On correlation numbers in Virasoro Minimal String Theory

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Non-critical string theory

- Formally, the action of non-critical string theory is a direct sum of three theories

$$S = S_{\mathrm{CFT}} + S_{\mathrm{L}} + S_{\mathrm{ghosts}}, \qquad c_{\mathrm{CFT}} + c_{\mathrm{L}} + c_{\mathrm{ghosts}} = 0.$$

- Physical observables correspond to BRST cohomologies

$$U_\Delta(x) = \Phi_\Delta(x) V_{1-\Delta}(x).$$

- The goal is to calculate the amplitudes, or correlation numbers

$$V_{g,n}(\Delta_1, \ldots, \Delta_n) = \int_{\mathcal{M}_{g,n}} Z_{\mathrm{gh}} \langle U_\Delta(x_1) \ldots U_\Delta(x_n) \rangle_g ,$$

where $\langle \ldots \rangle_g$ is the correlation function on a Riemann surface of the genus g and Z_{gh} is a contribution from the ghost sector.

Non-critical string theory

- The most well-known example is the minimal string theory, also known as minimal Liouville gravity. In this theory, the matter content consists of a single minimal model $\mathcal{M}_{p,p'}$.
- This theory is believed to be dual to a double-scaling limit of a certain matrix model and most of analytic results have been obtained within this duality on the matrix model side.
- Another model of 2D quantum gravity has been proposed recently (Collier, Eberhardt, Mühlmann and Rodriguez, The Virasoro minimal string 2024). In this case the spectrum of the underlying matter CFT, the so-called timelike Liouville CFT, is continuous.
- A remarkably simple polynomial formula for $V_{g,n}$ has been conjectured. In particular,

$$V_{0,4}(\Delta_1, \Delta_2, \Delta_3, \Delta_4) = \frac{c-13}{24} + \sum_{k=1}^{4} P_k^2, \quad \Delta_k = \frac{Q^2}{4} + P_k^2.$$

Liouville CFT

LFT is a CFT with central charge

$$c = 1 + 6Q^2, \qquad Q = b + \frac{1}{b}, \quad b \in \mathbb{C}.$$

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \end{aligned} \qquad n, m \in \mathbb{Z}.$$

The primary field satisfies

$$L_0 V_P(z) = \Delta(P) V_P(z), \qquad L_n V_P(z) = 0, \quad \text{for} \quad n > 0,$$

$$\Delta(P) = \bar{\Delta}(P) = \frac{Q^2}{4} + P^2, \qquad P \in \mathbb{R}.$$

Clearly, $\Delta(P) = \Delta(-P)$. We demand that $V_P(z) \sim V_{-P}(z)$

$$V_P(z) = R_P V_{-P}(z), \qquad \text{where} \qquad R_P R_{-P} = 1.$$

Liouville CFT

For conformal bootstrap we need to know

$$\langle V_{P_1}(z_1)V_{P_2}(z_2)\rangle \sim N(P_1)(\delta(P_1+P_2)+R_{P_1}\delta(P_1-P_2)),$$
$$\langle V_{P_1}(z_1)V_{P_2}(z_2)V_{P_3}(z_3)\rangle \sim C(P_1,P_2,P_3).$$

Higher order correlation functions are computed using OPE

$$V_{P_1}(z)V_{P_2}(w)=$$
$$=\int_{\mathbb{R}} dP\, C^{P}_{P_1,P_2}|z-w|^{2(\Delta(P)-\Delta(P_1)-\Delta(P_2))}\sum_{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(z-w)^{|\boldsymbol{\lambda}|}(\bar{z}-\bar{w})^{|\bar{\boldsymbol{\lambda}}|}\beta_{\boldsymbol{\lambda}}(P)\beta_{\bar{\boldsymbol{\lambda}}}(P)V_P^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(w)$$

Here $V_P^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}\equiv L_{-\boldsymbol{\lambda}}\bar{L}_{-\bar{\boldsymbol{\lambda}}}V_P\equiv(L_{-\lambda_1}L_{-\lambda_2}\dots)(\bar{L}_{-\bar{\lambda}_1}\bar{L}_{-\bar{\lambda}_2}\dots)V_P$ are descendant fields and all $\beta_{\boldsymbol{\lambda}}(P)$ are fixed unambiguously by conformal symmetry. They are rational functions of P with poles at

$$P=\pm P_{m,n}\equiv\pm\frac{i}{2}\left(mb+nb^{-1}\right),\qquad m,n\in\mathbb{Z}_{>0}.$$

