R-matrices from exterior algebras with braided Hopf structures

(Joint work with Vladimir Mangazeev)

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Motivation

Different Paths to the Yang–Baxter Equation

- **Fundamental role:** The **Yang–Baxter equation (YBE)** finds applications in both physics (integrable systems, statistical mechanics, quantum theory) and mathematics (representation theory, low-dimensional topology).
- **Multiple perspectives:** Different approaches highlight different aspects and applications of the YBE.

Outline:

1. Rigid R-matrices.
2. R-Matrices from Hopf algebras.
3. R-Matrices from Braided Hopf Algebras.
4. Exterior algebras and associated R-matrices.
5. Further perspectives.

Rigid R-Matrices

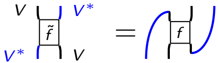
Algebraic Ingredient for Knot Invariant Constructions

- V: vector space
- R-matrix over V: $r \in \mathrm{GL}(V^{\otimes 2})$ satisfying the YBE (braid group form)

$$r'r''r' = r''r'r'', \quad r' = r \otimes \mathrm{id}_V, \; r'' = \mathrm{id}_V \otimes r$$

- $\dim V < \infty$, $\mathbb{B} \subset V$ a basis, $\{b^*\}_{b\in\mathbb{B}} \subset V^*$ the dual basis
- Partial transpose: $\mathrm{End}(V^{\otimes 2}) \ni f \mapsto \tilde{f} : V^* \otimes V \to V \otimes V^*$

$$\tilde{f}(a^* \otimes b) = \sum_{c,d\in\mathbb{B}} \langle a^* \otimes c^*, f(b \otimes d)\rangle c \otimes d^*$$



- Rigid R-matrix: $\widetilde{r^{\pm 1}}$ are invertible

Example:

- $\mathbb{B} \subset V$: linearly ordered basis
- Heaviside $\theta : \mathbb{B} \times \mathbb{B} \to \{0, 1\}$, with $(a, b) \mapsto \theta_{a,b} = 1$ iff $a > b$
- For $p \neq 0$:
  $\tau_p(a \otimes b) = (p^{\theta_{a,b}} - 1)a \otimes b - p^{\theta_{b,a}} b \otimes a, \quad \forall a, b \in \mathbb{B}$
- τ_p: R-matrix; rigid if $\dim(V) < \infty$

R-Matrices from Hopf Algebras

Drinfeld's Quantum Double Construction

- H: Hopf algebra with invertible antipode
- $\Rightarrow$ $D(H)$: quantum double with universal R-matrix
- λ: $D(H) \to \mathrm{End}(V)$: finite-dimensional representation
- Evaluating universal R-matrix at $\lambda \Rightarrow$ rigid R-matrix over V

Example:

- $H = B_q \subset U_q(\mathfrak{sl}_2)$: Borel Hopf subalgebra
  $B_q = \mathbb{C}\left\langle a^{\pm 1}, b \mid ab = qba,\ \Delta a = a \otimes a,\ \Delta b = a \otimes b + b \otimes 1 \right\rangle$
- λ: irreducible n-dim representation of $D(B_q)$
- $\Rightarrow$ rigid R-matrix $\Rightarrow$ n-coloured Jones polynomial $J_n(K; q)$

Braided Hopf Algebras

Hopf Objects in the Category of Braided Vector Spaces

A braided vector space is a vector space V with an R-matrix.

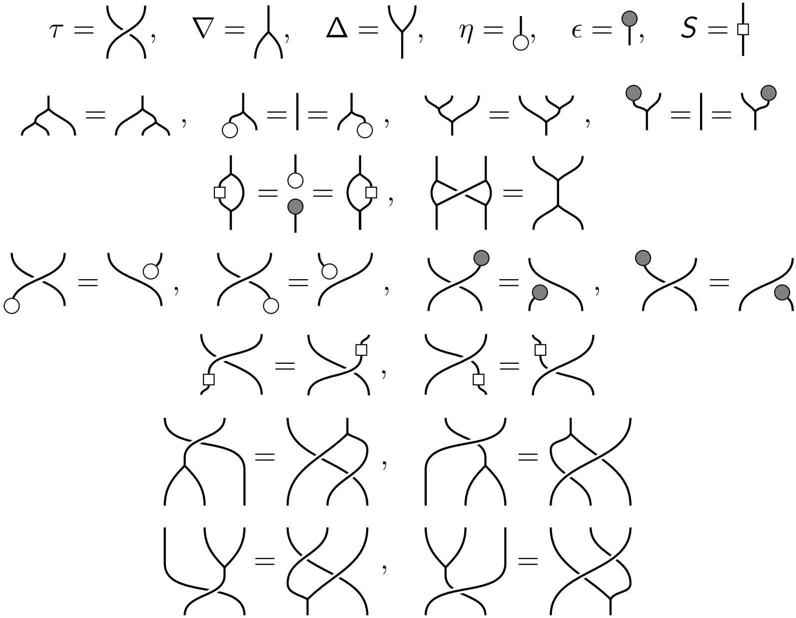
A braided Hopf algebra is a braided vector space (H, τ) with structural maps:

$$\nabla, \quad \Delta, \quad \eta, \quad \epsilon, \quad S$$

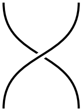
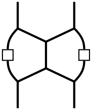
(product, coproduct, unit, counit, antipode), which are morphisms in the braided category and satisfy the usual Hopf algebra axioms (associativity, unitality, coassociativity, counitality, antipode invertibility) together with the braided compatibility condition

$$(\nabla \otimes \nabla)(\mathrm{id}_H \otimes \tau \otimes \mathrm{id}_H)(\Delta \otimes \Delta) = \Delta \nabla$$

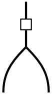
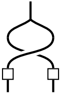
Graphical notation



Braiding in terms of structural maps

$$\tau = (\nabla \otimes \nabla)(S \otimes (\Delta\nabla) \otimes S)(\Delta \otimes \Delta) \Leftrightarrow$$  $=$ 

Relation between product, braiding and antipode

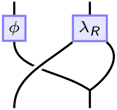
$$S\nabla = \nabla\tau(S \otimes S) \Leftrightarrow$$  $=$ 

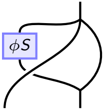
Example: Braided vector space (V, τ)

- $\Rightarrow\ T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$: braided Hopf algebra, deg 1 elements primitive, braiding $\hat{\tau}$ of $T(V)$ induced from τ
- $\Rightarrow\ \mathfrak{B}(V) = T(V)/\mathfrak{J}_V$ (Nichols algebra)
- $\mathfrak{J}_V$: Hopf ideal generated by primitive elements of degree ≥ 2

R-Matrices from Braided Hopf Algebras

Automorphisms Replace Representations

- H: braided Hopf algebra with invertible antipode
- $\phi \in \mathrm{Aut}(H)$: braided Hopf algebra automorphism
- $\Rightarrow$ R-matrix: $\rho =$ 

$\in \mathrm{GL}(H^{\otimes 2}), \quad \lambda_R =$ 

Example: Polynomial algebra: $\mathbb{C}[x] \simeq T(\mathbb{C}x)$

- $\tau(x \otimes x) = qx \otimes x$, $\phi x = q^{-n}x$, $n \in \mathbb{Z}_{\geq 0}$
- $\Rightarrow \rho_n = \rho|_{Y^{\otimes 2}}$: R-matrix over $Y = \mathbb{C}[x]_{\leq n}$
- $\Rightarrow$ $n+1$-coloured Jones polynomial $J_{n+1}(K; q)$ [Garoufalidis–K, 2023]

Exterior Algebras $\Lambda_p(V)$

Theorem (Mangazeev–K, 2025)

Let (V, τ_p) be a braided vector space with linearly ordered basis $\mathbb{B} \subset V$, and braiding

$$\tau_p(a \otimes b) = \left(p^{\theta_{a,b}} - 1\right) a \otimes b - p^{\theta_{b,a}} b \otimes a, \quad \forall a, b \in \mathbb{B},$$

where

$$\theta_{a,b} \in \{0, 1\}, \quad \theta_{a,b} = 1 \Leftrightarrow a > b.$$

Then the Nichols algebra $\mathfrak{B}(V) = T(V)/\mathfrak{J}_V$ is isomorphic to the exterior algebra $\bigwedge V$. It is a braided Hopf algebra, denoted $\Lambda_p(V)$, where all elements of V are primitive and the braiding $\hat{\tau}$ is induced by τ_p.

