Symbolic Dynamics and Complex Diophantine Approximation

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Joint with F. García-Ramos and M. Hussain

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Some authors that have worked on the topic

- Alexander Kechris
- Haseo Kim, Tom Linton
- Verónica Becher, Sergio Yuhjtman
- Oleksandr Sharkovsky, Andrei Sivak
- Dylan Airey, Steven Jackson, Dominik Kwietniak, and Bill Mance

Definition

The ***nearest Gaussian integer function*** $[\cdot] : \mathbb{C} \to \mathbb{Z}[i]$ *is given by*

$$\forall z \in \mathbb{C} \quad [z] := \left\lfloor \Re(z) + \frac{1}{2} \right\rfloor + i \left\lfloor \Im(z) + \frac{1}{2} \right\rfloor .$$

Define $\mathfrak{F} := \{z \in \mathbb{C} : [z] = 0\}$ *and* $T : \mathfrak{F} \to \mathfrak{F}$ *by*

$$\forall z \in \mathfrak{F} \quad T(z) = \begin{cases} z^{-1} - [z^{-1}], & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Define $a_1 : \mathfrak{F} \setminus \{0\} \to \mathbb{Z}[i]$ *by* $a_1(z) = [z^{-1}]$, $a_n(z) = a_1(T^n(z))$ *if* $T^n(z) \neq 0$, *and* $a_0 : \mathbb{C} \to \mathbb{Z}[i]$ *by* $a_0(z) = [z]$. *The* ***Hurwitz continued fraction*** *of a complex number* z *is*

$$a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \cfrac{1}{\ddots}}}.$$

Theorem (A. Hurwitz, 1887)

Let ζ be any complex number.

1. *If $\zeta \in \mathbb{Q}(i)$, the Hurwitz continued fraction algorithm associates to each a finite sequence $(a_j)_{j=0}^{n}$ in $\mathbb{Z}[i]$ such that*

$$\zeta = [a_0; a_1, \ldots, a_n].$$

2. *If $\zeta \in \mathbb{C} \setminus \mathbb{Q}(i)$, the algorithm associates an infinite sequence $(a_n)_{n\geq 1}$ in $\mathbb{Z}[i]$ such that*

$$\zeta = [a_0; a_1, \ldots].$$

Hurwitz Continued Fraction Process

For any $z \in \mathbb{C} \setminus \{0\}$, put $\iota(z) = z^{-1}$. The set $\iota[\mathfrak{F}]$, the image of $\mathfrak{F}$ under ι, is

$$\iota[\mathfrak{F}] = \mathbb{E}(1) \cap \overline{\mathbb{E}}(i) \cap \overline{\mathbb{E}}(-1) \cap \mathbb{E}(-i),$$

where

$$\mathbb{D}(z;\rho) = \{w \in \mathbb{C} : |z-w| < \rho\},$$
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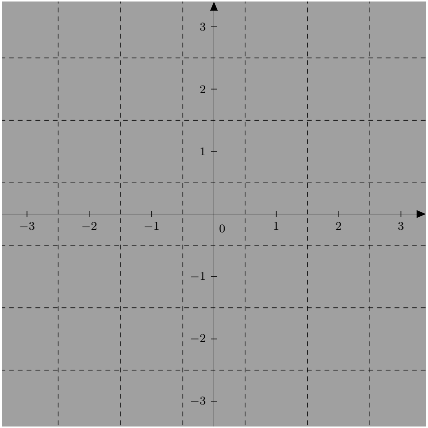

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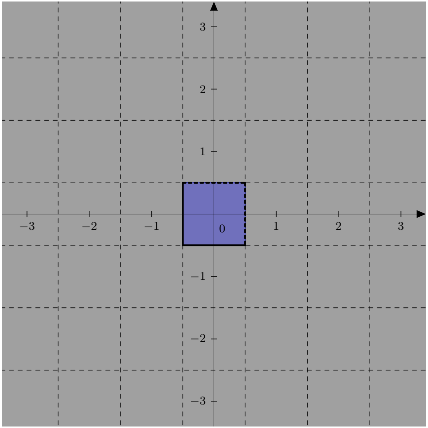

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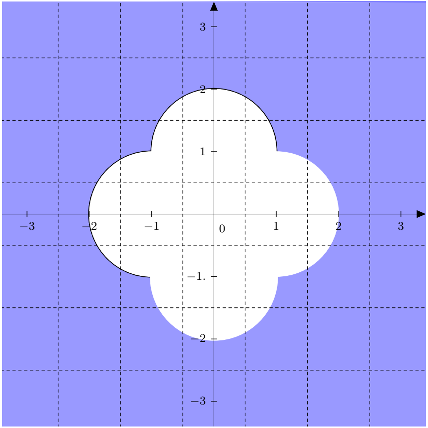

