Ocneanu Algebra of Seams: Critical E_6 RSOS Lattice Model

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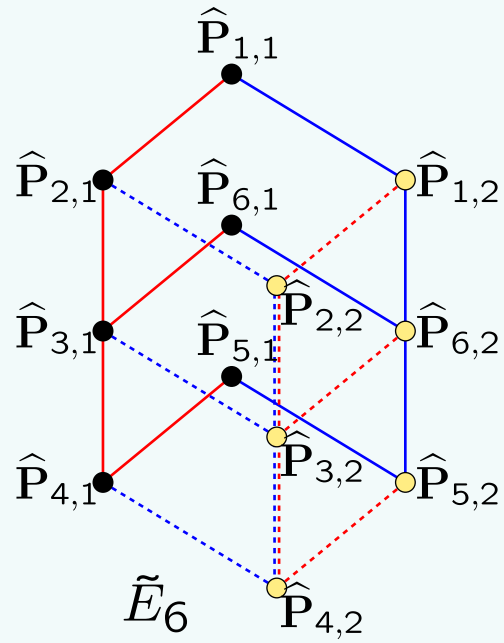

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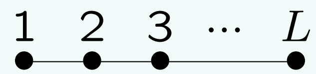
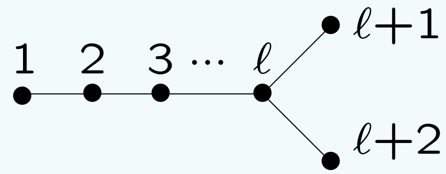
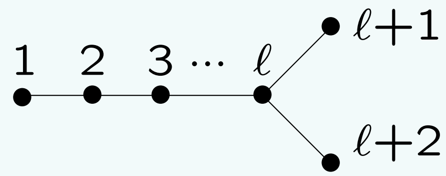
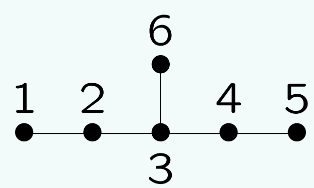
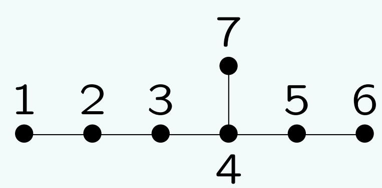
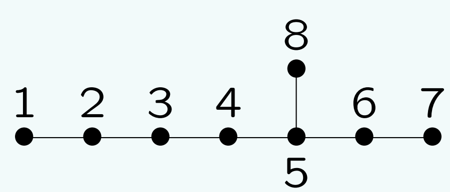
Simply Laced A-D-E Lie Algebras: Dynkin Diagrams

	Graph G	g	Exp(G)	Type/H	Γ
A_L		$L+1$	$1, 2, \cdots, L$	I	$\mathbb{Z}_2$
$D_{\ell+2}$ (ℓ odd)		$2\ell+2$	$1, 3, \cdots, 2\ell+1, \ell+1$	II/$A_{2\ell+1}$	$\mathbb{Z}_2$
$D_{\ell+2}$ (ℓ even)		$2\ell+2$	$1, 3, \cdots, 2\ell+1, \ell+1$	I	$\mathbb{Z}_2/\mathbb{S}_3$
E_6		12	$1, 4, 5, 7, 8, 11$	I	$\mathbb{Z}_2$
E_7		18	$1, 5, 7, 9, 11, 13, 17$	II/D_{10}	1
E_8		30	$1, 7, 11, 13, 17, 19, 23, 29$	I	1

- A-D-E Dynkin diagrams G, Coxeter numbers g, Coxeter exponents Exp(G), the type I or II, parent graphs $H \neq G$ and diagram automorphism group Γ. Nodes $1 =$ identity, $2 =$ fundamental.
- The eigenvalues of the adjacency matrix G are $2\cos\frac{m\pi}{g}$ with $m \in$ Exp(G).

Fusion Matrices (nimreps)

Verlinde Matrices: $G=A_L$ ($L\times L$ Matrices): $N_i=(N_i)_j{}^k,\qquad i,j,k\in A_L$

$$N_1=I,\quad N_2=A_L,\quad N_j=N_{j-1}N_2-N_{j-2},\quad N_L=\sigma\in\mathbb{Z}_2$$

$$N_i\,N_j=\sum_{k\in A_L} N_{ij}{}^k\,N_k,\quad L=g-1,\qquad N_j=U_{j-1}(\tfrac{1}{2}N_2)=\text{Chebyshev}$$

Tricritical Ising Model: $G=A_4,\quad g=5$

$$N_1=I,\quad N_2=\begin{pmatrix}0&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&0\end{pmatrix},\quad N_3=\begin{pmatrix}0&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&0\end{pmatrix},\quad N_4=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}=\sigma$$

