Heights of Drinfeld Modules

Zhenlin Ran

University of Newcastle

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Number Theory Down Under 11

Outline of this talk

1. Background
   - Absolute values and Weil heights
   - Overview on Drinfeld modules
   - Modular heights
2. Results on modular heights
   - Variation of Taguchi heights
   - Analogous result of Nakkajima and Taguchi's theorem
   - Lower bound of the set of the Weil heights of singular moduli

Extension of absolute values

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C : a projective, geometrically irreducible and smooth curve over a finite field $\mathbb{F}_q$, e.g. the projective line $\mathbb{P}^1$ over $\mathbb{F}_q$

k : the function field of C, e.g. $\mathrm{F}_q(t)$.

M_k : the set of places of k.

∞ : a fixed closed point of C, representing a place $\infty \in M_k$.

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To each $v \in M_k$, we associate an absolute value $|\cdot|_v$ as

$$|x|_v := |\mathbf{k}(v)|^{-v(x)}, \ \forall x \in k,$$

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where $\mathbf{k}(v)$ denotes the residue field of v.

If F/k is a field extension of finite degree and M_F is the set of places of F, for any $w \in M_F$ which lies over $v \in M_k$ we normalize the absolute value as

$$|y|_w := |N_{F_w/k_v}(y)|_v^{\frac{1}{[F:k]}},\ \forall y \in F.$$

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For any $y \in F$, we set:

$$|y| := \prod_{\substack{w \in M_F \\ w|\infty}} |y|_w.$$

It corresponds to the usual archimedean absolute value on the complex numbers $\mathbb{C}$.

Weil heights

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Definition

Let $\mathbf{x} = (x_0 : \cdots : x_n) \in \mathbb{P}^n(\overline{k})$ and F be a finite extension of k containing these coordinates. The *Weil height* of $\mathbf{x}$ is:

$$h(\mathbf{x}) := \sum_{w \in M_F} \max_j \log |x_j|_w.$$

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Theorem (Northcott)

For any $H \geq 0$ and $D \geq 0$, the set $\{\alpha \in k^{\text{sep}} : h(\alpha) \leq H, \deg(\alpha) \leq D\}$ is finite.

Drinfeld modules

Drinfeld modules

A : ring of functions in k that are regular outside ∞.

k_∞: the completion of k with respect to the absolute value $|\cdot|_\infty$, analogue of real numbers

$\mathbb{C}_\infty$: the completion of an algebraic closure of k_∞ with respect to the absolute value extended from $|\cdot|_\infty$, analogue of complex numbers.

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Let L be an A-field, i.e. a field L together with a ring homomorphism $\gamma : A \to L$ which is called the *characteristic* of L. If $\ker(\gamma) = (0)$, we say γ is a *generic characteristic*. So k_∞ and $\mathbb{C}_\infty$ are A-fields with a natural generic characteristic.

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Let $\mathbb{G}_a = \operatorname{Spec} L[X]$ be the additive group over L. It is known that

$$\operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_a) = L\{\tau\} := \left\{ \sum_{i=0}^{n} a_i \tau^i : a_i \in L, n \in \mathbb{N}. \right\},$$

where $L\{\tau\}$ is the ring of twisted polynomials over L and τ is the q-th Frobenius such that $\tau \cdot x = x^q \cdot \tau$, $\forall x \in L$.

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Let L be an A-field with characteristic γ. A *Drinfeld A-module* ϕ over L is a ring homomorphism $\phi : A \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a) = L\{\tau\}$ such that

1. $\partial \circ \phi(a) = \gamma(a)$, where ∂ is the differentiation operator.
2. there exists some $0 \neq a$ such that $\phi(a) \neq \gamma(a)\tau^0$.

