On algebraic structures underlying the rational Kashiwara-Miwa-type models

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Baxter2025 Exactly Solved Models and Beyond: Celebrating the life and achievements of Rodney James Baxter

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Descendants of the six-vertex model

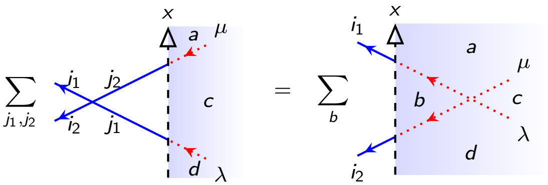
- V. Bazhanov and S. Sergeev, "*An Ising-type formulation of the six-vertex model*", Nuclear Physics B Vol 986 (2023) 116055
- V. Bazhanov and S. Sergeev, "*A distant descendant of the six-vertex model*", Nuclear Physics B Vol 1004 (2024) 116558

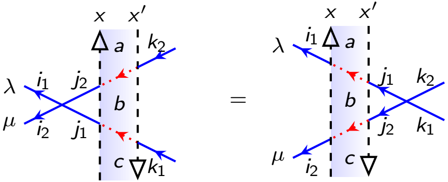
  S. Sergeev, "*On Faddeev's equation*", arxiv:2404.00499,

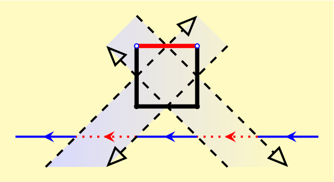
  S. Sergeev, "*Functional Bethe Ansatz for a sinh-Gordon model with real q*", Symmetry 16(8) (2024) 947

  $0 < |q| < 1$, spin variable $\in \mathbb{Z}$.
- To be continued ... S. Sergeev, "*On algebraic structures underlying the rational Kashiwara-Miwa-type models*", arXiv:2508.12537

Vertex-IRF correspondence, related L-operators and quantum intertwiner







Two types of Baxter's vectors

- Type I Baxter's vectors:
  - IRF model is equivalent to the 6-vertex R-matrix
  - L-operator corresponds to "sine-Gordon model" – evaluation representation of $\mathcal{U}_q(\widehat{sl}_2)$ with the help of Weyl algebra $\mathcal{W}_q \; : \quad \mathbf{uv} \; = \; q \, \mathbf{vu}$
  - Resulting Ising-type models include: Chiral Potts m., Faddeev-Volkov m., ... , six-vertex m. itself, and "distant descendant m.".
- Type II Baxter's vectors:
  - Eight vertex extension, Sklyanain algebra.
  - IRF model – a class of Andrews-Baxter-Forrester models.
  - L-operator corresponds to the Bose-gas – evaluation representation of $\mathcal{U}_q(\widehat{sl}_2)$ with the help of a *single* q-oscillator.
  - Resulting Ising-type models include the rational Kashiwara-Miwa model *and its relatives*.

The generalised q-oscillator algebra is defined by the following relations:

$$\mathcal{O}_q[\mathcal{K},\mathcal{K}'] \;:\; \left\{ \begin{array}{l} \mathcal{K}\mathcal{E}^{\pm} = q^{\pm 1}\mathcal{E}^{\pm}\mathcal{K}\,, \quad \mathcal{K}'\mathcal{E}^{\pm} = q^{\pm 1}\mathcal{E}^{\pm}\mathcal{K}'\,, \quad [\mathcal{K},\mathcal{K}'] = 0\,, \\ \mathcal{E}^{+}\mathcal{E}^{-} = 1 - q^{-1}\mathcal{K}\mathcal{K}'\,, \quad \mathcal{E}^{-}\mathcal{E}^{+} = 1 - q\mathcal{K}\mathcal{K}'\,. \end{array} \right.$$

The standard q-oscillator algebra is its subalgebra,

$$\mathcal{O}_q \;:\; q\mathcal{E}^{+}\mathcal{E}^{-} - q^{-1}\mathcal{E}^{-}\mathcal{E}^{+} = q - q^{-1}\,.$$

The generalised q-oscillator algebra has a center, $\omega^2 = \mathcal{K}\mathcal{K}'^{-1}$, if $\mathcal{K},\mathcal{K}'$ are invertible.