Fused Adjacency/Intertwiner Matrices:

$$|G|\times|G|\ \text{Matrices:}\qquad n_i=(n_i)_a{}^b,\quad i=1,2,\ldots,g-1;\quad a,b\in G$$

$$n_1=I,\quad n_2=G,\quad n_j=n_2n_{j-1}-n_{j-2},\quad 3\le j\le g-1$$

$$n_i\,n_j=\sum_{k\in A_{g-1}} N_{ij}{}^k\,n_k \qquad\qquad n_{g-1}=\begin{cases}I, & D_{2\ell},E_7,E_8\\ \sigma\in\mathbb{Z}_2, & A_L,D_{2\ell-1},E_6\end{cases}$$

Critical 3-State Potts Model: $G=D_4,\quad g=6$

$$n_1=n_5=I,\quad n_2=n_4=\begin{pmatrix}0&1&0&0\\1&0&1&1\\0&1&0&0\\0&1&0&0\end{pmatrix},\quad n_3=\begin{pmatrix}0&0&1&1\\0&2&0&0\\1&0&0&1\\1&0&1&0\end{pmatrix}$$

Graph Fusion Matrices (nimreps)

Graph Fusion Matrices: Entries of $\hat{N}_a$ are non-negative integers only for Type I models:

$$|G| \times |G| \text{ Matrices:} \quad \hat{N}_a = (\hat{N}_a)_b{}^c, \quad \hat{N}_1 = I, \quad \hat{N}_2 = G, \quad \hat{N}_2\hat{N}_a = \sum_{b\in G} G_{ab}\,\hat{N}_b, \quad a,b,c \in G$$

$$\hat{N}_a\hat{N}_b = \sum_{c\in G} \hat{N}_{ab}{}^c\,\hat{N}_c, \qquad n_i\hat{N}_a = \sum_{b\in G} n_{ia}{}^b\hat{N}_b, \qquad n_i = \sum_{a\in G} n_{i1}{}^a\hat{N}_a$$

For E_6, the rectangular fundamental intertwiner $n_{i1}{}^a$ admits a generalized left inverse giving

$$\hat{N}_a = n_a, \quad a = 1,2,3; \qquad \hat{N}_4 = n_6 - n_2, \quad \hat{N}_5 = n_5 - n_3, \quad \hat{N}_6 = n_2 + n_4 - n_6$$

Critical Ising: $G = A_3, \quad g = 4$ **Fusion Rules:** $\sigma^2 = I + \varepsilon, \quad \sigma\varepsilon = \varepsilon\sigma = \sigma, \quad \varepsilon^2 = I$

$$\hat{N}_1 = N_1 = I, \quad \hat{N}_2 = N_2 = \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} = \sigma, \quad \hat{N}_3 = N_3 = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} = \varepsilon$$

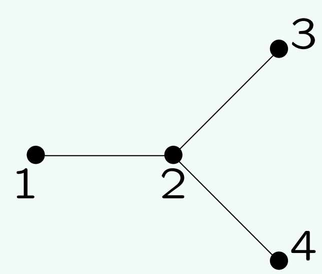
Critical 3-State Potts: $G = D_4, \quad g = 6, \quad \omega \in \mathbb{Z}_3$

$$\hat{N}_1 = I, \quad \hat{N}_2 = N_2 = \begin{pmatrix}0&1&0&0\\1&0&1&1\\0&1&0&0\\0&1&0&0\end{pmatrix} = G, \quad \hat{N}_3 = \begin{pmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{pmatrix} = \omega, \quad \hat{N}_4 = \begin{pmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{pmatrix} = \omega^2$$

Cayley Table/Fundamental Intertwiner

D_4 :

	I	G	ω	ω^2
I	I	G	ω	ω^2
G	G	$I+\omega+\omega^2$	G	G
ω	ω	G	ω^2	I
ω^2	ω^2	G	I	ω



$$\sigma = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \qquad C = (n_i)_1{}^a = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

Critical A-D-E Lattice Models

- The face weights of the critical A-D-E lattice models [ABF84,Pasquier87] are

$$W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = {}^{d}_{a}\boxed{u}^{c}_{b} = \frac{\sin(\lambda-u)}{\sin\lambda}\,\delta_{ac} + \frac{\sin u}{\sin\lambda}\sqrt{\frac{\psi_a\psi_c}{\psi_b\psi_d}}\,\delta_{bd}, \qquad \lambda = \frac{\pi}{g}, \qquad 0 \le u \le \lambda$$

where u is the spectral parameter. The face weights vanish if $G_{ab}G_{bc}G_{cd}G_{da} = 0$ and satisfy the Yang-Baxter equation implying commuting transfer matrices and integrability.