Given a Drinfeld A-module ϕ over L, the *rank* of ϕ is an integer r such that

$$\deg(\phi_a(\tau)) = r \cdot \deg(a),\ \forall a \in A.$$

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Example: Let $A = \mathbb{F}_q[t]$ and ϕ be a Drinfeld A-module of rank r over $\mathbb{C}_\infty$ which is an A-field equipped with a natural generic characteristic. Then ϕ is characterized by

$$\phi_t := t\tau^0 + g_1\tau + \cdots + g_r\tau^r.$$

Drinfeld modules

It can be easily checked that

$$\phi_a(x+y) = \phi_a(x) + \phi_a(y),\ \forall a \in A, \forall x, y \in \mathbb{C}_\infty,$$

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$$\begin{array}{ccc} \mathbb{G}_a & \xrightarrow{\quad \phi \quad} & \mathbb{G}_a \\ \downarrow & & \downarrow \\ T_0(\mathbb{G}_a) & \xrightarrow{\cdot \partial(\phi)} & T_0(\mathbb{G}_a) \end{array}$$

where $\phi \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a)$ and the vertical arrows mean taking the tangent space at the identity element 0 of the additive group, and $T_0(\mathbb{G}_a)$ is the tangent space of $\mathbb{G}_a$ at 0.

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A finitely generated discrete A-submodule Λ of a normed k_∞-vector space is called an *A-lattice*. The *rank* of a lattice Λ is defined to be the dimension of the k-vector space $\Lambda \otimes_A k$.

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Remark: Let F_∞ be a complete extension of k_∞ in $\mathbb{C}_\infty$. In this talk, we focus on A-lattices $\Lambda \subset F_\infty^{\text{sep}}$ such that Λ is invariant under $\text{Gal}(F_\infty^{\text{sep}}/F_\infty)$.

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For any such lattice, we have an associated function

$$e_\Lambda(z) := z \prod_{\substack{\lambda \in \Lambda \\ 0 \neq \alpha}} \left(1 - \frac{z}{\lambda}\right)$$

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Now $\phi_a^\Lambda(z) := az \prod_{0 \neq \lambda \in a^{-1}\Lambda/\Lambda} (1 - z/e_\Lambda(\lambda))$ gives a Drinfeld A-module over $\mathbb{C}_\infty$ of the rank same as that of Λ.

Drinfeld modules - analytic theory

Theorem (Uniformization theorem)

Let F_∞ be a complete extension of k_∞ in $\mathbb{C}_\infty$, and ϕ be a Drinfeld A-module over F_∞ of rank $r > 0$. Then there is an A-lattice $\Lambda := \Lambda_\phi$ over F_∞ of rank r such that ϕ is the associated Drinfeld A-module. Moreover, the association $\phi \mapsto \Lambda_\phi$ gives rise to an equivalence of categories between the category of Drinfeld A-modules of rank r over F_∞ and the category of A-lattices of rank r over F_∞.

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Remark: the morphisms in the category of Drinfeld A-modules over an A-field L are given by $L\{\tau\} \ni f : \phi \to \varphi$ such that

$$f \circ \phi_a = \varphi_a \circ f, \forall a \in A.$$

If $f \neq 0$, then we say f is an *isogeny*.

Graded heights

F: a field in $\mathbb{C}_\infty$ that is a finite extension of k. Let ϕ be a Drinfeld A-module over F. For any $w \in M_F$, we set

$$w(\phi) := -\min_a \min_i \left\{ \frac{w(a_i)}{q^i - 1} : 0 \neq a \in A, 1 \leq i \leq r \deg(a) \right\}.$$

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Remark: the graded height of ϕ does not depend on the choice of the field F and it is invariant under isomorphisms.

Graded heights

Example: let ϕ be a Drinfeld $\mathbb{F}_q[t]$-module of rank r over F. Then it is characterised by:

$$\phi_t = t\tau^0 + g_1\tau + \cdots + g_r\tau^r,\ g_i \in F, g_r \neq 0.$$

The graded height of ϕ is then given by:

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Proposition

Let ϕ be a Drinfeld A-module of rank r over F such that F/k is a separable extension. For any $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$, we denote by $\sigma(\phi)$ the Drinfeld A-module obtained by acting σ on the coefficients of a Drinfeld A-module ϕ. Then we have

$$h_G(\phi) = h_G(\sigma(\phi)).$$

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Definition

Let ϕ be a Drinfeld A-module over F with everywhere stable reduction. The *stable Taguchi height* of ϕ is defined by

$$h_{\mathrm{Tag}}^{\mathrm{st}}(\phi/F) = \frac{1}{[F:k]} \left(\sum_{w \in M_F^{\mathrm{fin}}} \deg(w) w(\phi) - \sum_{w \in M_F^{\infty}} \epsilon_w \log D_A(\Lambda_w) \right),$$

where M_F^{fin} (resp. M_F^{∞}) denotes the set of finite (resp. infinite) places of F and ϵ_w is the local degree at w, and $D_A(\Lambda_w)$ is the covolume of the A-lattice Λ_w.

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Remark: the stable Taguchi height is invariant under finite field extensions so that we will just write $h^{\mathrm{st}}_{\mathrm{Tag}}(\phi)$ instead of $h^{\mathrm{st}}_{\mathrm{Tag}}(\phi/F)$.

Taguchi heights

Definition

Let Λ be an A-lattice of rank r in $\mathbb{C}_\infty$, and let $\mathcal{O}_\infty$ be the ring of ∞-adic integers in k_∞. Choose a k_∞-basis $\{\lambda_i\}_{i=1}^r$ of $k_\infty \otimes \Lambda$ such that:

1. $\lambda_i \in \Lambda$ for $1 \leq i \leq r$;
2. $|a_1\lambda_1 + \cdots + a_r\lambda_r|_\infty = \max\{|a_i\lambda_i|_\infty : 1 \leq i \leq r\}$ for all $a_1, ..., a_r \in k_\infty$;
3. $k_\infty \otimes \Lambda = \Lambda + (\mathcal{O}_\infty\lambda_1 + \cdots + \mathcal{O}_\infty\lambda_r)$.

The *covolume* $D_A(\Lambda)$ of the A-lattice Λ is defined as follows:

$$D_A(\Lambda) := q^{1-g_k} \cdot \left(\frac{\prod_{i=1}^r |\lambda_i|_\infty}{\#\left(\Lambda \cap (\mathcal{O}_\infty\lambda_1 + \cdots + \mathcal{O}_\infty\lambda_r)\right)} \right)^{\frac{1}{r}}$$

$$= \left(\frac{\prod_{i=1}^r |\lambda_i|_\infty}{\#(\Lambda/(A\lambda_1 + \cdots + A\lambda_r))} \right)^{\frac{1}{r}}.$$

Variation of Taguchi heights

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Lemma

Let $f : \phi_1 \to \phi_2$ be an isogeny of Drinfeld A-modules over F with everywhere stable reduction. Then we have:

$$h^{\mathrm{st}}_{\mathrm{Tag}}(\phi_2) - h^{\mathrm{st}}_{\mathrm{Tag}}(\phi_1) = \frac{1}{r}\log|\deg(f)| - \frac{1}{[F:k]}\log\#(R/D_f),$$

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Theorem (Ran)

Let $f : \phi_1 \to \phi_2$ be an isogeny of Drinfeld A-modules over F with everywhere good reduction. Then we have:

$$h^{\mathrm{st}}_{\mathrm{Tag}}(\phi_2) - h^{\mathrm{st}}_{\mathrm{Tag}}(\phi_1) = \frac{1}{r}\log|\deg(f)| - \log|f_0| + h^{\mathrm{fin}}_{G}(\phi_2) - h^{\mathrm{fin}}_{G}(\phi_1),$$

where $f_0 = \partial(f)$ and $h^{\mathrm{fin}}_G(\phi_j) = \sum_{w\in M_F^{\mathrm{fin}}}\deg(w)w(\phi_j), j = 1, 2$.