Relations

$$\Delta(\mathcal{E}^{-}) = \mathcal{E}^{-}\otimes\mathcal{E}^{-} - \mathcal{K}\otimes\mathcal{K}'\,, \quad \Delta(\mathcal{K}) = \mathcal{E}^{-}\otimes\mathcal{K} + \mathcal{K}\otimes\mathcal{E}^{+}\,,$$

$$\Delta(\mathcal{K}') = \mathcal{K}'\otimes\mathcal{E}^{-} + \mathcal{E}^{+}\otimes\mathcal{K}'\,, \quad \Delta(\mathcal{E}^{+}) = \mathcal{E}^{+}\otimes\mathcal{E}^{+} - \mathcal{K}'\otimes\mathcal{K}$$

define a homomorphism $\mathcal{O}_q[\mathcal{K},\mathcal{K}'] \to \mathcal{O}_q[\mathcal{K},\mathcal{K}'] \otimes \mathcal{O}_q[\mathcal{K},\mathcal{K}']$.

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define a homomorphism $\mathcal{O}_q[\mathcal{K},\mathcal{K}'] \ \to \ \mathcal{O}_q[\mathcal{K},\mathcal{K}'] \ \otimes \ \mathcal{O}_q[\mathcal{K},\mathcal{K}']$.

q-oscillator [contd]

L-operator form:

$$\Delta(L) = L \dot{\otimes} L, \quad L \stackrel{def}{=} \begin{pmatrix} \mathcal{E}^- & -\mathcal{K} \\ \mathcal{K}' & \mathcal{E}^+ \end{pmatrix} \quad \text{vs} \quad L_{B.G.} = \begin{pmatrix} \lambda\mathcal{K}' & \mathcal{E}^+ \\ \mathcal{E}^- & -\lambda^{-1}\mathcal{K} \end{pmatrix}$$

Quantum intertwining relation:

$$\pi_{\lambda_1}(L) \dot{\otimes} \pi_{\lambda_2}(L) \check{S}(\lambda_1, \lambda_2) = \check{S}(\lambda_1, \lambda_2) \pi_{\lambda_2}(L) \dot{\otimes} \pi_{\lambda_1}(L) .$$

Remarkable homomorphism $\pi_{\omega,\mu} : \mathcal{O}_q[\mathcal{K}, \mathcal{K}'] \to \mathcal{W}_q$:

$$\begin{cases} \omega^{-1}\pi_{\omega,\mu}(\mathcal{K}) = \omega\,\pi_{\omega,\mu}(\mathcal{K}') = (\boldsymbol{v} - \boldsymbol{v}^{-1})(\boldsymbol{u} - \boldsymbol{u}^{-1})^{-1} , \\ \pi_{\omega,\mu}(\mathcal{E}^+) = \mu^{-1}(\boldsymbol{v}^{-1}\boldsymbol{u} - \boldsymbol{v}\boldsymbol{u}^{-1})(\boldsymbol{u} - \boldsymbol{u}^{-1})^{-1} , \\ \pi_{\omega,\mu}(\mathcal{E}^-) = \mu\,(\boldsymbol{v}\boldsymbol{u} - \boldsymbol{v}^{-1}\boldsymbol{u}^{-1})(\boldsymbol{u} - \boldsymbol{u}^{-1})^{-1} . \end{cases}$$

V. Bazhanov, V. Mangazeev and S. Sergeev, "*Quantum geometry of 3-dimensional lattices*", J.Stat.Mech.0807:P07004,2008,

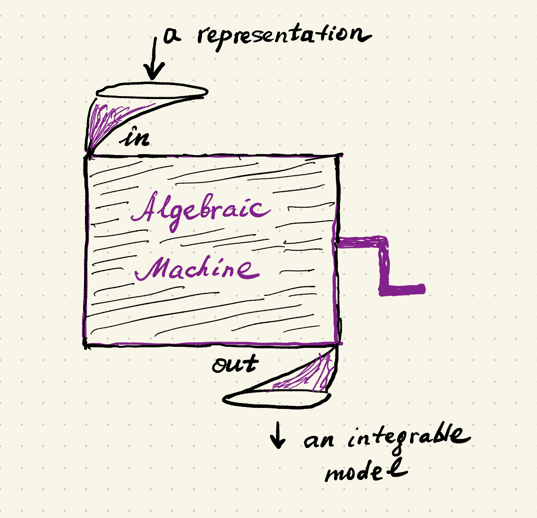
$\boldsymbol{u} = \exp(i\alpha)$, $\boldsymbol{v} = \exp(i\beta)$, α, β – angles of Ptolemy quadrilateral

Confer with an "elementary homomorphism" $\pi(\mathcal{K}^{\#}) = \boldsymbol{u}$, $\pi(\mathcal{E}^+) = \boldsymbol{v}$, $\pi(\mathcal{E}^-) = \boldsymbol{v}^{-1}(1 - q^{-1}\boldsymbol{u}^2)$