- The largest eigenvalue of the adjacency matrix G is $[2]_x = 2\cos\lambda$ with eigenvectors ψ

$$G\boldsymbol{\psi} = [2]_x\,\boldsymbol{\psi}, \qquad \boldsymbol{\psi} = (\psi_a)_{1\le a\le |G|} = \begin{cases} \big([1]_x, [2]_x, \ldots, [L]_x\big), & G = A_L \\ \big([1]_x, [2]_x, \ldots, [\ell]_x, \frac{[\ell]_x}{[2]_x}, \frac{[\ell]_x}{[2]_x}\big), & G = D_{\ell+2} \\ \big([1]_x, [2]_x, [3]_x, [2]_x, [1]_x, \frac{[3]_x}{[2]_x}\big), & G = E_6 \\ \big([1]_x, [2]_x, [3]_x, [4]_x, \frac{[6]_x}{[2]_x}, \frac{[4]_x}{[3]_x}, \frac{[4]_x}{[2]_x}\big), & G = E_7 \\ \big([1]_x, [2]_x, [3]_x, [4]_x, [5]_x, \frac{[7]_x}{[2]_x}, \frac{[5]_x}{[3]_x}, \frac{[5]_x}{[2]_x}\big), & G = E_8 \end{cases}$$

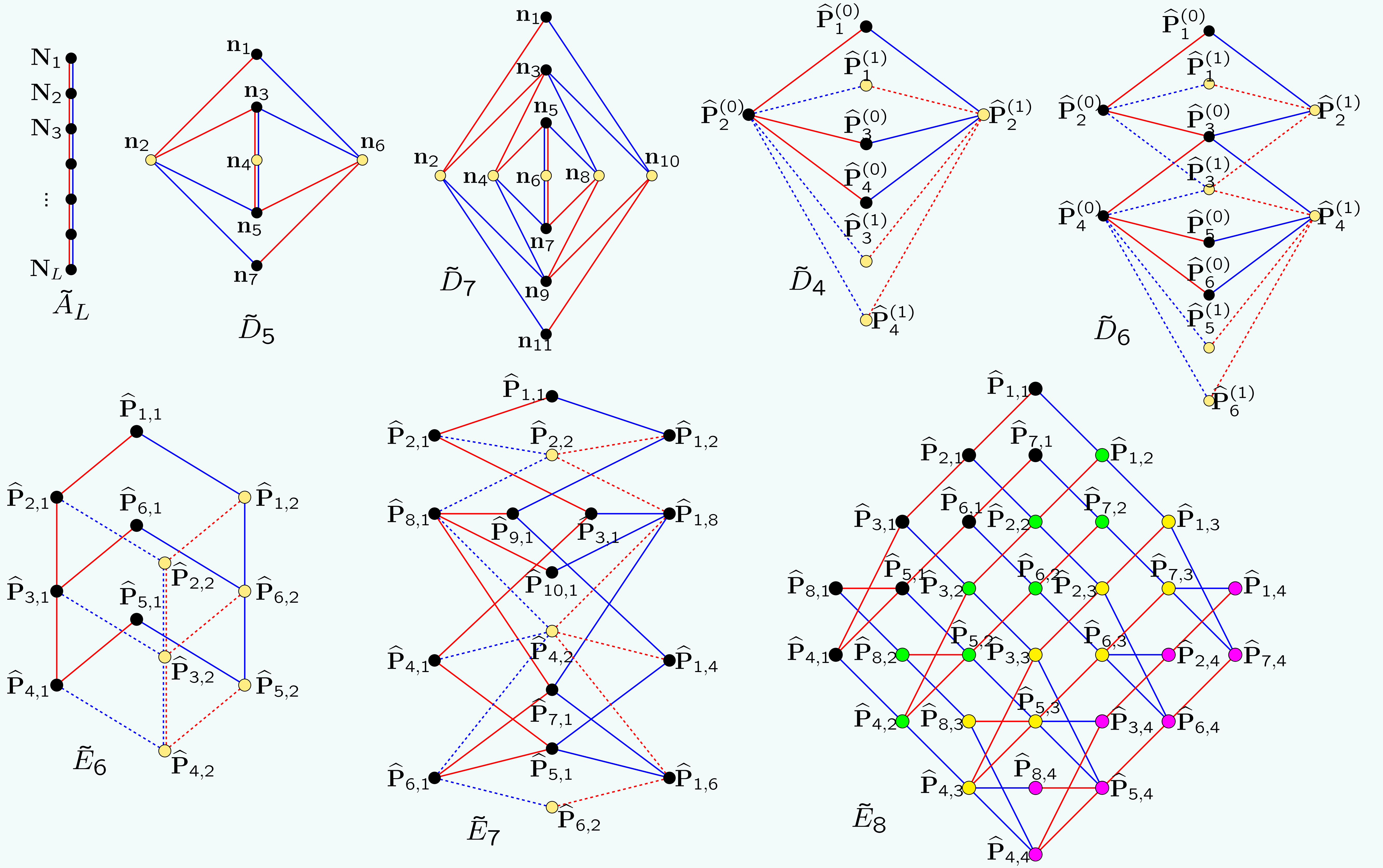
Here $[a]_x = \frac{x^a - x^{-a}}{x - x^{-1}}$ with $x = e^{i\lambda}$.