Liouville CFT

We postulate the existence of degenerate fields with

$$\Delta_{m,n} = \Delta(P_{m,n}), \quad P_{m,n} = \frac{i}{2}\left(mb + nb^{-1}\right), \quad m, n \in \mathbb{Z}_{>0}.$$

As one can see, for all $b \in \mathbb{C} \setminus i\mathbb{R}$ the momenta $P_{m,n}$ do not lie in the spectrum of LFT, they a priori form a different operator algebra. The vanishing descendant of $V_{m,n}$ has the form $D_{m,n}\bar{D}_{m,n}V_{m,n}$, where

$$D^{(b)}_{m,n} = L^{mn}_{-1} + c_1(b)L^{mn-2}_{-1}L_{-2} + \dots$$

Because of these vanishing descendants, correlation functions with degenerate fields satisfy BPZ differential equations. In particular OPE of $V_{m,n}$ with a field V_P in the spectrum has the form

$$V_{m,n}(z)V_P(w) = \sum_{k,l} C^{k,l}_{m,n}(P)|z-w|^{2(\Delta(P+P_{k,l})-\Delta(P)-\Delta_{m,n})}[V_{P+P_{k,l}}(w)+\dots]$$

Liouville CFT

Consider 4-point functions with degenerate fields $V_{1,2}$ and $V_{2,1}$:

$$\langle V_{2,1}(z) V_{P_2}(0) V_{P_3}(1) V_{P_4}(\infty)\rangle, \quad \langle V_{1,2}(z) V_{P_2}(0) V_{P_3}(1) V_{P_4}(\infty)\rangle.$$

In this case, there are only 2 conformal blocks in each channel.
This allows us to reduce the crossing symmetry constraint to

$$\frac{C_{2,1}^{1,0}(P_2+ib)C(P_2+ib,P_3,P_4)}{C_{2,1}^{-1,0}(P_2)C(P_2,P_3,P_4)} = \frac{\prod_{\pm_1,\pm_2}\gamma(\frac{1}{2}-\frac{b^2}{2}+ib(P_2\pm_1 P_3\pm_2 P_4))}{\gamma(-b^2+2ibP_2)\gamma(1-b^2+2ibP_2)},$$

$$\frac{C_{1,2}^{0,1}(P_2+ib^{-1})C(P_2+ib^{-1},P_3,P_4)}{C_{1,2}^{0,-1}(P_2)C(P_2,P_3,P_4)} = \frac{\prod_{\pm_1,\pm_2}\gamma(\frac{1}{2}-\frac{b^{-2}}{2}+ib^{-1}(P_2\pm_1 P_3\pm_2 P_4))}{\gamma(-b^{-2}+2ib^{-1}P_2)\gamma(1-b^{-2}+2ib^{-1}P_2)}.$$

Here $\gamma(z)=\Gamma(z)/\Gamma(1-z)$. Note that for $b\in\mathbb{R}$ and $b\in i\mathbb{R}$ these equations have unique solutions!

Spacelike Liouville CFT

The solution is

$$C(P_1, P_2, P_3) = \frac{\Upsilon_b'(0) \prod_{k=1}^3 \Upsilon_b(-2iP_k)}{\prod_{\pm_1,\pm_2} \Upsilon_b(Q/2 + iP_1 \pm_1 iP_2 \pm_2 iP_3)}.$$

In this normalization, we have

$$R_P = \frac{\Upsilon_b(-2iP)}{\Upsilon_b(2iP)}, \qquad C_{P_1,P_2}^P = C(-P, P_1, P_2).$$

The 3-point function has the following zeroes and poles in P_1:

$$\text{Poles}: \quad P_1 \pm_2 P_2 \pm_3 P_3 = \pm P_{2m-1,2n-1}.$$

$$\text{Zeroes}: \quad P_1 = \pm P_{m,n}, \quad P_1 = 0, \quad P_1 = -\frac{imb}{2}, \quad P_1 = -\frac{inb^{-1}}{2}.$$

