On $T\bar{T}$ and higher-spin deformations of integrable field theories

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$T\bar{T}$-like collaborators since 2018:
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The aim of the talk is to give a short overview of how to construct new infinite families of $2d$ integrable field theories based on $T\bar{T}$ deformations and its extensions.

The tools will be deformations driven by conserved currents engineered by appropriate couplings with auxiliary fields via analogies with results in $4d$ self-dual electrodynamics.

What and when $T\bar{T}$

and higher-spin generalisations?

When $T\bar{T}$?

Alexander Zamolodchikov: hep-th/0401146 "Expectation value of composite field T anti-T in two-dimensional quantum field theory"

What and when $T\bar{T} = \det[T_{\mu\nu}]$

Given a generic $2d$ QFT.

Thanks to translational symmetry: $\exists\ T_{\mu\nu}(x)$, local Energy-Momentum tensor, such that

$$\partial^{\mu} T_{\mu\nu}(x) = 0$$

THEN:

Always exist the local $T\bar{T}$ operator

$$\mathcal{O}_{T\bar{T}}(x) := \det[T_{\mu\nu}(x)]$$

For a CFT_2: $\mathcal{O}_{T\bar{T}}(x) \propto T\bar{T}$

- Can be proven to be quantum well-defined irrelevant operator

  remarkably against standard lore of Renormalisation in QFT

What and when $T\bar{T} = \det[T_{\mu\nu}]$

$T\bar{T}$ can then be used as irrelevant deformation of QFT_2

$$\mathcal{L}^{(0)} \to \mathcal{L}^{(\lambda)} = \mathcal{L}^{(0)} + \lambda\, \mathcal{O}^{(0)}_{T\bar{T}}(x) + \cdots$$

iterating the deformation, one obtains $T\bar{T}$-flow equation

$$\frac{\partial S^{(\lambda)}}{\partial \lambda} = \int \mathrm{d}^2 x\, \mathcal{O}^{(\lambda)}_{T\bar{T}}(x) = \int \mathrm{d}^2 x \, \det\left[T^{(\lambda)}_{\mu\nu}\right](x)$$

This will be the type of deformations we will look at and generalise

What and when $T\bar{T} = \det[T_{\mu\nu}]$:

After 2004, no developments... till 2016 when
[Smirnov-Zamolodchikov] [Cavaglià-Negro-Szécsényi-Tateo]
noticed (among other results) that:

- If $\mathcal{L}^{(0)}$ is an integrable $2D$ QFT (IQFT) $\Longrightarrow$ all $\mathcal{L}^{(\lambda)}$ are integrable!

It was also shown that the spectrum on cylinder is "solvable" for any λ if one knows E_n and P_n at a specific value of λ_0. For CFT_2:

$$E_n(R,\lambda) = \frac{R}{2\lambda}\left(\sqrt{1+\frac{4\lambda E_n(R,0)}{R}+\frac{4\lambda^2(P_n(R,0))^2}{R^2}}-1\right)$$

First example of solvability of $T\bar{T}$-deformed QFTs

Smirnov-Zamolodchikov operators (2016)

There were also discovered generalization of $T\bar{T}$ operators in $2d$

Given any $2d$ theory with a local conserved spin-s current,

$$\mathcal{J}_{s\pm} = \mathcal{J}_{\underbrace{\pm\pm\ldots\pm}_{s \text{ times}}}\,, \qquad \mathcal{J}_{(s-2)\pm} = \mathcal{J}_{\mp\underbrace{\pm\ldots\pm}_{s-1 \text{ times}}}\,, \qquad \partial^{\mu_1}\mathcal{J}_{\mu_1\ldots\mu_s} = 0\,,$$

there exists a quantum-mechanically well-defined local operator

$$\mathcal{O}_s(x) = \lim_{y\to x}\left(\mathcal{J}_{s+}(x)\mathcal{J}_{s-}(y) - \mathcal{J}_{(s-2)+}(x)\mathcal{J}_{(s-2)-}(y)\right)\,,$$

in the spectrum of the theory. When $s = 2$ and the conserved current is the stress tensor, this is the $T\bar{T}$ deformation.

- One can study new integrable deformations also driven by $\mathcal{O}_s(\sigma)$,

$$\frac{\partial\mathcal{L}^{(\lambda_s)}}{\partial\lambda_s} = \mathcal{O}_s(\sigma)$$

see, e.g., [Zamolodchikov, Negro, Tateo, ... (2016 —)],
[Bielli-Ferko-Galli-Huang-Smith-GTM (2024 —)]

but harder compared to $T\bar{T}$...