- Setting $\rho(u) = \sin(\lambda-u)/\sin\lambda$, the braid limits of the A-D-E face weights are given by the complex conjugate weights

$$B\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\right) = {}^{d}_{a}\boxed{-i\infty}^{c}_{b} = \lim_{u\to -i\infty} \frac{x^{-\frac{1}{2}}}{i\rho(u)}\, W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = -i\left(x^{-\frac{1}{2}}\,\delta_{ac} - x^{\frac{1}{2}}\sqrt{\frac{\psi_a\psi_c}{\psi_b\psi_d}}\,\delta_{bd}\right)$$

$$\overline{B}\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\right) = {}^{d}_{a}\boxed{i\infty}^{c}_{b} = \lim_{u\to i\infty} \frac{i\,x^{\frac{1}{2}}}{\rho(u)}\, W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = i\left(x^{\frac{1}{2}}\,\delta_{ac} - x^{-\frac{1}{2}}\sqrt{\frac{\psi_a\psi_c}{\psi_b\psi_d}}\,\delta_{bd}\right)$$

Ocneanu Fusion Graphs $\tilde{G}$



- The red/blue lines show the action of the left/right chiral fundamentals labelled by $a = 2, \bar{2}$.

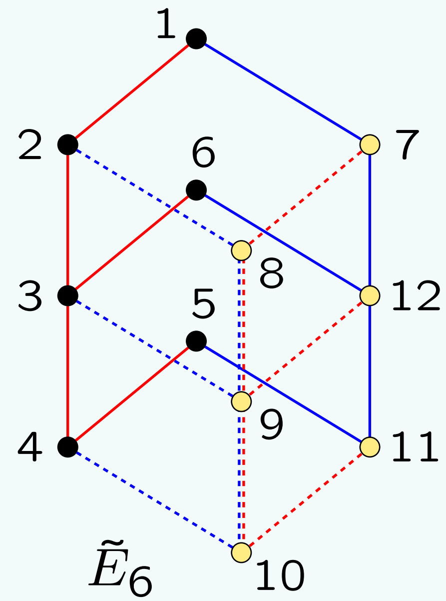
$\tilde{E}_6$ Ocneanu nimrpes

- The 12 Ocneanu graph fusion matrices (nimreps) $\tilde{N}_a$ for $\tilde{E}_6$ are

$$\begin{pmatrix}100000000000\\010000000000\\001000000000\\000100000000\\000010000000\\000001000000\\000000100000\\000000010000\\000000001000\\000000000100\\000000000010\\000000000001\end{pmatrix}\begin{pmatrix}010000000000\\101000000000\\010101000000\\001010000000\\000100000000\\001000000000\\000000010000\\000000101000\\000000010101\\000000001010\\000000000100\\000000001000\end{pmatrix}\begin{pmatrix}001000000000\\010101000000\\102010000000\\010101000000\\001000000000\\010100000000\\000000001000\\000000010101\\000000102010\\000000010101\\000000001000\\000000010100\end{pmatrix}\begin{pmatrix}000100000000\\001010000000\\010101000000\\101000000000\\010000000000\\001000000000\\000000000100\\000000001010\\000000010101\\000000101000\\000000010000\\000000001000\end{pmatrix}\begin{pmatrix}000010000000\\000100000000\\001000000000\\010000000000\\100000000000\\000001000000\\000000000010\\000000000100\\000000001000\\000000010000\\000000100000\\000000000001\end{pmatrix}\begin{pmatrix}000001000000\\001000000000\\010100000000\\001000000000\\000001000000\\100010000000\\000000000001\\000000001000\\000000010100\\000000001000\\000000000001\\000000100010\end{pmatrix}$$

$$\begin{pmatrix}000000100000\\000000010000\\000000001000\\000000000100\\000000000010\\000000000001\\100000000001\\010000001000\\001000010100\\000100001000\\000010000001\\000001100010\end{pmatrix}\begin{pmatrix}000000010000\\000000101000\\000000010101\\000000001010\\000000000100\\000000001000\\010000001000\\101000010101\\010101102010\\001010010101\\000100001000\\001000010100\end{pmatrix}\begin{pmatrix}000000001000\\000000010101\\000000102010\\000000010101\\000000001000\\000000010100\\001000010100\\010101102010\\102010020202\\010101102010\\001000010100\\010100002000\end{pmatrix}\begin{pmatrix}000000000100\\000000001010\\000000010101\\000000101000\\000000010000\\000000001000\\000100001000\\001010010101\\010101102010\\101000010101\\010000001000\\001000010100\end{pmatrix}\begin{pmatrix}000000000010\\000000000100\\000000001000\\000000010000\\000000100000\\000000000001\\000010000001\\000100001000\\001000010100\\010000001000\\100000000001\\000001100010\end{pmatrix}\begin{pmatrix}000000000001\\000000001000\\000000010100\\000000001000\\000000000001\\000000100010\\000001100010\\001000010100\\010100002000\\001000010100\\000001100010\\100010000002\end{pmatrix}$$

