Reminiscences relating to Rodney's work in 1983 and 1984

Peter Forrester @UniMelb

- StatMech@Unimelb 1980-1982
- With Rodney@ANU 1983-1984: Eight vertex solid-on-solid model
- With Rodney@ANU 1983-1984: Zamolodchikov model
- Legacy in Australia of the Baxter School of exact solutions in Stat Mech

Baxter 2025: Exactly Solved Models and Beyond, Sept 9th

StatMech@Unimelb 1980-1982

- Physics department: Ken Hines, Norm Frankel (plasma physics)

  Bruce McKellar (3rd year Stat Mech lecturer 1980)

- Maths department: Colin Thompson (text book: Mathematical Statistical Mechanics, then researching models of chaos),

  Ed Smith (fluids, systems with long range forces)

- Honours project supervised by Ed Smith on "Lattice sums" 1981

  (slowly convergent series giving the electrostatic potential in crystals)

  ★ Made extensive use of Jacobi theta functions.

e.g.

$$\left(\sum_{l=-\infty}^{\infty}(-1)^l q^{(3/2)l^2+l/2}\right)^3=\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}$$

★ $\Longrightarrow$ Exact result

$$\sum_{(n_1,n_2,n_2)\in\mathbb{Z}^3} \frac{(-1)^{n_1+n_2+n_3}}{\sqrt{(n_1-1/6)^2+(n_2-1/6)^2+(n_3-1/6)^2}} = \sqrt{3}$$

(Electrostatic potential at the point $(1/6,1/6,1/6)$ of the NaCl crystal)

★ Studying lattice sums and identities lead to the work of Ramanujan

$\in \mathbb{Z}_{\geq 0}$ (being edited by B. Berndt).

e.g. $$\sum_{n=1}^{\infty} \frac{n^{2M+1}}{e^{2\pi n}-1} = \int_0^{\infty} \frac{x^{4M+1}}{e^{2\pi x}-1}\,dx$$ (published in Rocky M. J. Math '83).

(no application to physics: instead mathematical aesthetics)

★ The Rogers-Ramanujan identities featured prominently in my reading

e.g. $$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)^n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

★ Browsing the physics library's new journal shelves I came across Rodney's J. Stat. Phys. paper:

Rogers–Ramanujan Identities in the Hard Hexagon Model

R. J. Baxter[1,2]

Received March 27, 1981

The hard hexagon model in statistical mechanics is a special case of a solvable class of hard-square-type models, in which certain special diagonal interactions are added. The sublattice densities and order parameters of this class are obtained, and it is shown that many Rogers–Ramanujan-type identities naturally enter the working.

KEY WORDS: Statistical mechanics; lattice statistics; Rogers–Ramanujan identities; hard hexagon model; combinatorial identities; basic hypergeometric series.

★ In 1982 I wrote my MSc thesis on the topic of "Soluble Coulomb systems" relying on methods in random matrix theory due to Dyson, Gaudin, Mehta supervised by Ed Smith.

Later in 1982 Ed Smith suggested I contact Rodney about a PhD with him.

The Australian National University

Institute of Advanced Studies

19th October, 1982.

Dear Mr. Forrester,

Perhaps I should warn you that 90% of my time is spent exploring ideas that lead nowhere, 9% is spent doing tedious algebra on the tractable problems, and 1% (right at the end, when the thermodynamic limit is taken) involves any analysis.

You are probably only too well aware of the present state of the job market, but let me warn you that a Ph.D. in exactly solved statistical mechanical models will probably lead to a post-doc, and then another post-doc, until you are in your thirties (or mid-thirties) and still without a permanent job. In fact, exactly solved models may be even worse in this regard than other Ph.D. topics (except that mathematical skills do perhaps give one a bit of flexibility).