What and when $T\bar{T} = \det[T_{\mu\nu}]$:

Since 2016, for $T\bar{T}$ deformations, it was proven that:

- Many observables can be solved in terms of the undeformed models
  Torus partition function (preserves modular invariance of CFT_2)
  S-matrix (CDD factors), see integrable QFTs
  correlation functions and algebra
  Exact (classical) Lagrangian flows

- Fundamentally, it became clear that $T\bar{T}$ deformations preserve (deform in a controlled way) symmetries of the original theory even Virasoro for CFTs! see Guica et.al.

  We did prove that supersymmetry is preserved.
  [GTM-Ferko-Sfondrini-Sethi-. . . (2018)–]

- Punchline: $T\bar{T}$ deformations give a new window to nonlocal QFT and surprising connections with theories of quantum gravity with a wealth of applications. See the nice reviews:
  [Yunfeng Jiang (2019)]
  [Christian Ferko PhD thesis (2021)]
  [He-Li-Ouyang-Sun, (2025)]

I will now focus on constructing Lagrangian flows and underline some dynamical features preserved by deformations

What and when $T\bar{T} = \det[T_{\mu\nu}]$: Nambu-Goto, strings

Examples of (classical) Lagrangian's $T\bar{T}$-flow.
Given a boundary/seed theory $\mathcal{L}^{(0)}$, seek solution of

$$\frac{\partial \mathcal{L}^{(\lambda)}}{\partial \lambda} = \det\left[T^{(\lambda)}_{\mu\nu}\right]$$

- [Cavaglià-Negro-Szécsényi-Tateo (2016)]: free real scalars

  $\mathcal{L}^{(0)} = \frac{1}{2}\partial_+\phi^i\partial_-\phi^i$, $(i = 1, \cdots, N)$ $\Longrightarrow$

  $$\mathcal{L}^{(\lambda)} = \frac{\sqrt{1 + 2\lambda\,\partial_+\phi^i\partial_-\phi^i - \lambda^2\,\partial_+\phi^i\partial_-\phi^j\partial_+\phi^{[j}\partial_-\phi^{i]}} - 1}{2\lambda}$$

  is Nambu-Goto string action! in a static gauge
  (more precisely in uniform light cone gauge)

  possibly simplest example of the relation of $T\bar{T}$ and strings/gravity

  Known to be integrable QFT

A toy model for higher-spin Smirnov-Zamolodchikov

Deform a single free boson with a spin-s operator
[Rosenhaus-Smolkin (2020)]

$$\mathcal{L}^{(\lambda)} = -\frac{1}{2}\partial_+\phi\,\partial_-\phi + \lambda\,(\partial_+\phi)^s\,(\partial_-\phi)^s + O(\lambda^2).$$

Appropriate ansatz ($z = \lambda(\partial_+\phi\,\partial_-\phi)^{s-1}$), $\frac{\partial\mathcal{L}^{(\lambda)}}{\partial\lambda} = \mathcal{O}_s$

$\Rightarrow \mathcal{L}^{(\lambda)} = \partial_+\phi\,\partial_-\phi\, f(z), \quad \tau_s = (\partial_+\phi)^n\, t_s(z), \quad \theta_{s-2} = (\partial_+\phi)^{s-1}\partial_-\phi\, h_s(z),$

Result $\longrightarrow$ constrained system of ODEs

$$zt_s' = (s-1)h_s + zh_s, \qquad f' = t_s^2 - h_s^2,$$

$$2\frac{f' + xf''}{f''} = st_s + h_s + zt_s' + zh_s'.$$

No closed form solution $\longrightarrow$ series solution in λ.

$$f = -\frac{1}{2}\partial_+\phi\,\partial_-\phi + \lambda(\partial_+\phi\,\partial_-\phi)^s - s^2\lambda^2(\partial_+\phi\,\partial_-\phi)^{2s-1} + \dots$$

Need better techniques: Auxiliary fields, manifest integrability, see later.

