&\frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times\left[\left(\boldsymbol{u} - \frac{\kappa}{\rho}\nabla\times\boldsymbol{B}\right)\times\boldsymbol{B}\right], && \text{Induction}\\
&\nabla\cdot(\rho\boldsymbol{u}) = 0, && \text{Continuity}\\
&\nabla\cdot\boldsymbol{B} = 0. && \text{Solenoidal}
\end{aligned}\tag{10}$$

Remarks:

- When $\boldsymbol{u} = \mathbf{0}$, eq. (10) reduces to MHD equilibria (1) with barotropic pressure $P = P(\rho)$.
- Dotting the momentum eq. with $\boldsymbol{u}$, and using the eq. of state, one finds that

$$\boldsymbol{u} = \alpha\nabla\times\boldsymbol{B}\times+\beta\boldsymbol{B}, \qquad \alpha = \alpha(\boldsymbol{x},t), \qquad \beta = \beta(\boldsymbol{x},t). \tag{11}$$

Approximate Flux Surfaces

Using generalized Ohm's law (6), ordering (8), and induction equation $\partial \boldsymbol{B}/\partial t = -\nabla \times \boldsymbol{E}$,

$$\frac{T^2}{L^2\sqrt{\mu_0\rho_0}}\left[\left(\alpha - \frac{\kappa}{\rho}\right)(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \frac{\epsilon}{T}\nabla\Psi\right] \sim \epsilon^2. \tag{12}$$

Since $\nabla \cdot \boldsymbol{B} = 0$, eq. (12) implies that there is a single valued Θ in a small neighborhood $U \subset \Omega$ such that

$$\boldsymbol{B} = \nabla\Psi \times \nabla\Theta + \frac{L}{T}\sqrt{\mu_0\rho_0}\, o(\epsilon) \qquad \text{in } U. \tag{13}$$

Allowing Θ to be multivalued, at leading order we may therefore set

$$\boldsymbol{B} = \nabla\Psi \times \nabla\Theta \qquad \text{in } \Omega. \tag{14}$$

Remarks:

- $\boldsymbol{B}$ and the reference density $\rho_0 \in \mathbb{R}$ can be large in the ordering (8). Only the ratio $T\boldsymbol{B}/L\sqrt{\mu_0\rho_0}$ is constrained.

Reduction at Leading Order – Induction Equation

Introducing the vector field

$$\boldsymbol{\xi} = \boldsymbol{u} - \frac{\kappa}{\rho}\nabla \times \boldsymbol{B} = \left(\alpha - \frac{\kappa}{\rho}\right)\nabla \times \boldsymbol{B} + \beta \boldsymbol{B} = A\nabla \times \boldsymbol{B} + \beta \boldsymbol{B}, \tag{15}$$

and substituting the Clebsch form (14) into the induction equation, one finds

$$\nabla\left(\frac{\partial \Psi}{\partial t} + \boldsymbol{\xi} \cdot \nabla \Psi\right) \times \nabla \Theta = \nabla\left(\frac{\partial \Theta}{\partial t} + \boldsymbol{\xi} \cdot \nabla \Theta\right) \times \nabla \Psi. \tag{16}$$

When $\boldsymbol{B} \neq \boldsymbol{0}$, the vector fields $\nabla\Psi$ and $\nabla\Theta$ are linearly independent. Dotting (16) by $\nabla\Psi$ and $\nabla\Theta$ thus gives

$$\frac{\partial \Psi}{\partial t} + \boldsymbol{\xi} \cdot \nabla \Psi = f(\Psi, \Theta), \qquad \frac{\partial \Theta}{\partial t} + \boldsymbol{\xi} \cdot \nabla \Theta = g(\Psi, \Theta), \qquad \frac{\partial f}{\partial \Psi} = -\frac{\partial g}{\partial \Theta}. \tag{17}$$

The functions f and g can be chosen as follows. Equilibria with $\boldsymbol{u} = \boldsymbol{0}$ of the reduced system (10) satisfy

$$A_0 \frac{\partial P_0}{\partial \Psi}\nabla\Psi + A_0 \frac{\partial P_0}{\partial \Theta}\nabla\Theta = \frac{A_0}{\mu_0}\left[(\nabla \times \boldsymbol{B} \cdot \nabla\Theta)\nabla\Psi - (\nabla \times \boldsymbol{B} \cdot \nabla\Psi)\nabla\Theta\right] = \nabla\Phi. \tag{18}$$