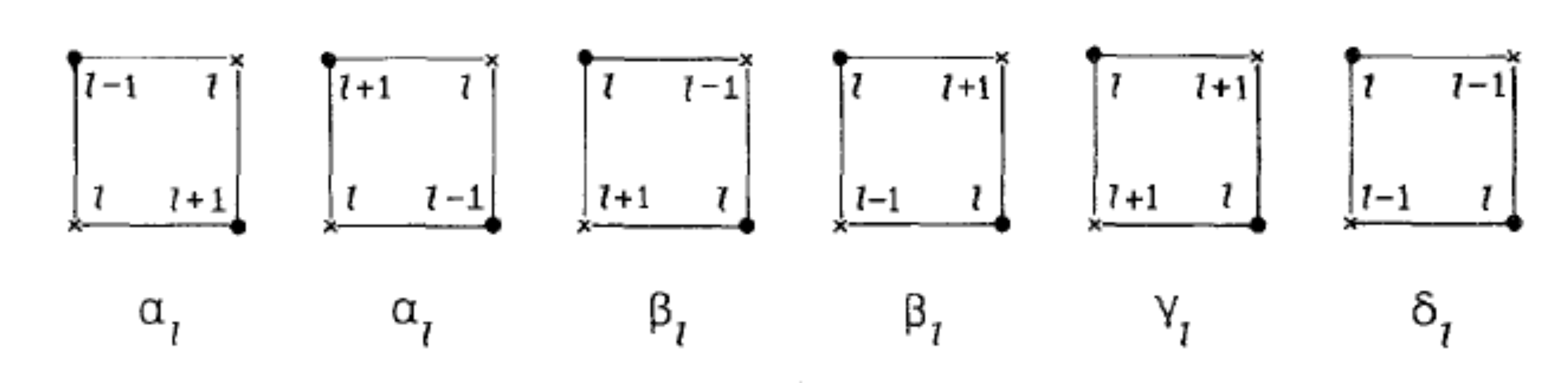
Having said that, I still hope I shall hear from you soon.

Yours sincerely,

R J Baxter

PROFESSOR R.J. BAXTER.

With Rodney@ANU 1983-1984: Eight vertex solid-on-solid model



$1 \leq l \leq r-1$

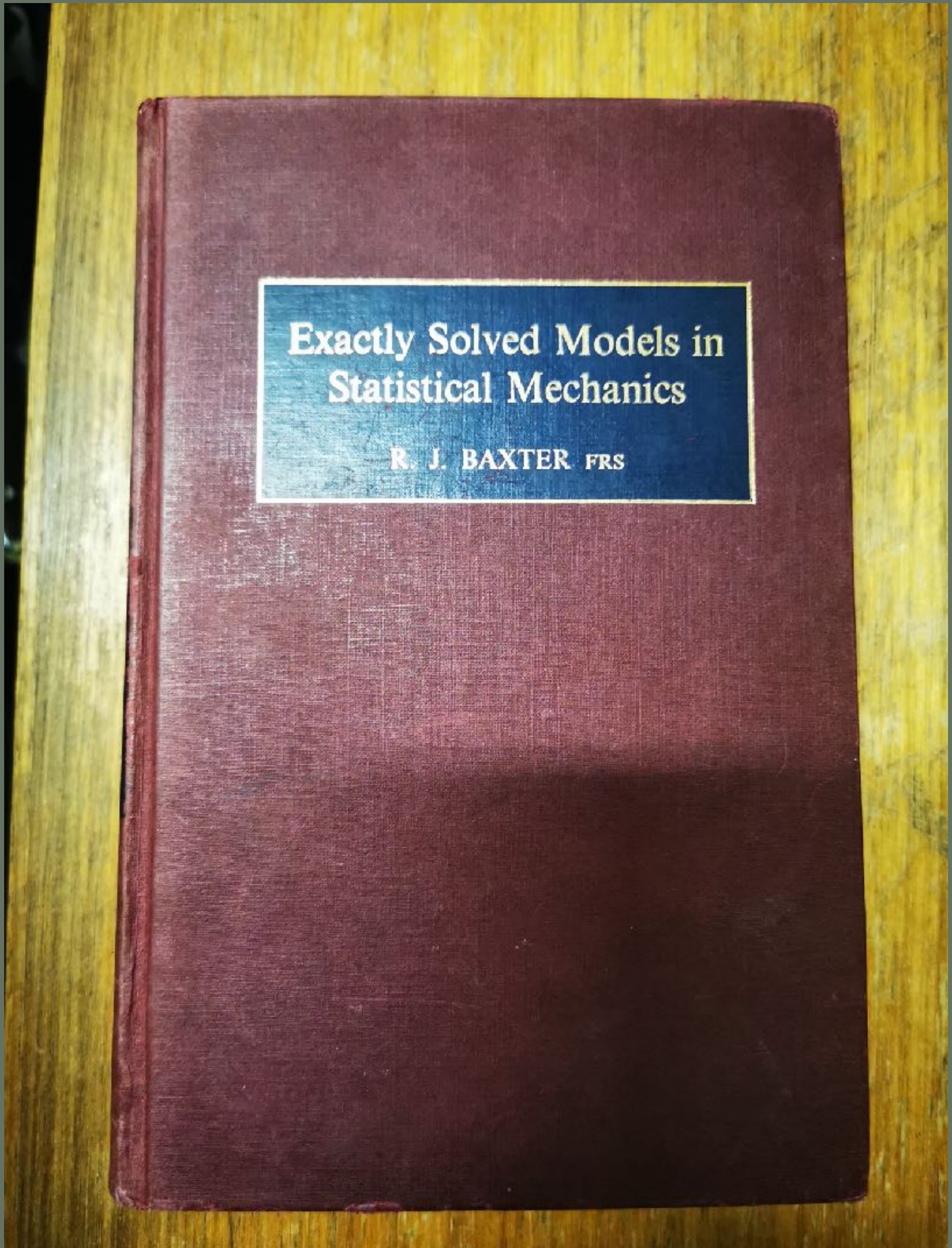
(four distinct regimes)