- Representation theory and co-product. E. g.

$$\mathbb{F}\otimes\mathbb{F}=\mathbb{T}\otimes\mathbb{F},\ \ \widetilde{\mathbb{F}}\otimes\widetilde{\mathbb{F}}=(\mathbb{Z}_2\otimes\mathbb{Z})\otimes\widetilde{\mathbb{F}},\ \ \mathbb{F}\otimes\widetilde{\mathbb{F}}=\sum_{n=0}^{\infty}\mathbb{N}_n$$

  where $\mathbb{F}$ is the Fock space representation ($\mathcal{E}^-|0\rangle=0$), $\widetilde{\mathbb{F}}$ is the anti-Fock space representation ($\mathcal{E}^+|0\rangle=0$), $\mathbb{N}_n$ is $n+1$ dimensional representation with $\mathcal{K}\mathcal{K}'$ – Jordan cell, etc.
- Clebsh-Gordan coefficients, characters,
- Etc.
- In particular, the Star-Triangle equation arises as a certain associativity condition in the co-product


a representation
in
Algebraic
Machine
out
an integrable
model

Kashiwara-Miwa rational weights are defined by

$$V_x(m,m') \;=\; \left(\frac{q}{x}\right)^{2m} \frac{(x;q^2)_{m-m'}}{(q^2/x;q^2)_{m-m'}} \frac{(\gamma^2 x;q^2)_{m+m'}}{(q^2/\gamma^2 x;q^2)_{m+m'}}, \quad S_m \;=\; \frac{[\gamma q^{2m}]}{[\gamma]}\;,$$

where

$$\gamma = \mathrm{i} q^{n/2}$$

They satisfy the Star-Triangle relation.

$$\sum_{d\in\mathbb{Z}} S_d\, V_x(a,d)\, V_y(b,d)\, V_z(c,d) \;=$$

$$=\; R_{x,y,z}\, V_{q/x}(b,c)\, V_{q/y}(a,c) V_{q/z}(a,b)\;, \quad xyz \;=\; q\;,$$

By algebraic construction, the weight is a character, $V_x \;\sim\; x^{\log(\mathcal{K}\mathcal{K}')/2}$,

$$\sum_{b\in\mathbb{Z}} V_x(a,b)\, S_b\, V_y(b,c) \;=\; \frac{\Phi(x)\Phi(y)}{\Phi(xy)}\, V_{xy}(a,c)\;.$$

V. Bazhanov, A. Kels and S. Sergeev, "*Quasi-classical expansion of the star-triangle relation and integrable systems on quad-graphs*", J. Phys. A: Math. Theor. 49 (2016) 464001

However, the primary weight is given by

$$V_{\mu/\mu'}(m,m') = q^{m(m+1)/2+m'(m'+1)/2} \times$$

$$\times \left(\frac{\mu'}{\mu}q^{2+m-m'}, \frac{\mu'}{\mu}q^{2-m+m'}, \frac{\mu'}{\mu}q^{3+m+m'}, \frac{\mu'}{\mu}q^{1-m-m'}; q^2\right)_\infty .$$

The Star-Triangle relation is

$$\sum_{m\in\mathbb{Z}} V_x(m_a,m)\,(-)^m[q^{m+1/2}]\,V_{-q/xy}(m,m_c)\,V_y(m_b,m) =$$

$$= R_{x,y,-q/xy}\,V_{-q/y}(m_a,m_c)\,V_{xy}(m_a,m_b)\,V_{-q/x}(m_b,m_c),$$

and the summation formula holds:

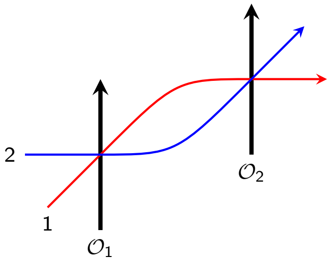
$$\sum_{m\in\mathbb{Z}} V_x(m,m')\,(-)^{m'}[q^{m'+\frac{1}{2}}]\,V_y(m,m'') = \frac{\Phi(x)\Phi(y)}{\Phi(xy)}V_{xy}(m,m'') .$$

Physical regime is $0 < -q < 1$. Partition functions per site and per edge coincide with that of the "distant descendant...".

Thank you

Discussion

$$L_{12}[\mathcal{O}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathcal{K}' & \mathcal{E}^{+} & 0 \\ 0 & \mathcal{E}^{-} & -\mathcal{K} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$



$$\Delta(L) = L_{12}[\mathcal{O}_1] L_{21}[\mathcal{O}_2] P_{12}$$