$$\Sigma = \begin{pmatrix}100000000000\\000000100000\\000000000001\\000000000010\\000010000000\\000001000000\\010000000000\\000000010000\\000000001000\\000000000100\\000100000000\\001000000000\end{pmatrix}$$



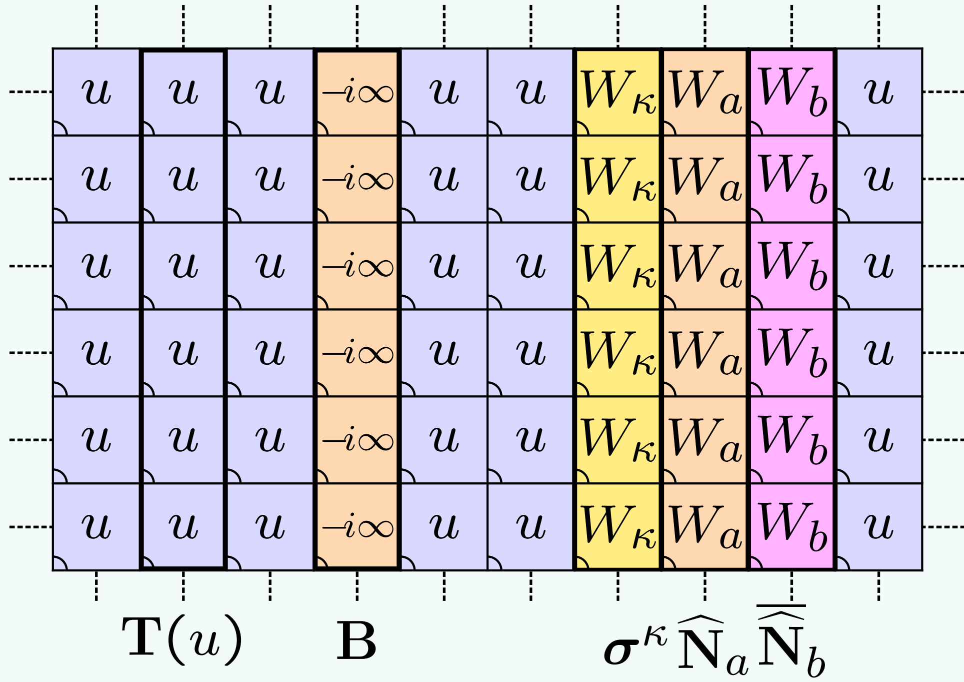
$$\tilde{N}_a\,\tilde{N}_b = \sum_{c=1}^{12} \tilde{N}_{ab}{}^{c}\,\tilde{N}_c$$

$$\tilde{N}_a = \begin{cases} \hat{N}_a \oplus \hat{N}_a, & a = 1, 2, \ldots, 6 \\ \tilde{N}_7 = \Sigma\,\tilde{N}_2 \Sigma, & \\ \tilde{N}_7 \tilde{N}_{a-6}, & a = 8, 9, \ldots, 12 \end{cases}$$

- $\Sigma = \mathbb{Z}_2$ chiral conjugation, $\Sigma\tilde{N}_a\Sigma = \tilde{N}_{\bar{a}}$, $\bar{2} = 7$, $\bar{3} = 12$, $\bar{4} = 11$ else $\bar{a} = a$ (ambichirals).

Commuting Column Transfer Matrices

$[\mathbf{T}(u), \mathbf{T}(v)] = 0$



- A periodic lattice showing (i) the column/seam transfer matrices $\mathbf{T}(u)$, $\widehat{\mathbf{N}}_a$, $\widehat{\overline{\mathbf{N}}}_b$ and (ii) the automorphism seam $\boldsymbol{\sigma}^\kappa$ with $\kappa = 0, 1$. The composite seam is the matrix product $\mathbf{T}_x = \boldsymbol{\sigma}^\kappa \widehat{\mathbf{N}}_a \widehat{\overline{\mathbf{N}}}_b$.