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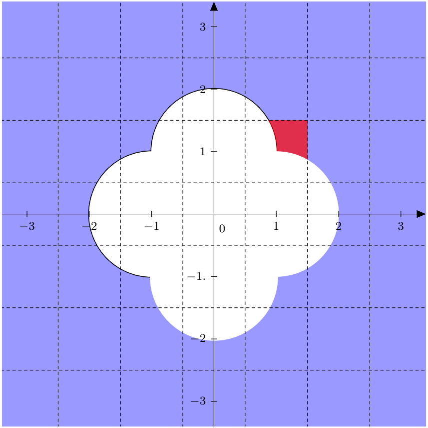

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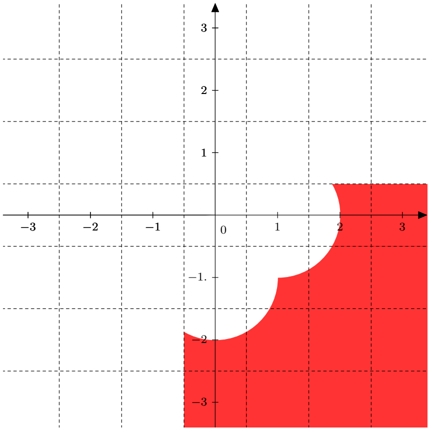

Definitions

Definition

Write $\mathcal{D} := \{a \in \mathbb{Z}[i] : |a| \geq \sqrt{2}\}$.

Definition

For each $n \in \mathbb{N}$ *and each* $\mathbf{b} \in \mathcal{D}^n$*, define*

$$\mathcal{C}_n(\mathbf{b}) := \{z \in \mathfrak{F} : a_1(z) = b_1, \ldots, a_n(z) = b_n\}.$$

Partition of $\mathfrak{F}$ by cylinders

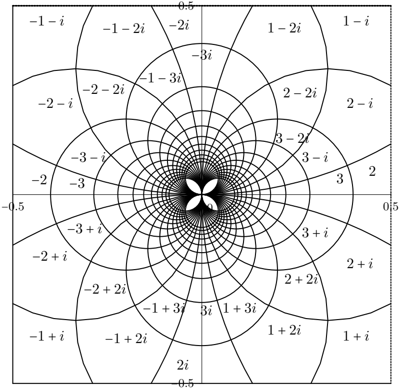

Figure 1: Partition of $\mathfrak{F}$.

Normal numbers

Theorem (H. Nakada, 1976)

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Definition

*A number $z \in \mathfrak{F}$ is **normal** if for all $n \in \mathbb{N}$ and all $\mathbf{a} \in \mathcal{D}^n$, we have*

$$\lim_{N\to\infty} \frac{\#\left\{0 \leq j \leq N-1 : T^j(z) \in \mathcal{C}_n(\mathbf{a})\right\}}{N} = \mu\left(\mathcal{C}_n(\mathbf{a})\right).$$

We denote the set of normal numbers by $\operatorname{Norm}(\mu)$.

A theorem on normal numbers

Theorem

The set $\mathrm{Norm}(\mu)$ *is* $\boldsymbol{\Pi}^0_3(\mathbb{C})$*-complete.*

Some symbolic dynamics

Let $\mathcal{A}$ be a non-empty and at most countable set.

Definition

*For $n \in \mathbb{N}$ and $\mathbf{v} = v_1 \cdots v_n$, $\mathbf{w} = w_1 \cdots w_n \in \mathcal{A}^n$, the **normalized Hamming distance** between $\mathbf{v}$ and $\mathbf{w}$ is*

$$d_H(\mathbf{v}, \mathbf{w}) = \frac{\#\{j \in \{1, \ldots, n\} : v_j \neq w_j\}}{n}.$$

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Definition

*A **subshift** X of $\mathcal{A}$ is a closed and shift invariant subset of $\mathcal{A}^{\mathbb{N}}$. Denote by $\mathcal{L}(X)$ the set of all finite words that appear as factors of some words in X.*
*A sub-shift X has the **right feeble specification property** if there is some $\mathcal{G} \subseteq \mathcal{L}(X)$ such that:*

1. *If $\mathbf{u}, \mathbf{v} \in \mathcal{G}$, then $\mathbf{uv} \in \mathcal{G}$;*
2. *For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $\mathbf{u} \in \mathcal{G}$ and any $\mathbf{v} \in \mathcal{L}(X)$ satisfying $|\mathbf{v}| \geq N$ there are $\mathbf{s}', \mathbf{v}' \in \mathcal{A}^{<\omega}$ satisfying*

$$|\mathbf{v}'| = |\mathbf{v}|, \quad 0 \leq |\mathbf{s}| \leq \varepsilon|\mathbf{v}|, \quad d_H(\mathbf{v}, \mathbf{v}') < \varepsilon, \quad \mathbf{usv}' \in \mathcal{G}.$$

Theorem (Airey, Jackson, Kwietniak, Mance (TAMS, 2020))

Let X be a sub-shift of $\mathbb{A}$ such that:

1. *X has the right feeble specification property,*
2. *X has at least two shift-invariant Borel probability measures.*

Let μ be a shift-invariant Borel probability measure. The set of μ-generic points G_μ is Π^0_3-hard.