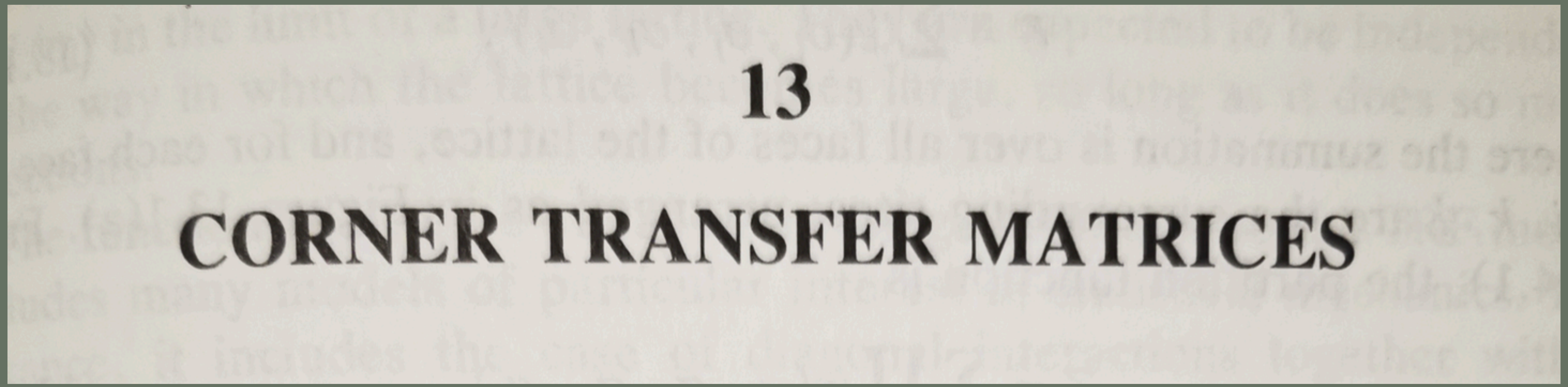
★ Began as a project between Rodney and George Andrews at the beginning of 1983 when George Andrews visited ANU for several months

★ The project was initially called “Decorated hard hexagon model”

The case $r = 5$ reclaims the hard hexagon model

★ Soon after Rodney noticed an “equivalence” with the eight vertex model (common transfer matrix eigenvalues). However the order parameters are distinct. Rodney’s theory of corner transfer matrices was crucial.