Spacelike Liouville CFT

We now consider analytic continuation of OPE

$$V_{P_1}(z)V_{P_2}(w)=$$
$$=\int_{\mathbb{R}} dP\, C^{P}_{P_1,P_2}|z-w|^{2(\Delta(P)-\Delta(P_1)-\Delta(P_2))}\sum_{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(z-w)^{|\boldsymbol{\lambda}|}(\bar{z}-\bar{w})^{|\bar{\boldsymbol{\lambda}}|}\beta_{\boldsymbol{\lambda}}(P)\beta_{\bar{\boldsymbol{\lambda}}}(P)V_P^{\boldsymbol{\lambda},\bar{\boldsymbol{\lambda}}}(w)$$

in the momentum P_1 away from the spectrum $\mathbb{R}$. The integrand is a meromorphic function of P_1 and P. Poles in P can originate from $C^{P}_{P_1,P_2}=C(-P,P_1,P_2)$ and from $\beta_{\boldsymbol{\lambda}}(P)$, $\beta_{\bar{\boldsymbol{\lambda}}}(P)$. Careful analysis shows that

$$V_{P_{r,s}}=V_{r,s}$$

This analysis is rather straightforward, but requires the so called HEM (Al. Zamolodchikov 2003)

Spacelike Liouville CFT

We identified degenerate fields $V_{m,n}$ with primary fields $V_{P_{m,n}}$. By definition of degenerate fields, they satisfy

$$D^{(b)}_{m,n}\bar{D}^{(b)}_{m,n}V_{m,n}(z) = 0, \qquad m, n \in \mathbb{Z}_{>0}.$$

Al. Zamolodchikov discovered the following "higher equations of motion" (HEM):

$$D^{(b)}_{m,n}\bar{D}^{(b)}_{m,n}V'_{m,n}(z) = B^{(b)}_{m,n}V_{m,-n}(z), \text{ where } V'_{m,n}(z) \equiv \frac{\partial}{\partial P}V_P(z)\bigg|_{P=P_{m,n}}$$

with (here Here by $V_{m,-n}(z) = V_{P_{m,-n}}(z)$)

$$B^{(b)}_{m,n} = \prod_{k,l}(P_1+P_2+P_{k,l})^2(P_1-P_2+P_{k,l})^2 \cdot \frac{\partial C(P, P_1, P_2)/\partial P|_{P=P_{m,n}}}{C(P_{m,-n}, P_1, P_2)}$$

Spacelike Liouville CFT

Consider the correlation function of N primary fields:

$$\langle V_{P_1}(z_1) V_{P_2}(z_2) \dots V_{P_N}(z_N) \rangle.$$

It can be proved that the only poles of this correlation function in P_1 are simple poles, which are known as "screening poles":

$$P_1 + \sum_{i=2}^{N} \pm_i P_i = \pm i \left(\frac{(N-2)Q}{2} + (r-1)b + (s-1)b^{-1} \right), \; r, s \in \mathbb{Z}_{>0}.$$

The proof goes by induction in N

$$\langle V_{P_1}(z_1) V_{P_2}(z_2) \ldots V_{P_N}(z_N) \rangle =$$
$$= \int_C dP \, |z_1 - z_2|^{2(\Delta(P) - \Delta(P_1) - \Delta(P_2))} C(-P, P_1, P_2) \left[\langle V_P(z_2) V_{P_3}(z_3) \ldots V_{P_N}(z_N) \rangle + \ldots \right].$$

Spacelike Liouville CFT

Last ingredient that we need is the triality transformation for 4−point correlation function

$$\langle V_{P_1}(z)V_{P_2}(0)V_{P_3}(1)V_{P_4}(\infty)\rangle.$$

It can be proven that it is covariant under the transformation

$$P_1 \to \tilde{P}_1 = \frac{P_1+P_2+P_3+P_4}{2}, \quad P_2 \to \tilde{P}_2 = \frac{P_1+P_2-P_3-P_4}{2},$$
$$P_3 \to \tilde{P}_3 = \frac{P_1-P_2+P_3-P_4}{2}, \quad P_4 \to \tilde{P}_4 = \frac{P_1-P_2-P_3+P_4}{2}.$$

$$\langle V_{\tilde{P}_1}(z)V_{\tilde{P}_2}(0)V_{\tilde{P}_3}(1)V_{\tilde{P}_4}(\infty)\rangle = |z|^{-(P_1-P_2-P_3-P_4)(P_1-P_2+P_3+P_4)}|1-z|^{-(P_1-P_2-P_3-P_4)(P_1}$$
$$\times \prod_{k=1}^{4} \frac{\Upsilon_b(-2i\tilde{P}_k)}{\Upsilon_b(-2iP_k)} \langle V_{P_1}(z)V_{P_2}(0)V_{P_3}(1)V_{P_4}(\infty)\rangle$$