$4d$ BI as a T^2-deformation [Conti-Negro-Tateo (2018)]

Abelian Maxwell-Born-Infeld:
(a fundamental ingredient in describing EFT of open strings)

$$
\begin{aligned}
S_{\rm BI} &= \frac{1}{\alpha^2}\int \mathrm{d}^4x \left[1-\sqrt{-\det(\eta_{\mu\nu}+\alpha F_{\mu\nu})}\right] \\
&= \frac{1}{\alpha^2}\int \mathrm{d}^4x \left[1-\sqrt{1+\frac{\alpha^2}{2}F^2-\frac{\alpha^4}{16}(F\tilde{F})^2}\right] \\
&= -\frac{1}{4}\int \mathrm{d}^4x\, F^2 + \text{higher derivative terms}\ ,
\end{aligned}
$$

$F_{\mu\nu}=(\partial_\mu A_\nu-\partial_\nu A_\mu)$ is field strength for an Abelian gauge field A_μ, and

$$
F^2 \equiv F_{\mu\nu}F^{\mu\nu}, \qquad F\tilde{F} \equiv F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}
$$

Then:

$$
\frac{\partial \mathcal{L}_{\rm BI}}{\partial \alpha^2} = \frac{1}{8}\mathcal{O}_{T^2}\ , \qquad \mathcal{O}_{T^2} = T^{\mu\nu}T_{\mu\nu}-\frac{1}{2}[T^\mu{}_\mu]^2
$$

BI is a T^2-flows, with boundary condition, $\lambda=0$, given by Maxwell

Extends to all models of duality-invariant electrodynamics.

$4d$ duality invariant theories and $T\bar{T}$-like flows

Specifically, consider $\mathcal{L} = \mathcal{L}^{(\lambda)}(S, P)$

$$S = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad P = -\frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu}, \quad \widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\tau}F_{\lambda\tau}$$

Then imposing non-linear EM-duality implies the following PDE [Gaillard-Zumino, Kuzenko-Theisen,$\cdots$]

$$\mathcal{L}_S^2 - \frac{2S}{P}\mathcal{L}_S\mathcal{L}_P - \mathcal{L}_P^2 = 1$$

Extend Maxwell's duality-invariance of EOM $\partial^\mu F_{\mu\nu} = 0$, $\partial^\mu \tilde{F}_{\mu\nu} = 0$ under duality $F \leftrightarrow \tilde{F}$. BI: example of non-linear duality-invariant theory.

Theorem: non-linear duality-invariance if and only if $T\bar{T}$-like flow

$$\frac{\partial \mathcal{L}^{(\lambda)}}{\partial \lambda} = \mathcal{O}^{(\lambda)}$$

for some operator $\mathcal{O}^{(\lambda)} = \mathcal{O}^{(\lambda)}(T_{\mu\nu}, \lambda)$. [Ferko-Kuzenko-Smith-GTM (2023)].

Important comment for later: duality invariance can be efficiently imposed by using auxiliary field formulations of [Ivanov-Zupnik (2002)] (no time for details)

Remarkably, ideas developed for $d = 4$ duality invariant theories lead to construction of new infinite families of $2d$ integrable field theories

Let's see what and how

Integrable deformations of Principal Chiral Model (PCM)

PCM: Well known to be an integrable field theory
Fields $g(x) \in G$ are maps between $2d$ space-time and a Lie group G with Lie algebra $\mathfrak{g}$. (Flat) Maurer-Cartan form and PCM Lagrangian are:

$$j = g^{-1}dg, \quad j_\mu = g^{-1}\partial_\mu g \ , \qquad \mathcal{L}_{PCM} = -\frac{1}{2}\,\mathrm{tr}(j_+ j_-).$$

Next, define

$$S = -\frac{1}{2}\,\mathrm{tr}(j_+ j_-), \quad P^2 = \frac{1}{4}\left(\mathrm{tr}(j_+ j_+)\,\mathrm{tr}(j_- j_-) - (\mathrm{tr}(j_+ j_-))^2\right).$$

A Lagrangian $\mathcal{L}(S, P)$ that extends the PCM can be shown to be integrable (EOM are flatness of a Lax connection) if

$$\mathcal{L}_S^2 - \frac{2S}{P}\mathcal{L}_S\mathcal{L}_P - \mathcal{L}_P^2 = 1,$$

same PDE as $4d$ duality-invariant Electrodynamics!
[Borsato-Ferko-Sfondrini (2022)][Ferko-Kuzenko-Smith-GTM (2023)].

Questions: is there an auxiliary field formulation? Are all integrable deformation in 1-1 correspondence with $T\bar{T}$-like flows? More? Yes!

Systematizing deformations: $2d$ vector auxiliary fields

One can present and extend all such deformations using an **auxiliary field prescription**. Introduce another field $v_\pm \in \mathfrak{g}$ and consider the action

$$S = \int d^2x \left(\frac{1}{2}\mathrm{tr}(j_+ j_-) + \mathrm{tr}(v_+ v_-) + \mathrm{tr}(j_+ v_- + j_- v_+) + E(v_\pm) \right) ,$$

When $E = 0$, algebraically integrating out the auxiliary field $v_\pm$ using its equation of motion returns us to the PCM.