- The braid seams $\mathbf{B}, \overline{\mathbf{B}}$ are the limits $\lim_{u\to\mp i\infty} \mathbf{T}(u)$, that is, the face weights are replaced by the complex braid face weights $B, \overline{B}$. The seams $\mathbf{B}, \overline{\mathbf{B}}$ commute with $\mathbf{T}(u)$ and with each other. The fused s-type braid seams $\mathbf{n}_s$ (and their complex conjugates $\bar{\mathbf{n}}_s$) are defined recursively by

$$\mathbf{n}_0 = 0, \qquad \mathbf{n}_1 = I, \qquad \mathbf{n}_2 = \mathbf{B}, \qquad \mathbf{n}_s = \mathbf{n}_2\mathbf{n}_{s-1} - \mathbf{n}_{s-2}, \qquad 3 \le s \le g-1$$

The $\widehat{\mathbf{N}}_a$ seams are given by the same linear combinations of $\mathbf{n}_s$ as the graph fusion matrices.

- For each A-D-E lattice model, the seams $\mathbf{n}_s, \bar{\mathbf{n}}_s$ and $\widehat{\mathbf{N}}_a, \widehat{\overline{\mathbf{N}}}_b$ satisfy the Verlinde and graph fusion algebras respectively for arbitrary system sizes:

$$\mathbf{n}_i\, \mathbf{n}_j = \sum_{k\in A_{g-1}} {N_{ij}}^k\, \mathbf{n}_k, \quad 1 \le i, j \le g-1, \qquad \widehat{\mathbf{N}}_a \widehat{\mathbf{N}}_b = \sum_{c\in G} {\hat{N}_{ab}}{}^c\, \widehat{\mathbf{N}}_c, \quad a, b \in G$$

$\tilde{E}_6$ Ocneanu Algebra of Seams

- Following the relations for $\widehat{N}_a$:

$$\widehat{\mathbf{N}}_a = \mathbf{n}_a, \quad a = 1,2,3; \qquad \widehat{\mathbf{N}}_4 = \mathbf{n}_6 - \mathbf{n}_2, \quad \widehat{\mathbf{N}}_5 = \mathbf{n}_5 - \mathbf{n}_3 = \boldsymbol{\sigma}, \quad \widehat{\mathbf{N}}_6 = \mathbf{n}_2 + \mathbf{n}_4 - \mathbf{n}_6$$

with $\overline{\widehat{\mathbf{N}}}_b = \widehat{\mathbf{N}}_{\bar{b}}$, $b = 1,2,\ldots,6$. The 12 $\tilde{E}_6$ seams $\hat{\mathbf{P}}_{a,b} = \widehat{\mathbf{N}}_a \overline{\widehat{\mathbf{N}}}_b$ ($a = 1,2,\ldots,6$; $b = 1,2$) are

$$\{\hat{\mathbf{P}}_1, \hat{\mathbf{P}}_2, \ldots, \hat{\mathbf{P}}_{12}\} = \{\hat{\mathbf{P}}_{1,1}, \hat{\mathbf{P}}_{2,1}, \ldots, \hat{\mathbf{P}}_{6,1}, \hat{\mathbf{P}}_{1,2}, \hat{\mathbf{P}}_{2,2}, \ldots, \hat{\mathbf{P}}_{6,2}\} \qquad \text{(Basis Seams)}$$

- The seams satisfy the Ocneanu algebra (checked in Mathematica for system sizes $M \le 12$)

$$\hat{\mathbf{P}}_\eta \hat{\mathbf{P}}_\mu = \sum_{\nu=1}^{12} \tilde{N}_{\eta\mu}{}^{\nu} \hat{\mathbf{P}}_\nu, \qquad \tilde{N}_\eta \tilde{N}_\mu = \sum_{\nu=1}^{12} \tilde{N}_{\eta\mu}{}^{\nu} \tilde{N}_\nu, \qquad \eta,\mu = 1,2,\ldots,12$$

- Cayley table of the commutative $\tilde{E}_6$ Ocneanu algebra using the notation $\eta = \hat{\mathbf{P}}_\eta$:

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1+3	2+4+6	3+5	4	3	8	7+9	8+10+12	9+11	10	9
3	3	2+4+6	1+2(3)+5	2+4+6	3	2+4	9	8+10+12	7+2(9)+11	8+10+12	9	8+10
4	4	3+5	2+4+6	1+3	2	3	10	9+11	8+10+12	7+9	8	9
5	5	4	3	2	1	6	11	10	9	8	7	12
6	6	3	2+4	3	6	1+5	12	9	8+10	9	12	7+11
7	7	8	9	10	11	12	1+12	2+9	3+8+10	4+9	5+12	6+7+11
8	8	7+9	8+10+12	9+11	10	9	2+9	1+3+8 +10+12	2+4+6 +7+2(9)+11	3+5+8 +10+12	4+9	3+8+10
9	9	8+10+12	7+2(9)+11	8+10+12	9	8+10	3+8+10	2+4+6 +7+2(9)+11	1+2(3)+5 +2(8)+2(10)+2(12)	2+4+6 +7+2(9)+11	3+8+10	2+4+2(9)
10	10	9+11	8+10+12	7+9	8	9	4+9	3+5+8 +10+12	2+4+6 +7+2(9)+11	1+3+8 +10+12	2+9	3+8+10
11	11	10	9	8	7	12	5+12	4+9	3+8+10	2+9	1+12	6+7+11
12	12	9	8+10	9	12	7+11	6+7+11	3+8+10	2+4+2(9)	3+8+10	6+7+11	1+5+2(12)