Timelike Liouville CFT

For $b = i\hat{b} \in i\mathbb{R}$

$$\hat{c} = 1 + 6\hat{Q}^2, \quad \hat{Q} = b + b^{-1} = i\hat{b} - i\hat{b}^{-1}$$

functional equations have another solution

$$\hat{C}_{(i\hat{b})}(\hat{P}_1, \hat{P}_2, \hat{P}_3) = \frac{\prod_{\pm_1,\pm_2} \Upsilon_{\hat{b}}((\hat{b} + \hat{b}^{-1})/2 + \hat{P}_1 \pm_1 \hat{P}_2 \pm_2 \hat{P}_3)}{\Upsilon'_{\hat{b}}(0) \prod_{k=1}^{3} \Upsilon_{\hat{b}}(\hat{b} + \hat{b}^{-1} + 2\hat{P}_k)}.$$

with

$$\begin{aligned} \text{Poles}: &\quad \hat{P}_1 = \pm\hat{P}_{m,-n}, \quad \hat{P}_1 = 0, \quad \hat{P}_1 = \frac{m\hat{b}}{2}, \quad \hat{P}_1 = \frac{n\hat{b}^{-1}}{2}, \\ \text{Zeroes}: &\quad \hat{P}_1 \pm_2 \hat{P}_2 \pm_3 \hat{P}_3 = \pm\hat{P}_{2m-1,-2n+1}, \end{aligned}$$

where $\hat{P}_{m,n} = i\left(\frac{mb}{2} + \frac{nb^{-1}}{2}\right) = -\frac{m\hat{b}}{2} + \frac{n\hat{b}^{-1}}{2}$.

Timelike Liouville CFT

OPE in timelike LFT is slightly ill-defined. Since we have $b \in i\mathbb{R}$, the momenta that correspond to degenerate fields now lie in the spectrum: $\hat{P}_{m,n} \in \mathbb{R}$. This leads to a problem in using OPE, because the double poles of the structure constant and the poles of $\beta_{\boldsymbol{\lambda}}$, $\beta_{\bar{\boldsymbol{\lambda}}}$ now lie on the real axis. Moreover, the set of all poles of the latter $\{\hat{P}_{m,n} | \, m, n \in \mathbb{Z}_{>0}\}$ is dense in the real line.

This problem is solved by changing the contour of integration to $\mathbb{R} + i\varepsilon$ and taking the limit $\varepsilon \to 0$ after computing the correlation function.

The analytic properties of $\langle \Phi_{\hat{P}_1}(z_1)\Phi_{\hat{P}_2}(z_2)\ldots\Phi_{\hat{P}_n}(z_n)\rangle$ as a function of $\hat{P}_1$ is much simpler than in spacelike LFT. Namely, the only poles are

$$\hat{P}_1 = \pm\hat{P}_{m,-n}, \quad \hat{P}_1 = 0, \quad \hat{P}_1 = \frac{m\hat{b}}{2}, \quad \hat{P}_1 = \frac{n\hat{b}^{-1}}{2}, \; m, n \in \mathbb{Z}_{>0}.$$

Timelike Liouville CFT

Note that in timelike LFT the degenerate field

$$\Phi_{m,n}(z) = \Phi_{\hat{P}_{m,n}}(z)$$

has a non-vanishing null-vector. Instead, one has HEM

$$D^{(ib)}_{m,n}\bar{D}^{(ib)}_{m,n}\Phi_{m,n}(z) = B^{(i\hat{b})}_{m,n}\Phi_{m,-n}(z),$$

with

$$B^{(i\hat{b})}_{m,n} = 2\frac{\Upsilon'_{\hat{b}}(-2\hat{P}_{m,-n})}{\Upsilon_{\hat{b}}(-2\hat{P}_{m,n})}.$$

Note that if $\hat{b} = b$, we have

$$B^{(ib)}_{m,n} = iB^{(b)}_{m,n}.$$

Virasoro minimal string

Formally

VMS = Spacelike LFT $\otimes$ Timelike LFT $\otimes$ Fadeev−Popov ghosts.