Reduction at Leading Order – Induction Equation

Comparing (18) with steady states of (17), one finds the solution

$$f_0 = 0, \qquad g_0 = \mu_0 A_0(P_0)\frac{dP_0}{d\Psi}, \qquad P_0 = P_0(\Psi). \tag{19}$$

The induction equation for the magnetic field can thus be written as

$$\frac{\partial \Psi}{\partial t} + A\nabla \cdot [\nabla\Psi \times (\nabla\Theta \times \nabla\Psi)] = 0, \qquad \frac{\partial \Theta}{\partial t} - A\nabla \cdot [\nabla\Theta \times (\nabla\Psi \times \nabla\Theta)] = \mu_0 A_0 \frac{dP_0}{d\Psi}. \tag{20}$$

Remarks:

- Solutions of system (20) produce exact time-dependent solutions of system (10) such that steady states without flow have pressure $P_0(\Psi)$ and $A_0(P_0)$.
- The two equations appearing in (20) can be regarded as a dynamical system describing the nonlinear evolution of $\boldsymbol{B}$. Here, the function A is evaluated through α, ρ, and P, which are determined from the solution of the reduced ideal MHD system (10) for the variables $\boldsymbol{B}$, $\boldsymbol{u}$, ρ, P.

Closure of the Induction Equation

- Consider an MHD equilibrium (1) at $t = t_0$. From (18) we have $A_0 = A_0(P_0)$.
- We may identify $P_0 = \lambda\Psi$ with $\lambda \in \mathbb{R}$ without loss of generality.
- We perturb the system at some $t = t_1$, and suppose that the fields $\boldsymbol{u}$, ρ, and P react passively to changes in $\boldsymbol{B}$ so that the functional form of α, ρ, and P is preserved for $t \geq t_1$. This amounts to assuming

$$A = A_0(P_0) = A_0(\Psi) \qquad \forall\, t \geq t_0. \tag{21}$$

Then, system (20) reduces to an independent nonlinear system of two coupled PDEs for the variables Ψ and Θ,

$$\begin{aligned} \frac{\partial\Psi}{\partial t} + A_0(\Psi)\nabla\cdot[\nabla\Psi\times(\nabla\Theta\times\nabla\Psi)] &= 0, \\ \frac{\partial\Theta}{\partial t} - A_0(\Psi)\nabla\cdot[\nabla\Theta\times(\nabla\Psi\times\nabla\Theta)] &= \mu_0\lambda A_0(\Psi). \end{aligned} \tag{22}$$

$\Theta = M\mu + N\nu + \chi(\boldsymbol{x}, t)$

A possible set of boundary conditions is

$$\Psi = \text{constant}, \qquad \nabla\Theta\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega. \tag{23}$$

Ordering II

Let $\epsilon > 0$ be a small ordering parameter, $L \sim \Omega^{1/3}$ the typical size of the system (e.g. the linear size of a stellarator), and T a reference time scale (for example, a small fraction of the confinement time scale). Assuming $\rho > 0$, we order

$$\frac{T}{L}\frac{\boldsymbol{B}}{\sqrt{\mu_0\rho}} \sim \frac{T}{L}\boldsymbol{u} \sim T\nabla\times\boldsymbol{u} \sim \frac{T}{\sqrt{\mu_0\rho}}\nabla\times\boldsymbol{B} \sim \frac{T^2}{\rho L^2}P \sim 1, \qquad v_A = \frac{B}{\sqrt{\mu_0\rho}}$$

$$\frac{T}{|\boldsymbol{u}|}\frac{\partial\boldsymbol{u}}{\partial t} \sim \frac{T}{\rho}\frac{\partial\rho}{\partial t} \sim \frac{T}{|\boldsymbol{B}|}\frac{\partial\boldsymbol{B}}{\partial t} \sim \frac{T}{|\boldsymbol{B}|}\nabla\times\boldsymbol{E} \sim \epsilon. \tag{24}$$