Back to Hurwitz continued fractions

Proposition (Dani, Nogueira (TAMS, 2014))

Consider any $n \in \mathbb{N}$ *and any* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathrm{R}(n)$.

1. *If* $\mathrm{Pm}(a_n) \geq 3$ *and* $\mathbf{b} \in \Omega(m)$ *for some* $m \in \mathbb{N}$*, then* $\mathbf{ab} \in \Omega(m+n)$.
2. *If* $\mathbf{a} \in \mathrm{R}(n)$*, there exists some* $b \in \mathbb{Z}[i]$ *such that* $\mathrm{Pm}(b) \geq 3$ *and* $\mathbf{a}b \in \mathrm{F}(n+1)$.

Yes, but . . .

Consider the sequences

$$\mathbf{a} = (-2, 1+2i, -2+i, 1+2i, -2+i, 1+2i, -2+i, \ldots),$$
$$\mathbf{b} = (-2, 2i, 2, -2i, -2, 2i, 2, -2i, \ldots).$$

We have

- $\mathbf{a} \in \Omega$.
- $\mathbf{b} \notin \Omega$, but $(b_1, \ldots, b_n) \in \mathrm{R}(n)$ for all $n \in \mathbb{N}$.
- If $\zeta = -\frac{1}{2} + i\left(\frac{2-\sqrt{3}}{2}\right)$, then

$$\begin{aligned}\zeta &= [0; -2, 1+2i, -2+i, 1+2i, -2+i, 1+2i, -2+i, \ldots] \\ &= [0; -2, 2i, 2, -2i, -2, 2i, 2, -2i, \ldots].\end{aligned}$$

Symbolic extension of $(T, \mathfrak{F})$

Let $\Lambda : \Omega \to \mathfrak{F}$ be given by

$$\Lambda(a_1, a_2, a_3, \ldots) = [0; a_1, a_2, a_3, \ldots].$$

Let $\overline{\Lambda}$ be the unique continuous extension of $\Lambda|_{\mathsf{R}}$ to $\overline{\mathsf{R}} \subseteq \mathcal{D}^{\mathbb{N}}$.

Theorem (García-Ramos, G.R., Hussain, 2023)

$$\overline{\Lambda}\left[\overline{\mathsf{R}}\right] = \overline{\mathfrak{F}} \setminus \mathbb{Q}(i).$$

Theorem (García-Ramos, G.R., Hussain, 2023)

The function $\overline{\Lambda}$ is at most six-to-one.

Algorithm

- Input. $z = [0; a_1, a_2, \ldots] \in \mathfrak{F}$ with $(a_n)_{n\geq 1} \in \Omega \setminus \mathrm{R}$.
- Output. A sequence $\mathbf{b} = (b_j)_{j\geq 1} \in \mathcal{D}^{\mathbb{N}}$ such that: $(b_1, \ldots, b_n) \in \mathrm{R}(n)$ and $z \in \overline{\mathcal{C}}_n(a_1, \ldots, a_n)$ for all $n \in \mathbb{N}$.

1. Put $\mathbf{b}^0 :\ = (b_j^0)_{j\geq 1} :\ = (a_j)_{j\geq 1}$ and $N = 0$.
2. If possible, pick $j_N \in \mathbb{N}$ such that

$$(b_1, \ldots, b_{j_N}) \in \mathrm{R}(j_N) \quad \text{and} \quad (b_1, \ldots, b_j, b_{j_N+1}) \notin \mathrm{R}(j_N + 1). \tag{1}$$

   Put

$$b_{j_N+1}^{N+1} :\ = S(b_{j_N+1}^N), \qquad b_l^{N+1} :\ = b_l^N \text{ for all } l \in \{1, \ldots, j_N\}.$$

   If $b_{j+1}^N = 1 + im$ or $b_{j+1}^N = -1 + im$, establish $b_{j+1+n}^{N+1} :\ = -\overline{b_{j+1+n}^N}$ for every $n \in \mathbb{N}$.
   If $b_{j_N+1}^N = m + i$ or $b_{j+1}^N = m - i$, establish $b_{j_N+1+n}^{N+1} :\ = \overline{b_{j+1+n}^N}$ for every $n \in \mathbb{N}$.
   If there is no $j_N \in \mathbb{N}$ such that (1) holds, put $\mathbf{b}^{N+1} :\ = \mathbf{b}^N$.
3. Repeat step 2
4. Take $\mathbf{b} := \lim_{N\to\infty} \mathbf{b}^N$. ■

Thank you for your attention.