In order for the conformal anomaly to vanish, the total central charge of the three CFTs should be equal to 0

$$\hat{c} = 26 - c = 1 - 6\left(b - b^{-1}\right)^2 \implies \hat{b} = b.$$

$$V_P(z)\Phi_{\hat{P}}(z), \quad \Delta(P) + \hat{\Delta}(\hat{P}) = 1, \quad \implies \quad \hat{P} = iP.$$

$$V_{g,n}(P_1, \ldots, P_n) = \\ = \int_{\mathcal{M}_{g,n}} Z_{\mathrm{gh}} \cdot \langle V_{P_1}(z_1) \ldots V_{P_n}(z_n) \rangle \langle \Phi_{iP_1}(z_1) \ldots \Phi_{iP_n}(z_n) \rangle.$$

Virasoro minimal string

The first nontrivial volume on a sphere is the one with 4 field insertions: $V_{0,4}(P_1, P_2, P_3, P_4)$. The moduli space of the sphere with 4 marked points $\mathcal{M}_{0,4}$ is parametrized by the position z_1 of one of the points: $z_1 \in \mathbb{C} \setminus \{z_2, z_3, z_4\}$

$$V_{0,4}(P_1,P_2,P_3,P_4)=$$
$$=\int_{\mathbb{C}} d^2z \, \langle V_{P_1}(z) V_{P_2}(0) V_{P_3}(1) V_{P_4}(\infty)\rangle \langle \Phi_{iP_1}(z)\Phi_{iP_2}(0)\Phi_{iP_3}(1)\Phi_{iP_4}(\infty)\rangle.$$

Note that because of the relation $\Delta(P) + \hat{\Delta}(\hat{P}) = 1$ fields in at least one of the LFT correlation functions do not lie in the spectrum. Therefore this equation should be understood as an analytic continuation.

Virasoro minimal string

Note that the integrand in

$$V_{0,4}(P_1,P_2,P_3,P_4) = \\ = \int_{\mathbb{C}} d^2z \, \langle V_{P_1}(z) V_{P_2}(0) V_{P_3}(1) V_{P_4}(\infty) \rangle \langle \Phi_{iP_1}(z) \Phi_{iP_2}(0) \Phi_{iP_3}(1) \Phi_{iP_4}(\infty) \rangle .$$

is a complicated meromorphic function of the momenta. Nevertheless, the proposed answer

$$V_{0,4}(\Delta_1, \Delta_2, \Delta_3, \Delta_4) = \frac{c-13}{24} + \sum_{k=1}^{4} P_k^2$$

is a polynomial. Thus all residue terms must be exact forms! Thee two types of poles: at $P_1 = P_{m,n}$ and the screening poles. But they are related by triality transformation.

Virasoro minimal string

At $P_1 \to P_{m,n}$

$$V_{P_1}\Phi_{iP_1}(z) \to V_{m,n}\Phi_{m,-n}(z) \overset{HEM}{\sim} V_{m,n}D^{(ib)}_{m,n}\bar{D}^{(ib)}_{m,n}\Phi_{m,n}(z)$$

The key observation is the "integration by parts" formula

$$\langle V_{m,n}D^{(ib)}_{m,n}\bar{D}^{(ib)}_{m,n}\Phi_{m,n}(z)\ldots\rangle = \partial\bar{\partial}\mathcal{H}_{m,n}\bar{\mathcal{H}}_{m,n}\langle V_{m,n}\Phi_{m,n}(z)\ldots\rangle,$$

where $\mathcal{H}_{m,n}$ and $\bar{\mathcal{H}}_{m,n}$ are combinations of differential operators. Thus the pole gets integrated to 0 and we consider the next order:

$$V_{0,4}(P_{m,n},P_2,P_3,P_4) = i\int d^2z\langle V'_{m,n}(z)\Phi_{m,-n}(z)\ldots\rangle + \\ + \int d^2z\langle V_{m,n}(z)\Phi'_{m,-n}(z)\ldots\rangle,$$

where

$$V_P = V_{m,n} + (P - P_{m,n})V'_{m,n} + \ldots, \quad \Phi_{\hat{P}} = \frac{\Phi_{m,-n}}{\hat{P} - \hat{P}_{m,-n}} + \Phi'_{m,-n} + \ldots$$