We close the ideal MHD system (2) via the equation of state (generalized Bernoulli principle)

$$\frac{T^3}{L^3}\boldsymbol{u}\cdot\left(\frac{1}{\rho}\nabla P + \frac{1}{2}\nabla\boldsymbol{u}^2\right) \sim \epsilon. \tag{25}$$

- When $\rho = \rho_c \in \mathbb{R}$ is constant, eq. (9) can be satisfied through the Bernoulli principle $\nabla(P + \rho_c\boldsymbol{u}^2 + h) = \boldsymbol{0}$ where h is any function such that $\boldsymbol{u}\cdot\nabla h = 0$.
- Eq. (25) arises from the plasma momentum equation when the system is steady and $\nabla\times\boldsymbol{u}$ and $\nabla\times\boldsymbol{B}$ are small (stellarator regime).

Conservation Laws

Invariant	Expression	Field Conditions	Boundary Conditions
Magnetic energy M_Ω	$\frac{1}{2\mu_0}\int_\Omega \boldsymbol{B}^2\,dV$		$\boldsymbol{B}\cdot\boldsymbol{n}=\nabla\times\boldsymbol{B}\cdot\boldsymbol{n}=0$
Magnetic helicity K_Ω	$\frac{1}{2}\int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}\,dV$	$\frac{\partial\boldsymbol{q}}{\partial t}=\frac{\partial\boldsymbol{A}}{\partial t}-A(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=\boldsymbol{0}$	$\boldsymbol{B}\cdot\boldsymbol{n}=\nabla\times\boldsymbol{B}\cdot\boldsymbol{n}=0$

Tab 1. Invariants of system (10). Field conditions for the conservation of magnetic helicity K_Ω specify the gauge $\partial\boldsymbol{q}/\partial t$ of the vector potential $\boldsymbol{A}$.

Invariant	Expression	Field Conditions	Boundary Conditions
Magnetic energy M_Ω	$\frac{1}{2\mu_0}\int_\Omega \boldsymbol{B}^2\,dV$	$\boldsymbol{B}=\nabla\Psi\times\nabla\Theta$	$\Psi=\text{constant}$
Magnetic helicity K_Ω	$\frac{1}{2}\int_\Omega \boldsymbol{A}\cdot\boldsymbol{B}\,dV$	$\boldsymbol{B}=\nabla\Psi\times\nabla\Theta$, $\boldsymbol{A}=\boldsymbol{q}_0(\boldsymbol{x})+\Psi\nabla\Theta$	$\Psi=\text{constant}$, $\partial\Omega$ not connected
Magnetic flux F_Ω	$\int_\Omega f(\Psi)\,dV$	$\boldsymbol{B}=\nabla\Psi\times\nabla\Theta$	$\Psi=\text{constant}$

Tab 2. Invariants of system (22). The magnetic helicity K_Ω degenerates to a trivial invariant $K_\Omega=0$ when $\partial\Omega$ is a connected surface. Here, $\boldsymbol{q}_0(\boldsymbol{x})\in\ker(\text{curl})$.

Proposition 1. *System (22) is a Hamiltonian system with Poisson bracket*

$$\{F, G\} = \mu_0 \int_\Omega A_0(\Psi)\left(\frac{\delta F}{\delta \Psi}\frac{\delta G}{\delta \Theta} - \frac{\delta F}{\delta \Theta}\frac{\delta G}{\delta \Psi}\right) dV, \tag{26}$$

and Hamiltonian

$$H_\Omega = \int_\Omega \left(\frac{1}{2\mu_0}|\nabla\Psi \times \nabla\Theta|^2 - \lambda\Psi\right) dV\,. \tag{27}$$

The noncanonical Hamiltonian form of system (22) is

$$\begin{aligned}\frac{\partial \Psi}{\partial t} &= \{\Psi, H_\Omega\} = \mu_0 A_0(\Psi)\frac{\delta H_\Omega}{\delta \Theta},\\ \frac{\partial \Theta}{\partial t} &= \{\Theta, H_\Omega\} = -\mu_0 \lambda A_0(\Psi)\frac{\delta H_\Omega}{\delta \Psi}.\end{aligned} \tag{28}$$