Set-theoretic basis of $\Lambda_p(V)$

Denote $\mathcal{P}_{\mathrm{fin}}(\mathbb{B}) := \{A \mid A \subseteq \mathbb{B},\ |A| < \infty\} = \bigsqcup_{k=0}^{\infty} \binom{\mathbb{B}}{k}$, where

$$\binom{\mathbb{B}}{k} := \{A \in \mathcal{P}_{\mathrm{fin}}(\mathbb{B}) \mid |A| = k\}.$$

Set-theoretic basis: $\{f_A \mid A \in \mathcal{P}_{\mathrm{fin}}(\mathbb{B})\}$,

$$A = \{a, b, \ldots, c\}, \quad a < b < \cdots < c \Rightarrow f_A = ab \cdots c$$

Denoting

$$\theta_{A,B} := \sum_{a \in A} \sum_{b \in B} \theta_{a,b}, \quad \gamma_k := (-1)^k p^{k(k-1)/2},$$

we have

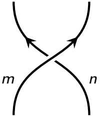
$$f_E f_F = \delta_{|E \cap F|,0} (-1)^{\theta_{E,F}} f_{E \cup F}$$

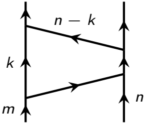
$$\Delta f_E = \sum_{A \subseteq E} (-p)^{\theta_{A,E \setminus A}} f_A \otimes f_{E \setminus A}$$

$$S f_E = \gamma_{|E|} f_E$$

Braiding of $\Lambda_p(V)$ via MOY calculus

Braiding $\hat{\tau}$ of $\Lambda_p(V)$ is given by MOY diagrammatic formula

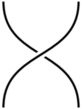
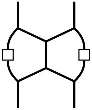
$$\hat{\tau}(\pi_m \otimes \pi_n) =$$ 

$$= \sum_{k=0}^{\min(m,n)} \gamma_k$$ 

with projections to homogeneous subspaces

$$\pi_k \colon \Lambda_p(V) \to \Lambda_p^k(V), \quad f_A \in \Lambda_p^k(V) \Leftrightarrow |A| = k$$

Similar formula is derived by [Murakami–Ohtsuki–Yamada, 1998] and [Cautis–Kamnitzer–Morrison, 2014].

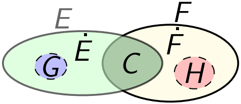
New proof is based on equality $\hat{\tau} =$  $=$ 

Matrix coefficients of $\hat{\tau}$

Let E, F, G, H be such that $G \subseteq E \setminus F,\ H \subseteq F \setminus E$.
Define

$$C := E \cap F, \quad \dot{E} := (E \setminus F) \setminus G, \quad \dot{F} := (F \setminus E) \setminus H,$$

$$E' := (E \setminus G) \cup H, \quad F' := (F \setminus H) \cup G,$$



and

$$\beta_{E,G;F,H} = (-1)^{\theta_{F,E}+\theta_{F',E'}} p^{\theta_{G \sqcup C,E}+\theta_{\dot{F},E'}} \prod_{A \in \binom{G}{1}} (p^{\theta_{A,H}-\theta_{A,G}} - 1)$$

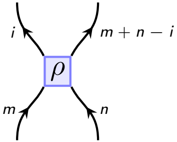
We have

$$\hat{\tau}(f_E \otimes f_F) = (-1)^{|E||F|} \sum \beta_{E,G;F,H} f_{F'} \otimes f_{E'}$$

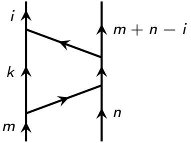
where summation runs over G and H satisfying $|G| = |H|$

R-matrix ρ

Let $\phi \in \mathrm{Aut}(\Lambda_p(V))$ be defined by $\phi f_A = t^{|A|} f_A$, $t \neq 0$. Then the associated R-matrix ρ has the form



$$= t^i (tp^m; p)_{n-i} \sum_{k=0}^{\min(i,m)} \gamma_k$$



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Furthermore,

$$\rho(f_E \otimes f_F) = (-1)^{|E||F|} \sum r_{E,G;F,H} f_{(F \setminus H) \sqcup G} \otimes f_{(E \setminus G) \sqcup H}$$

with summation running over $G \subseteq E \setminus F$ and $H \subseteq F \setminus E$ satisfying $|G| \leq |H|$, and

$$r_{E,G;F,H} = t^{|F|-|H|+|G|} (tp^{|E|}; p)_{|H|-|G|} \beta_{E,G;F,H}$$

Further perspectives

Let $\dim(V) = N$. Then the knot invariant associated with ρ is conjecturally the invariant $\mathrm{LG}^{(N)}(p, t)$ of [Links–Gould, 1993] associated with 2^N-dim representations of super quantum group $U_q(\mathfrak{gl}(N|1))$.

Calculations for few knots with $N = 2, 3, 4$ are consistent with results of [De Wit, 2001]

Future directions include:

- Computation of knot invariants without fixing N;
- Representation theory of $U_q(\mathfrak{gl}(N|1))$ and construction of coloured versions of Links–Gould invariants.