Virasoro minimal string

In the first term in

$$V_{0,4}(P_{m,n},P_2,P_3,P_4)=i\int d^2z\langle V'_{m,n}(z)\Phi_{m,-n}(z)\ldots\rangle+ \\ +\int d^2z\langle V_{m,n}(z)\Phi'_{m,-n}(z)\ldots\rangle,$$

we use timelike HEM, the "integration by parts" and then spacelike HEM and get the following (up to exact terms):

$$v_{0,4}^{m,n}(P_2,P_3,P_4)\equiv V_{0,4}(P_{m,n},P_2,P_3,P_4)-V_{0,4}(P_{m,-n},P_2,P_3,P_4)= \\ =\int d^2z\left\langle V_{m,n}\Phi'_{m,-n}(z)V_{P_2}\Phi_{iP_2}(0)V_{P_3}\Phi_{iP_3}(1)V_{P_4}\Phi_{iP_4}(\infty)\right\rangle$$

Virasoro minimal string

Using modified version of "integration" by parts formula, one can prove that

$$v_{0,4}^{m,n}(P_{r,s}, P_3, P_4) = v_{0,4}^{m,n}(P_{r,-s}, P_3, P_4).$$

Assuming that $V_{0,4}$ depends on $\{P_k\}$ polynomially, this constraint implies that

$$\frac{\partial}{\partial P_2} v_{0,4}^{m,n}(P_2, P_3, P_4) = 0$$

Thus

$$\frac{\partial}{\partial P_2} V_{0,4}(P_{m,n}, P_2, P_3, P_4) = \frac{\partial}{\partial P_2} V_{0,4}(P_{m,-n}, P_2, P_3, P_4)$$

and hence

$$\frac{\partial^2}{\partial P_1 \partial P_2} V_{0,4}(P_1, P_2, P_3, P_4) = 0.$$

Thus $V_{0,4}(P_1, P_2, P_3, P_4)$ is a symmetric polynomial of the form

$$V_{0,4}(P_1, P_2, P_3, P_4) = f(P_1) + f(P_2) + f(P_3) + f(P_4).$$

On the other hand, from the triality symmetry of the volume we get

$$\begin{aligned} f(P_1) + f(P_2) + f(P_3) + f(P_4) = \\ = f\left(\frac{P_1+P_2+P_3+P_4}{2}\right) + f\left(\frac{P_1+P_2-P_3-P_4}{2}\right) + \\ + f\left(\frac{P_1-P_2+P_3-P_4}{2}\right) + f\left(\frac{P_1-P_2-P_3+P_4}{2}\right) \end{aligned}$$

The only polynomial that satisfies this equation is a quadratic one:

$$V_{0,4}(P_1, P_2, P_3, P_4) = c_1(b) + c_2(b)(P_1^2 + P_2^2 + P_3^2 + P_4^2).$$

Virasoro minimal string

We can recover two coefficients $c_1(b)$ and $c_2(b)$ from two exactly solvable cases (Onofri, Fateev ... 2009)

$$V_{0,4}\left(\frac{ib^{-1}}{2}+\frac{ib}{4},\frac{ib}{4},\frac{ib}{4},\frac{ib}{4}\right)=-\frac{1}{4},$$
$$V_{0,4}\left(\frac{ib^{-1}}{2}-\frac{ib}{4},\frac{ib}{4},\frac{ib}{4},\frac{ib}{4}\right)=\frac{1}{4}.$$

The result is

$$V_{0,4}(\Delta_1,\Delta_2,\Delta_3,\Delta_4)=\frac{c-13}{24}+\sum_{k=1}^{4}P_k^2.$$

Conclusion

- Easy to generalize for $V_{1,1}$
- No proof of polynomial growth
- Triality symmetry is special for $V_{0,4}$ (for $V_{1,1}$)
- Generalization for $V_{g,n}$
- Analog of Mirzakhani recursion formula
- Relation to intersection theory