Analogous result of Nakkajima and Taguchi's theorem

Theorem

Let $A = \mathbb{F}_q[t]$, and ϕ_1, ϕ_2 be two Drinfeld A-modules of rank 2 with CM by $\mathcal{O}_K$ and $\mathcal{O}$ respectively, where K is an imaginary quadratic field and $\mathcal{O}_K$ (resp. $\mathcal{O}$) is the maximal (resp. an arbitrary) order. We write $\mathcal{O} = A + f_0\mathcal{O}_K$ for some $f_0 \in A$. Then

$$h^{\mathrm{st}}_{\mathrm{Tag}}(\phi_2) - h^{\mathrm{st}}_{\mathrm{Tag}}(\phi_1) = \frac{1}{2}\log|f_0| - \frac{1}{2}\sum_{v|f_0} \deg(v) e_{f_0}(v),$$

where v runs over all monic prime factors of f_0 and for $l := q^{\deg(v)}$

$$e_{f_0}(v) = \frac{(1-\chi(v))(1-l^{-v(f_0)})}{(l-\chi(v))(1-l^{-1})},$$

and $\chi(v) = 1$ if v splits in K; $\chi(v) = 0$ if v ramifies in K; $\chi(v) = -1$ if v is inert in K.

Variation of graded hehgits

Corollary

Assume the same conditions as the previous theorem. The following formula is true

$$h_G(\phi_2) - h_G(\phi_1) = \log|f_0| - \frac{1}{2}\sum_{v|f_0} \deg(v)e_{f_0}(v) + h_G^{\infty}(\phi_2') - h_G^{\infty}(\phi_1'),$$

where ϕ_1' is the Drinfeld A-module given by the lattice $\mathcal{O}_K$ and ϕ_2' is given by $\mathcal{O}$, and for $j = 1, 2$

$$h_G^{\infty}(\phi_j') = \sum_{w|\infty} \deg(w)w(\phi_j').$$

Lower bound of the Weil heights of singular moduli

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Some setups: we fix $A = \mathbb{F}_q[t]$ and ϕ a rank 2 A-Drinfeld module over $\mathbb{C}_\infty$. Then it is characterized by

$$\phi_t = t\tau^0 + g\tau + \Delta\tau^2.$$

The *j-invariant* of ϕ is given by $\frac{g^{q+1}}{\Delta}$. A *singular modulus* is the j-invariant of a Drinfeld module over $\mathbb{C}_\infty$ with complex multiplication.

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Theorem (Ran)

Let J be a singular modulus of a rank 2 CM Drinfeld A-module ϕ. Let δ be the discriminant of the endomorphism ring of ϕ with conductor f_0. There exists some computable constant C_q with respect to q such that

$$h(J) \geq (q^2-1)\left(\frac{1}{2} - \frac{1}{\sqrt{q}+1}\right)\log\sqrt{|\delta|} + \left(\frac{1}{2} + \frac{1}{\sqrt{q}+1}\right)\log|f_0| - \frac{9}{4}\log\log|f_0| - C_q.$$

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3. Apply a result from Breuer-Pazuki-Razafinjatovo, which essentially tells us

$$|h_G^{\infty}(\phi) - h_G^{\infty}(\phi')| \leq \frac{q}{q-1} - \frac{q^r}{q^r-1}.$$

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3. Apply a result from Breuer-Pazuki-Razafinjatovo, which essentially tells us

$$|h_G^{\infty}(\phi) - h_G^{\infty}(\phi')| \leq \frac{q}{q-1} - \frac{q^r}{q^r-1}.$$

4. Apply Wei's work on the the Colmez conjecture for Drinfeld modules.

Lower bound of the Weil heights of singular moduli

Sketch of the proof:

1. Notice that $(q^2-1)h_G(\phi) = h(J)$.
2. Do the following estimate:

$$\frac{1}{2}\sum_{v|f_0} \deg(v) e_{f_0}(v) \leq \frac{9}{4}\log\log|f_0| + C_q,$$

   where C_q is a computable constant depending on q.
3. Apply a result from Breuer-Pazuki-Razafinjatovo, which essentially tells us

$$|h_G^{\infty}(\phi) - h_G^{\infty}(\phi')| \leq \frac{q}{q-1} - \frac{q^r}{q^r-1}.$$

4. Apply Wei's work on the the Colmez conjecture for Drinfeld modules.
5. Note that $h_G(\phi) \geq h_{\mathrm{Tag}}^{\mathrm{st}}(\phi)$.

End

Thank you!