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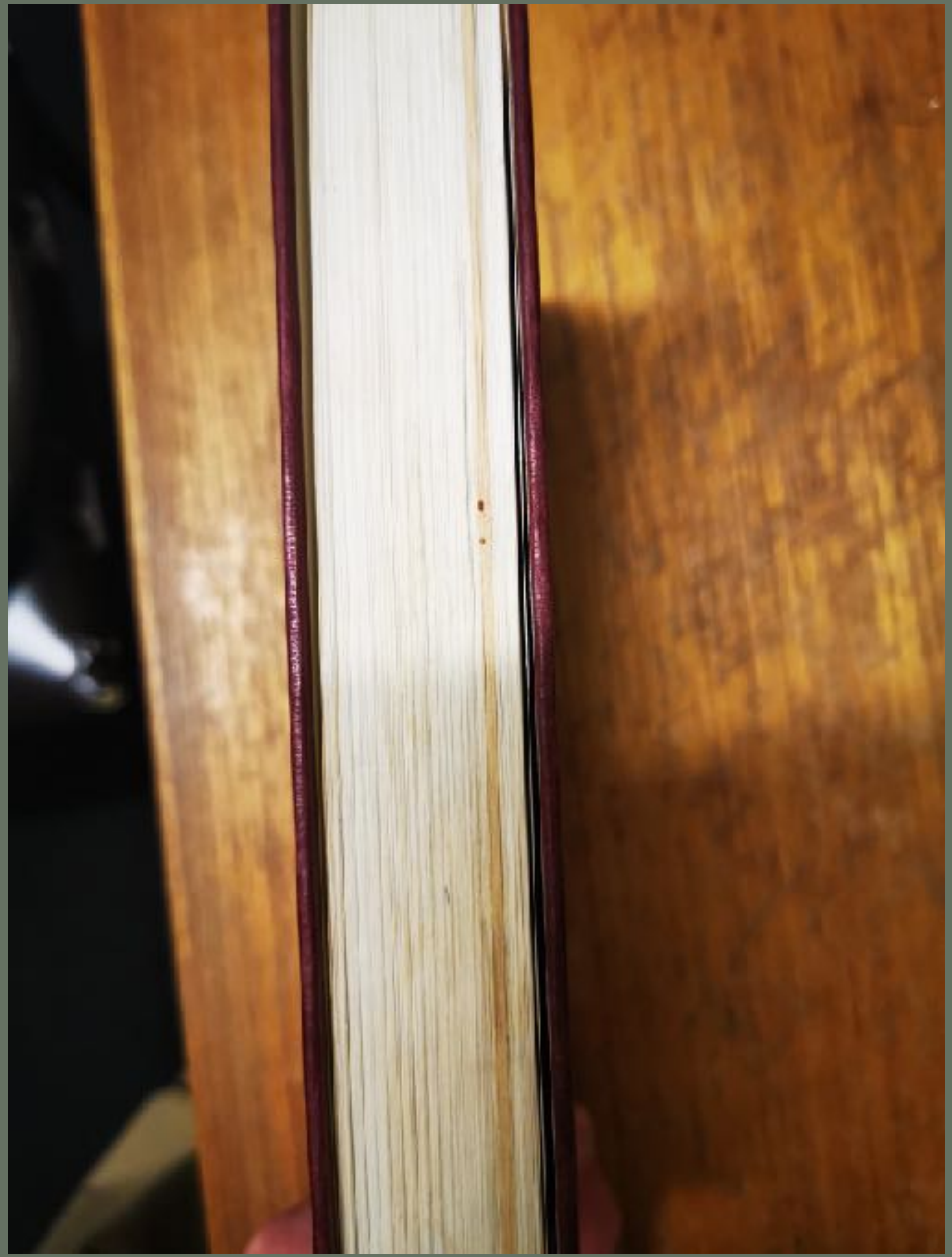
CORNER TRANSFER MATRICES