Nonlinear Stability

Critical points $\chi_0 = (\Psi_0, \Theta_0) \in \mathfrak{X}$ of the Hamiltonian such that $\delta H[\chi_0] = 0$ correspond to steady states of (22). χ_0 is nonlinearly stable if norms $\|\cdot\|_1 : \mathfrak{X} \to \mathbb{R}$ and $\|\cdot\|_2 : \mathfrak{X} \to \mathbb{R}$ and constants $C, C' > 0$ can be found such that

$$C\|\chi(t) - \chi_0\|_1^2 \le |H_\Omega[\chi(t)] - H_\Omega[\chi_0]| = |H_\Omega[\chi(0)] - H_\Omega[\chi_0]| \le C'\|\chi(0) - \chi_0\|_2^2 \qquad \forall t \ge 0. \tag{29}$$

Proposition 2. *Critical points $\chi_0 = (\Psi_0, \Theta_0) \in \mathfrak{X}$ of system (22) are nonlinearly stable against perturbation of Ψ_0 in the distance (seminorm) $\|\Psi\|_\Theta^2 = \frac{1}{2}\int_\Omega |\nabla\Psi \times \nabla\Theta|^2 \mathrm{dV}$. In particular, for all $t \ge 0$*

$$\|\Psi(t) - \Psi_0\|_{\Theta_0}^2 = \frac{1}{2\mu_0}\int_\Omega |\nabla(\Psi(t) - \Psi_0) \times \nabla\Theta_0|^2 dV = |H_\Omega[\Psi(t), \Theta_0] - H_\Omega[\Psi_0, \Theta_0]| = |H_\Omega[\Psi(0), \Theta_0] - H_\Omega[\Psi_0, \Theta_0]| = \frac{1}{2\mu_0}\int_\Omega |\nabla(\Psi(0) - \Psi_0) \times \nabla\Theta_0|^2 dV = \|\Psi(0) - \Psi_0\|_{\Theta_0}^2. \tag{30}$$

Proposition 2. *Critical points $\chi_0 = (\Psi_0, \Theta_0) \in \mathfrak{X}$ of system (22) are nonlinearly stable against perturbation of Θ_0 in the distance (seminorm) $\|\Theta\|_\Psi^2 = \frac{1}{2}\int_\Omega |\nabla\Psi \times \nabla\Theta|^2 \mathrm{dV}$. In particular, for all $t \ge 0$*

$$\|\Theta(t) - \Theta_0\|_{\Psi_0}^2 = \frac{1}{2\mu_0}\int_\Omega |\nabla\Psi_0 \times \nabla(\Theta(t) - \Theta_0)|^2 dV = \cdots = \|\Theta(0) - \Theta_0\|_{\Psi_0}^2. \tag{31}$$

Construction of MHD Equilibria by Double Bracket Dissipation

Given an n-dimensional Hamiltonian system $\dot{z}^i = \mathcal{J}^{ij} H_j$, double bracket dissipation is obtained as follows:

$$\dot{z}^i = \mathcal{J}^{ij} g_{jk} \mathcal{J}^{km} H_m \qquad \Rightarrow \qquad \dot{H} = -\mathcal{J}^{ji} H_i g_{jk} \mathcal{J}^{km} H_m \le 0. \tag{32}$$

System (22) can be written in the form

$$\begin{bmatrix} \Psi_t \\ \Theta_t \end{bmatrix} = \mu_0 A_0(\Psi) \mathcal{J}_s \begin{bmatrix} \delta_\Psi H_\Omega \\ \delta_\Theta H_\Omega \end{bmatrix} = \mu_0 A_0(\Psi) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_\Psi H_\Omega \\ \delta_\Theta H_\Omega \end{bmatrix}. \tag{33}$$

The corresponding double bracket dissipation system is

$$\begin{bmatrix} \Psi_t \\ \Theta_t \end{bmatrix} = \mu_0 A_0(\Psi) \mathcal{J}_s \Pi \mathcal{J}_s \begin{bmatrix} \delta_\Psi H_\Omega \\ \delta_\Theta H_\Omega \end{bmatrix} = \mu_0 A_0(\Psi) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_\Psi H_\Omega \\ \delta_\Theta H_\Omega \end{bmatrix}, \tag{34}$$

where the constant diagonal covariant 2-tensor Π serves the purpose of keeping the consistency of physical units.