- For $1 \le \eta \le 6$ and $\mu \ge 3$, $\hat{\mathbf{P}}_{\eta,\mu}$ is given in terms of the Basis Seams by the Quantum Symmetry

$$\hat{\mathbf{P}}_{\eta,\mu} = \widehat{\mathbf{N}}_\eta \overline{\widehat{\mathbf{N}}}_\mu = \widehat{\mathbf{N}}_\eta \widehat{\mathbf{N}}_{\bar{\mu}} = \sum_{\nu=1}^{12} \tilde{N}_{\eta\bar{\mu}}{}^{\nu} \hat{\mathbf{P}}_\nu, \qquad \eta,\mu = 1,2,\ldots,12$$

$\tilde{E}_6$ Ocneanu Algebra as a Quotient

- There is a ring homomorphism

$$\phi : \left\langle \widehat{\mathbf{P}}_{a,1}, \widehat{\mathbf{P}}_{a,2} \,|\, a = 1,\ldots,6 \right\rangle \to \mathbb{Z}[x,y] \Big/ \left\langle p_1(x), p_2(x,y) \right\rangle$$

$$p_1(x) = \prod_{s \in \mathrm{Exp}(E_6)} \left(x - 2\cos\tfrac{s\pi}{g}\right) = x^6 - 5x^4 + 5x^2 - 1, \qquad \mathrm{Exp}(E_6) = \{1,4,5,7,8,11\}$$

$$p_2(x,y) = y^2 + (x^5 - 5x^3 + 4x)y - 1, \qquad p_2(y,y) = p_1(y)$$

$$\widehat{\mathbf{P}}_{1,1} \mapsto 1, \quad \widehat{\mathbf{P}}_{2,1} \mapsto x, \quad \widehat{\mathbf{P}}_{3,1} \mapsto x^2 - 1, \quad \widehat{\mathbf{P}}_{4,1} \mapsto x^5 - 4x^3 + 2x$$

$$\widehat{\mathbf{P}}_{5,1} \mapsto x^4 - 4x^2 + 2, \quad \widehat{\mathbf{P}}_{6,1} \mapsto -x^5 + 5x^3 - 4x, \quad \widehat{\mathbf{P}}_{a,2} \mapsto \phi(\widehat{\mathbf{P}}_{a,1})\, y, \quad a = 1,2,\ldots,6$$

- The $\tilde{E}_6$ integrable seams mutually commute and are simultaneously diagonalizable.

The eigenvalues yield a 1-d representation of the $\tilde{E}_6$ Ocneanu algebra in terms of the quantum dimensions. Set $x = q^s + q^{-s} = 2\cos\frac{s\pi}{12}$, $y = q^{\bar{s}} + q^{-\bar{s}} = 2\cos\frac{\bar{s}\pi}{12}$ with $q = e^{\pi i/12}$

- Solving

$$p_1(x) = 0, \qquad p_2(x,y) = 0$$

yields $6 \times 2 = 12$ solutions for x, y:

$$(s,\bar{s}) \in \mathcal{S} = \{(1,1),(4,4),(5,5),(7,7),(8,8),(11,11),(1,7),(7,1),(4,8),(8,4),(5,11),(11,5)\}$$

These indices coincide with the operator content of the E_6 modular invariant partition function

$$Z(q) = \sum_{r=1}^{g-2} \sum_{(s,\bar{s}) \in \mathcal{S}} \chi_{r,s}(q)\chi_{r,\bar{s}}(\bar{q})$$

$\tilde{E}_6$ Structure Constants

For $\tilde{E}_6$: $\quad p_2(\mathbf{B},\overline{\mathbf{B}}) = \overline{\mathbf{B}}^2 - \widehat{\mathbf{P}}_{6,1}\overline{\mathbf{B}} - \mathbf{I} = \mathbf{0} \quad \Rightarrow \quad \overline{\widehat{\mathbf{N}}}_2^{\,2} = \mathbf{I} + \widehat{\mathbf{N}}_6\overline{\widehat{\mathbf{N}}}_2$