- Publication of this work in J. Stat. Phys.:

Eight-Vertex SOS Model and Generalized Rogers–Ramanujan-Type Identities

George E. Andrews,[1,2] R. J. Baxter,[3] and P. J. Forrester[3]

Received December 21, 1983

coincided with the classification of two-dimensional critical exponents, assuming conformal invariance and unitarity, by Frieda, Qiu and Shenker (in turn inspired by Belavin, Polyakov, Zamolodchikov)

- The critical exponents in Regime III realised all those in the classification.

(first pointed out by D.A. Huse)

With Rodney@ANU 1983-1984: Zamolodchikov model

★ A particular interation-round-a-cube model, proposed by Zamolodchikov (CMP ’81) as satisfying the 3D Yang-Baxter equations (tetrahedron equations) Has the (drawback) that some weights are negative.

★ Proved to satisfy the tetrahedron equations by Rodney (CMP ’83). Spherical trigonometry was important.

★ The weights are trigonometric functions, and Rodney speculated it is critical.. He introduced a temperature like variable. This model was studied using a variational approach (3D, but scalar version, of that underlying the corner transfer matrix formalism).

★ (published in JPhyA)

Is the Zamolodchikov model critical?

R J Baxter and P J Forrester

Department of Theoretical Physics, Research School of Physical Sciences, The Australian National University, Canberra, Australia 2601

Received 27 November 1984

Abstract. Evidence is presented in favour of the hypothesis that the Zamolodchikov model (an exactly solvable three-dimensional lattice model without any temperature-like parameters) is critical. The evidence is obtained by generalising the Zamolodchikov model to include a temperature-like variable. The magnetisation curve of this model is then studied using a modified form of a variational approximation formulated earlier. Also we show the two-layer Zamolodchikov model corresponds to a critical free-fermion model.

★

Free energy of the Zamolochikov model was computed by Rodney, Friday, July 13th 1984

ZFE   Friday 13th July 1984 (1)

Zamolodchikov Free Energy

Take weights as in Baxter, Comm. Math. Phys. 88, 185.

e.g.
$$W(+,\dots,+) = P_0 - Q_0 = 1 - \sqrt{\tan\tfrac{\alpha_0}{2}\tan\tfrac{\alpha_1}{2}\tan\tfrac{\alpha_2}{2}\tan\tfrac{\alpha_3}{2}}$$

$$W(+,\dots,+,-) = R_0 = \sqrt{\frac{\sin\alpha_0/2}{\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\cos\frac{\alpha_3}{2}}}$$

Then commutation + factorization + symmetry give

$$\frac{1}{N}\ln Z = \ln \kappa = \frac{1}{2}\ln\left(\frac{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2}}{c_0 c_1 c_2 c_3}\right) - \frac{1}{2\pi}\left(a\ln T_1 + b\ln T_2 + c\ln T_3\right)$$
$$+ \frac{\sin\theta_2\sin\theta_3}{\pi}\int_0^a \frac{x\sin x \, dx}{1 - [\sin\theta_2\sin\theta_3\cos x - \cos\theta_2\cos\theta_3]^2}$$

Deriv w.r.t. θ_1 $\propto \frac{a}{\sin\theta_1}$

Legacy in Australia of the Baxter School of exact solutions in Stat Mech

- Rodney's great individual achievements on the international stage and the far reaching consequences of his work has given inspiration, and a high standing, to the broader field of solvable and integrable systems in the Australian mathematical sciences. $\Longrightarrow$ jobs, funding, positions of influence.

- The critical mass of researchers so assembled has given an internationally. recognised theme for a large body of research in mathematical physics in . Australia, and a branding.

Cf. Kyoto School (Jimbo and Miwa)
Leningrad School (Fadeev)
Stony Brook Yang ITP School (Yang)