Construction of MHD Equilibria by Double Bracket Dissipation

System (34) can be explicitly written as coupled diffusion equations

$$\begin{aligned}\frac{\partial \Psi}{\partial t} &= \gamma A_0(\Psi)\nabla\cdot[\nabla\Theta\times(\nabla\Psi\times\nabla\Theta)] + \gamma\mu_0\lambda A_0(\Psi),\\ \frac{\partial \Theta}{\partial t} &= \sigma A_0(\Psi)\nabla\cdot[\nabla\Psi\times(\nabla\Theta\times\nabla\Psi)].\end{aligned} \tag{35}$$

Proposition 4. *Steady states of system (35) correspond to MHD equilibria*

$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \mu_0\lambda\nabla\Psi, \qquad \nabla\cdot\boldsymbol{B} = 0,$$

with $\boldsymbol{B} = \nabla\Psi\times\nabla\Theta$. Furthermore, the energy H_Ω is progressively dissipated,

$$\frac{dH_\Omega}{dt} \le 0 \qquad \forall t \ge 0.$$

Remarks:

- If solutions of (35) exist in the limit $t\to\infty$ they are nontrivial critical points of H_Ω.

Construction of MHD Equilibria by Iteration

Nontrivial steady solutions of system (22) are given by the system of coupled PDEs for Ψ and Θ:

$$\nabla \cdot [\nabla\Psi \times (\nabla\Theta \times \nabla\Psi)] = 0, \qquad \nabla \cdot [\nabla\Theta \times (\nabla\Psi \times \nabla\Theta)] = \mu_0\lambda. \tag{36}$$

Take $\Psi_0(\boldsymbol{x})$ as initial condition. Construct the sequence $\Theta_0, \Psi_1, \Theta_1, \Psi_2, \Theta_2, \ldots$ by iteratively solving (36), i.e.

$$\begin{gathered} \nabla \cdot [\nabla\Psi_0 \times (\nabla\Theta_0 \times \nabla\Psi_0)] = 0 \\ \nabla \cdot [\nabla\Theta_0 \times (\nabla\Psi_1 \times \nabla\Theta_0)] = \mu_0\lambda \\ \nabla \cdot [\nabla\Psi_1 \times (\nabla\Theta_1 \times \nabla\Psi_1)] = 0 \\ \nabla \cdot [\nabla\Theta_1 \times (\nabla\Psi_2 \times \nabla\Theta_1)] = \mu_0\lambda \\ \vdots \\ \nabla \cdot [\nabla\Theta_{i-1} \times (\nabla\Psi_\mathrm{i} \times \nabla\Theta_{i-1})] = \mu_0\lambda \\ \nabla \cdot [\nabla\Psi_\mathrm{i} \times (\nabla\Theta_\mathrm{i} \times \nabla\Psi_\mathrm{i})] = 0 \\ \vdots \end{gathered} \tag{37}$$

Hopefully, the limit $\lim_{i\to\infty}(\Theta_{i-1}, \Psi_i) = (\Theta_\infty, \Psi_\infty)$ converges to a regular solution of system (36).

Construction of MHD Equilibria by Iteration

Theorem 1. *Assume* $\mu_0\lambda \neq 0$ *and consider an iterative scheme in which the 2 equations is system (36) are solved alternately in* Ω,

$$\nabla \cdot [\nabla\Theta_{i-1} \times (\nabla\Psi_{\mathrm{i}} \times \nabla\Theta_{i-1})] = \mu_0\lambda,$$
$$\nabla \cdot [\nabla\Psi_{\mathrm{i}} \times (\nabla\Theta_{\mathrm{i}} \times \nabla\Psi_{\mathrm{i}})] = 0, \qquad i = 1,2,3,\dots,$$

starting from a given pair $\left(\Theta_0(\boldsymbol{x}), \Psi_0(\boldsymbol{x})\right) \in \mathfrak{X}$ *such that*