- Using this, the product of any two seams $\widehat{\mathbf{P}}_{a,a'}$ and $\widehat{\mathbf{P}}_{b,b'}$ is obtained using commutativity and associativity. Let $a,b,c = 1,2,\ldots,6$ and $a',b',c' = 1,2$ and consider the four quadrants (a',b') of the Cayley table. By commutativity the $(1,2)$ and $(2,1)$ quadrants agree leaving 3 cases:

$$a'b' = 1: \quad \widehat{\mathbf{N}}_a\widehat{\mathbf{N}}_b = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{N}}_c = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{P}}_{c,1}$$

$$a'b' = 2: \quad (\widehat{\mathbf{N}}_a\overline{\widehat{\mathbf{N}}}_2)\widehat{\mathbf{N}}_b = \widehat{\mathbf{N}}_a(\widehat{\mathbf{N}}_b\overline{\widehat{\mathbf{N}}}_2) = \sum_c \widehat{N}_{ab}{}^c(\widehat{\mathbf{N}}_c\overline{\widehat{\mathbf{N}}}_2) = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{P}}_{c,2}$$

$$a'b' = 4: \quad (\widehat{\mathbf{N}}_a\overline{\widehat{\mathbf{N}}}_2)(\widehat{\mathbf{N}}_b\overline{\widehat{\mathbf{N}}}_2) = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{N}}_c\,\overline{\widehat{\mathbf{N}}}_2^{\,2} = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{N}}_c\,(\mathbf{I} + \widehat{\mathbf{N}}_6\overline{\widehat{\mathbf{N}}}_2)$$

$$= \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{N}}_c + \sum_{c,d} \widehat{N}_{ab}{}^c\,\widehat{N}_{c6}{}^d(\widehat{\mathbf{N}}_d\overline{\widehat{\mathbf{N}}}_2) = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{P}}_{c,1} + \sum_{c,d} \widehat{N}_{ab}{}^c\,\widehat{N}_{c6}{}^d\widehat{\mathbf{P}}_{d,2}$$

This gives the $\tilde{E}_6$ Ocneanu algebra structure constants in terms of $\widehat{N}_{ab}{}^c$ as

$$\widetilde{N}_{a,a';\,b,b'}{}^{c,c'} = (\delta_{a',b'}\delta_{c',1} + \delta_{a'b',2}\delta_{c',2})\widehat{N}_{ab}{}^c + \delta_{a'b',4}\delta_{c',2}\sum_{d=1}^{6} \widehat{N}_{ab}{}^d\widehat{N}_{d6}{}^c = \text{twelve } 12\times 12 \text{ matrices}$$

- Let us prove $p_2(\mathbf{B},\overline{\mathbf{B}}) = \mathbf{0}$. For a given u-independent eigenvector of $\mathbf{T}(u)$, the eigenvalues of $\mathbf{B},\overline{\mathbf{B}}$ are $2\cos\frac{s\pi}{12}$, $2\cos\frac{\bar{s}\pi}{12}$ with $(s,\bar{s}) \in \mathcal{S}$. But it is readily verified that

$$p_2(2\cos\tfrac{s\pi}{12}, 2\cos\tfrac{\bar{s}\pi}{12}) = 0, \qquad (s,\bar{s}) \in \mathcal{S}$$

The proof that the integrable seams satisfy the $\tilde{E}_6$ Ocneanu algebra, for arbitrary system sizes M, thus follows by simultaneous diagonalization since the seam eigenvalues (quantum dimensions) satisfy the $\tilde{E}_6$ Ocneanu algebra.

Conclusion

- It is argued that, for all critical A-D-E lattice models and arbitrary system sizes M:

(i) the s-type braid transfer matrices $\mathbf{n}_s$ satisfy the Verlinde algebra,

(ii) the a-type integrable seams $\widehat{\mathbf{N}}_a$ satisfy the graph fusion algebra,

(iii) the integrable seams $\widehat{\mathbf{P}}_{a,b}$ satisfy the Ocneanu algebra and quantum symmetry.

All of these statements have been checked in Mathematica for $M \leq 12$.

- Expressions can be obtained for the local face weights of each of the integrable seams. For Type I theories, explicit expressions are obtained for the Ocneanu algebra structure constants.

- Conformal topological defects result from the integrable seams in the continuum scaling limit.

- The algebra of seams gives a natural physical interpretation of the Ocneanu algebra — it implements the local operator product expansion on conformal defects. The Ocneanu algebra encodes how, for twisted boundaries, the left and right chiral halves of a CFT are glued together.

- Since the seams commute with the transfer matrix $\mathbf{T}(u)$, they are in some sense symmetries. But some seams admit zero eigenvalues, so these seams are generalized noninvertible symmetries.

- The A-D-E models are $s\ell(2)$ spin-$\frac{1}{2}$ models. The new insights give tools to systematically investigate the Ocneanu algebras of higher spin and higher rank models, eg. spin-1 and $s\ell(3)$ models as well as $s\ell(2)$ affine A-D-E models.