For more general E, interaction functions, integrating out $v_\pm$ gives a complicated (higher-derivative) interacting Lagrangians for $j_\pm$.

We refer to this theory as the **auxiliary field sigma model** or AFSM.

Theorem: Consider the interaction function

$$E(v_\pm) = E(\mathrm{tr}(v_+^k), \mathrm{tr}(v_-^l)) , \qquad \forall k, l \text{ integers} ,$$

the resulting infinite family of models is classically integrable

First [Ferko-Smith (2024)] for $k = l = 2$, then various (higher-spin) extensions [Bielli-Ferko-Smith-GTM (2024)] [Bielli-Ferko-Galli-GTM (2025)] [Ferko-Galli-Huang-GTM (2025)]

Key to Integrability.

When the auxiliary field equation of motion is satisfied, the g-field equation of motion can be written as the modified conservation condition

$$\partial_+ \mathfrak{J}_- + \partial_- \mathfrak{J}_+ \doteq 0\,, \qquad \mathfrak{J}_\pm = -\left(j_\pm + 2v_\pm\right)\,.$$

Define the Lax connection

$$\mathfrak{L}_\pm = \frac{j_\pm \pm z\mathfrak{J}_\pm}{1-z^2}\,.$$

Then one has

$$\partial_+ \mathfrak{L}_- - \partial_- \mathfrak{L}_+ + [\mathfrak{L}_+, \mathfrak{L}_-] \doteq 0 \iff \partial_+ \mathfrak{J}_- + \partial_- \mathfrak{J}_+ \doteq 0\,.$$

Despite the complicated interactions, and the large number of models in this family, all theories of this type are classically integrable!

Intuition for relation to current deformations.

The AFSMs are related to (generalized) $T\bar{T}$ and Smirnov-Zamolodchikov deformations! Consider an interaction function

$$E(\lambda, v_\pm) = \lambda\, \mathrm{tr}(v_+^s)\mathrm{tr}(v_-^s) = \lambda\, \nu_s\,,$$

for some $s \geq 2$. Working to leading order in λ and eliminating $v_\pm$ using its equation of motion gives

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}(j_+ j_-) + \lambda \mathrm{tr}(j_+^s)\mathrm{tr}(j_-^s) + \cdots = \mathcal{L}^{PCM} + \lambda \mathcal{O}_s + \cdots$$

This is exactly a current bilinear deformation, since in the PCM one has the higher-spin conserved currents

$$\mathcal{J}_{s\pm} = \mathrm{tr}(j_\pm^s)\,.$$

More general interaction functions implement deformations by non-trivial functions of the conserved currents. For instance, the class of interaction functions $E(\nu_2)$ includes all deformations by functions of $T_{\mu\nu}$, like $T\bar{T}$.

Recap of results from 7 recent papers [Bielli-Ferko-Galli-Huang-Smith-GTM (2024–)]

- For any interaction function $E(v_\pm) = E(\mathrm{tr}(v_+^k), \mathrm{tr}(v_-^l))$, the auxiliary field sigma model exhibits a Lax representation for its equations of motion, and the Poisson bracket of the Lax takes the Maillet form (infinite number charges in involution from monodromy matrix).

- Possible to construct not only Lax etc, but exist also procedures to obtain local higher-spin currents (to our knowledge, no general prescription without auxiliary fields).

- With small modifications, one can deform with auxiliaries the PCM with WZ term, bi-Yang-Baxter deformed PCM, (semi-)symmetric space sigma models. All examples studied in, e.g., AdS/CFT. All integrable, the EoM have a Lax representation.

- Moreover, auxiliary fields can be used to study non-abelian T-Duality, the geometry of deformed soliton surfaces, prove that instanton-like solutions are preserved in stress tensor flows (see also [Ferko-Hue-Morone-GTM-Tateo (2024)] for instantons in $4d$)

Conclusion and Outlook

The community is looking at several applications of $T\bar{T}$-like deformations: integrability, RG-flows, quantum gravity, holography ...

We are still exploring new tools to learn more about $T\bar{T}$ and Smirnov-Zamolodchikov type deformations.

Auxiliary fields techniques have provided new infinite family of integrable deformations where duality and integrability are remarkably simple.

Outlook

- Current algebras? Symmetries? Yangians?
- Auxiliaries for deformations of more general (non-conformal) models?
- Applications to (higher-spin) AdS/CFT?
- $d > 2$ CS and self-dual YM theories?
- Quantum properties? E.g., Partition function and correlation functions for $T\bar{T}$ using auxiliary fields? What about SZ?

Thanks!