$$\nabla \cdot [\nabla\Psi_0 \times (\nabla\Theta_0 \times \nabla\Psi_0)] = 0,$$

with $\nabla\Psi_0 \times \nabla\Theta_0 \neq \mathbf{0}$. *Suppose that during the iteration solutions exist and are nontrivial, i.e.* $\nabla\Psi_i \times \nabla\Theta_i \neq \mathbf{0}$ *for* $i \geq 1$. *Further assume that the limit*

$$(\Theta_\infty, \Psi_\infty) = \lim_{i\to+\infty} (\Theta_{i-1}, \Psi_i)$$

exists. Then, the pair $(\Theta_\infty, \Psi_\infty)$ *solves system (36). Furthermore, the vector field* $\boldsymbol{B} = \nabla\Psi_\infty \times \nabla\Theta_\infty$ *defines a nontrivial MHD equilibrium.*

Proof: see arXiv:2311.03095

Remarks on the Existence of 3D MHD Equilibria

- *If Ω is a hollow toroidal volume with boundary $\partial\Omega$ corresponding to 2 distinct level sets of a smooth function $\Psi_0 \in C^\infty(\Omega)$, with $\nabla\Psi_0 \neq \mathbf{0}$ in Ω, and if level sets of Ψ_0 foliate Ω with nested toroidal surfaces, theorem 1 of [*JMP **64** 081505 (2023)*] ensures that equation $\nabla \cdot [\nabla\Psi_0 \times (\nabla\Theta_0 \times \nabla\Psi_0)] = 0$ always has a nontrivial solution Θ_0 such that $\nabla\Psi_0 \times \nabla\Theta_0 \neq \mathbf{0}$. Furthermore, the angle variable Θ_0 is not unique, but solutions exist in the form $\Theta_0 = M\mu + N\nu + \chi_0$, where μ, ν are toroidal and poloidal angle variables, the functions $M(\Psi)$, $N(\Psi)$ determine the rotational transform of the vector field $\nabla\Psi_0 \times \nabla\Theta_0$, and the function $\chi_0(\boldsymbol{x})$ is single-valued. The same result applies when solving for Θ_i at any step of the iteration provided that Ψ_i satisfies the same properties listed above for Ψ_0 in Ω.*

- *An argument analogous to that used in the proof of theorem 1 in [*JMP **64** 081505 (2023)*] shows that for a given angle variable Θ_{i-1} in $\nabla \cdot [\nabla\Theta_{i-1} \times (\nabla\Psi_i \times \nabla\Theta_{i-1})] = \mu_0\lambda$, a solution Ψ_i can be obtained by reducing the equation to a 2-dimensional elliptic equation on each level set of Θ_{i-1} and by joining solutions corresponding to adjacent level sets.*

- *In light of the 2 remarks above, if one could show that at each step of the iteration the solutions Θ_{i-1} and Ψ_i, $i \geq 1$ preserve their properties (in particular, Θ_i remains an angle variable and Ψ_i foliates Ω with nested toroidal surfaces) then, combining this result with theorem 1 proved in this section, one would have obtained a proof of the existence of MHD equilibria in hollow toroidal volumes of arbitrary shape. In such construction, although no control is available on the form of the flux surfaces Ψ_∞ within Ω, one can conjecture that, if they exist, solutions $\boldsymbol{B} = \nabla\Psi_\infty \times \nabla\Theta_\infty$ with different rotational transforms can be obtained by appropriate choice of M and N.*

Concluding Remarks

- Reduced iMHD induction eq. (22) for nonlinear evolution of $\boldsymbol{B}$ by Clebsch potentials.
- System (22) is a toy model of turbulence that can be useful to assess dynamical accessibility and stability of MHD equilibria in physically relevant regimes.
- Ordering I relevant for stellarator plasmas (small flow, small electric current, and approximate flux surfaces). Ordering II is more general.
- System (22) preserves magnetic energy, magnetic helicity, and total magnetic flux.
- System (22) has a noncanonical Hamiltonian structure
- Steady solutions are nonlinearly stable against perturbations involving a single Clensch potential.
- Double bracket dissipation gives a diffusive dynamical system (35) that can be used to compute nontrivial MHD equilibria by minimizing the Hamiltonian.
- Iterative scheme to compute nontrivial MHD equilibria. May serve as basis for proof of existence of MHD equilibria with a non-vanishing pressure gradient in general hollow tori.

Thank you for your attention!